Homological Algebra and Differential Linear Logic

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Overview

- Differential linear logic axiomatizes a category in which objects are some sort of space with a smooth structure, i.e. a way of defining smooth maps between objects.
- Linear maps appear as the maps in the base category and smooth maps in the coKleisli category.
- The category CVS of convenient vector spaces and linear maps completely captures this intuition.
- Now that we have some good models, let's try to apply ideas from differential topology.
- In particular, let's define differential forms, de Rham cohomology, integration, etc.
- Ultimately, we wish to lift this to manifolds (Cockett, Cruttwell) which locally look like objects in our category.

Kähler Differentials

- One source of abstract differential forms is Algebraic Geometry. In AG, they are interested in solutions to systems of polynomial equations.
- Even if the field is $\mathbb{R}$ or $\mathbb{C}$, the solution set may or may not be a manifold, due to the existence of singular points.
- One can define forms anyway, via Kähler differentials.
- Instead of considering the solution set directly, it is more useful to examine the *coordinate ring*, i.e. $A = k[x_1, x_2, \ldots, x_n]/I$, where I is the ideal generated by the polynomials.
- Note that A is a commutative, associative algebra, so this makes sense in monoidal categories.

Kähler Differentials II: Definitions

The traditional notion of Kähler differentials defines the notion of a module of A-differential forms with respect to A, where A is a commutative k-algebra. Let M be a (left) A-module.

Definition

An *A-derivation* from A to M is a k-linear map $\partial : A \to M$ such that $\partial(aa') = a\partial(a') + a'\partial(a)$.

Definition

Let A be a k-algebra. A *module of A-differential forms* is an A-module Ω_A together with an A-derivation $\partial : A \to \Omega_A$ which is universal in the following sense: for any A-module M, for any A-derivation $\partial' : A \to M$, there exists a unique A-module homomorphism $f : \Omega_A \to M$ such that $\partial' = \partial f$.

Kähler Differentials III: Existence Theorem

$$
\begin{tikzcd}
A \arrow[r, "\partial"] \arrow[dr, "\partial'"'] & \Omega_A \arrow[d, dotted, "h"] \\
& M
\end{tikzcd}
$$

Lemma

For any commutative k-algebra A, a module of A-differential forms exists.

One approach is to construct the free A-module generated by the symbols $\{da \mid a \in A\}$ divided out by the evident relations, most significantly $d(aa') = ad(a') + a'd(a)$.

Kähler Differentials IV: Example

For the key example, let $A = k[x_1, x_2, \ldots, x_n]$, then Ω_A is the free A-module generated by the symbols $dx_1, dx_2, \ldots, dx_n$, so a typical element of Ω_A looks like

$$f_1(x_1, x_2, \ldots, x_n)dx_1 + f_2(x_1, x_2, \ldots, x_n)dx_2 + f_n(x_1, x_2, \ldots, x_n)dx_n.$$

Then we have

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \ldots + \frac{\partial f}{\partial x_n}dx_n$$

If V is an n-dimensional space and S(V) is the free symmetric algebra construction, then there are canonical isomorphisms:

$$\Omega_A \cong \Omega_{S(V)} \cong S(V) \otimes V.$$

Kähler Categories I: Algebra Modalities

Definition

- A symmetric monoidal category $\mathcal{C}$ is *additive* if every HomSet is an abelian group, and this is preserved by composition.
- An additive symmetric monoidal category has an *algebra modality* if it is equipped with a monad (T, μ, η) such that for every object C in $\mathcal{C}$, the object, $T(C)$, has a commutative associative algebra structure

$$m : T(C) \otimes T(C) \to T(C), \qquad e : I \to T(C)$$

and this family of associative algebra structures satisfies evident naturality conditions.

The key example is the !-functor in $\mathcal{C}^{op}$ where $\mathcal{C}$ is a model of linear logic, in particular a differential category.

Kähler Categories II: Definition (RB, Cockett, Porter, Seely)

Definition

A *Kähler category* is an additive symmetric monoidal category with

- a monad T,
- a (commutative) algebra modality for T,
- for all objects C, a module of $T(C)$-*differential forms* $\partial_C : T(C) \to \Omega_C$, *i.e.* a $T(C)$-module Ω_C, and a $T(C)$-derivation, $\partial_C : T(C) \to \Omega_C$, which is universal in the following sense: for every $T(C)$-module M, and for every $T(C)$-derivation $\partial' : T(C) \to M$, there exists a unique $T(C)$-module map $h : \Omega_C \to M$ such that $\partial; h = \partial'$.

$$\begin{tikzcd} T(C) \arrow[r, "\partial"] \arrow[dr, "\partial'"'] & \Omega_C \arrow[d, dotted, "h"] \\ & M \end{tikzcd}$$

Kähler Categories III: Examples

Theorem

The category of vector spaces over an arbitrary field is a Kähler category, with structure as described above. The monad is the free symmetric algebra monad, and the map d is the usual differential as applied to polynomials.

- Note that Vec^{op} is a differential category, and the map d is the canonical deriving transform in the definition of differential category.
- It is reasonable to ask if the opposite of every differential category is Kähler. We don't know. At the moment, we need an extra condition. The condition is minor and every example satisfies it.

Kähler Categories IV: Property and Theorem

Definition

- Let F denote the free, associative algebra monad.
- The monad T satisfies *Property* K if the natural transformation $\varphi : F \to T$ is a componentwise epimorphism.

Theorem (RB, Cockett, Porter, Seely)

If $\mathcal{C}$ is a codifferential storage category, whose monad satisfies Property K*, then $\mathcal{C}$ is a Kähler category, with $\Omega_C = T(C) \otimes C$, and the differential being the map $d\colon T(C) \to T(C) \otimes C$, the canonical differential arising from Differential Linear Logic.*

Kähler Categories V: Convenient Vector Spaces

Theorem (RB,Ehrhard,Tasson)

The category CVS *of convenient vector spaces and linear continuous maps is a differential storage category.*

One can check that in this case the (co)monad ! satisfies property K, and so:

Theorem

The opposite of the category CVS *is a Kähler category.*

A very interesting question is to see if the HKR-theorem (discussed below) holds conveniently.

Wedges in Monoidal Categories

To proceed any further with homological techniques, we will need constructions not available in arbitrary monoidal categories.

- We will need the coequalizer of the maps:
  $id\colon V \otimes V \to V \otimes V$ and
  $-c\colon V \otimes V \to V \otimes V$ (where c is the symmetry.)

  The result will be denoted $V \wedge V$.

- Similarly, one forms arbitrary wedge products $\wedge^n V$. It is the coequalizer of all of the possible symmetries of $\otimes^n V$, multiplied by (-1) to the appropriate power.

Constructing n-forms

- Now that we have constructed an object of one-forms, we wish to combine them and obtain an algebra of forms.
- Let $\Omega^2 = \Omega^1 \wedge_{TC} \Omega^1$ be the object of Kähler 2-forms, etc. Notice that the functor $\wedge_{TC}$ is defined to allow elements of the algebra to pass back and forth across the $\wedge$.
- So in a codifferential category, we get a general formula:

$$\Omega^n_C = TC \otimes (\wedge^n C)$$

- Then we take the coproduct of all forms:

$$\Omega^\bullet_C = \bigoplus_n \Omega^n_C$$

Differential Graded Algebras I: Definition

Definition

- An associative algebra is *graded* if it can be written $A = \bigoplus_{n \in \mathbb{N}} A_n$, and multiplication respects the grading, i.e. $A_p \cdot A_q \subseteq A_{p+q}$.
- A graded algebra is a *differential graded algebra* if equipped with a family of maps $d \colon A^p \to A^{p+1}$ such that
  - $d^2 = 0$
  - $d(ab) = d(a)b + (-1)^{|a|} a d(b)$ (Graded Leibniz)

Note that we are writing formulas using elements, but these can be written categorically as well.

Differential Graded Algebras II: Examples

The primary example is the de Rham cohomology of an n-dimensional manifold M. Here the 0-forms are the smooth functionals on M, and one defines the differential for 0-forms on a local chart

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots + \frac{\partial f}{\partial x_n} dx_n$$

Then the d operator lifts to arbitrary forms.
We also have:

Theorem

In any codifferential storage category (not necessarily Kähler), the object $\Omega^\bullet_C$ is a differential graded algebra for all objects C.

Differential Graded Algebras III: Formula for the differential

We need a map $d\colon \Omega^p \to \Omega^{p+1}$.

$$TC\otimes(\wedge^p C)\underbrace{\longrightarrow}_{\mathbf{1}} TC\otimes(\bigotimes^p C)\underbrace{\longrightarrow}_{\mathbf{2}} TC\otimes(\bigotimes^{p+1} C)\underbrace{\longrightarrow}_{\mathbf{3}} TC\otimes(\wedge^{p+1} C)$$

1. This map exists when the coequalizer splits, as it does in most examples.
2. This is just $d \otimes id$ followed by associativity.
3. This is the identity in the first component and the quotient in the second component.

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Differential Graded Algebras IV: Why this works

In de Rham cohomology, we get $d^2 = 0$ because partial derivatives commute and the corresponding forms anticommute. We have:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \qquad \text{BUT} \qquad dxdy = -dydx$$

We get the same equation in a (co)differential storage category:

$$d; d \otimes id \colon TC \to TC \otimes C \to TC \otimes C \otimes C$$

is unchanged when followed by a symmetry:

$$d; d \otimes id; id \otimes c \colon TC \to TC \otimes C \to TC \otimes C \otimes C \to TC \otimes C \otimes C$$

This follows from the cocommutativity of the coalgebra structure of TC.

Differential Graded Algebras V: Cohomology

Whenever you have a DGA, you have a *cochain complex*, i.e. a sequence of modules and maps:

$$\mathrm{C} = \cdots C^{-2} \xrightarrow{d_{-2}} C^{-1} \xrightarrow{d_{-1}} C^{0} \xrightarrow{d_{0}} C^{1} \xrightarrow{d_{1}} C^{2} \cdots$$

satisfying $d^2 = 0$, one can associate its cohomology:

$$H_n(\mathrm{C}) = \frac{kernel(d_n)}{image(d_{n-1})}$$

So every DGA has a cohomology.

Differential Graded Algebras VI: Ideas From Noncommutative Geometry

Consider Alain Connes' interpretation of the result:

Theorem (Gelfand-Naimark)

Every commutative unital C^*-algebra is the algebra of continuous complex-valued functions on a compact Hausdorff space.

Connes argues that general (noncommutative) C^*-algebras should be viewed as a space of functions on a *noncommutative space*. A great deal of traditional analysis is then redone noncommutatively.

By the same argument, Connes argues that one should view a DGA as the cohomology of a noncommutative manifold. This is his theory of *noncommutative differential geometry*. In particular, one should be able to **integrate**.

Differential Graded Algebras VII: Cycles

Definition

An *n*-dimensional *cycle* on a differential graded algebra $\Omega^\bullet = \bigoplus_{k=0}^n \Omega_k$ is a linear map

$$\int : \Omega^\bullet \to \mathbb{R} \qquad \text{such that}$$

- $\int \omega_k = 0$ if $k \neq n$, with $\omega_k \in \Omega_k$.
- $\int \omega_k \omega_l = (-1)^{kl} \int \omega_l \omega_k$
- $\int d\omega_{n-1} = 0 \qquad$ (Stokes' Theorem)

Examples come from integration, as suggested by the syntax. This definition generalizes nicely to the differential category setting.

Differential Graded Algebras VIII: Cool example of a cycle

Consider smooth functions of compact support on $\mathbb{R}^n$ which take values in $\mathcal{M}^m(\mathbb{R})$, the space of $m \times m$ matrices.
Then define:

$$\int \omega_n = \int_{\mathbb{R}^n} trace(\omega_n)$$

There is the more general theory of *de Rham currents*, which gives the same structure. In a differential category, we have a large class of DGAs. We would be looking at arrows:

$$\int : \Omega^\bullet \to I \qquad \text{the monoidal unit}$$

We could use associative, traced algebras i.e. algebras (A, μ) with a map

$$\tau : A \to I, \qquad \text{such that } \mu; \tau = c; \mu; \tau$$

Hochschild Homology I: Preliminaries

A great deal of structure is contained in the (co)algebra modality of a model of differential linear logic.

- Since each object $T(C)$ has a (co)algebra structure, we can associate to it a homology theory, the *Hochschild homology*.
- There is a canonical comparison between the Hochschild homology and the Kähler differentials.
- In nice cases, the *smooth algebras*, this comparison is an isomorphism. In the finitely generated case, smooth algebras correspond to those ideals of polynomials for which the solution set has no singularities.

We'll work with the algebras rather than coalgebras since it is the more classic theory. We will also write formulas using elements, as this is much easier to read. And again, we need to assume quotients and (-1), etc.

Hochschild Homology II: Boundary Map

We let A be an associative, unital algebra over a commutative ring k. Let M be an A-bimodule. Define the k-module of n-chains by $C_n(A, M) = M \otimes A^{\otimes n}$. The *Hochschild boundary map* $b\colon C_n(A, M) \to C_{n-1}(A, M)$ is given by:

$$\begin{aligned} b(m, a_1, a_2, \ldots, a_n) = (ma_1, a_2, \ldots, a_n) + \\ \sum_{i=1}^{n-1} (-1)^i (m, a_1, a_2, \ldots, a_i a_{i+1}, \ldots, a_n) \\ +(-1)^n (a_n m, a_1, a_2, \ldots, a_{n-1}) \end{aligned}$$

One readily checks that $b^2 = 0$ and the resulting homology is called the *Hochschild homology of A with coefficients in M*.

Hochschild Homology III: Example

- Let $R = k[x_1, x_2, \ldots, x_n]$. One can verify that $H_1(R,R) \cong \bigoplus_1^n R$, and that $H_p(R) = \wedge^p H_1(R)$, if$p \leq n$.
- Also remember that for Kahler differentials, we showed that a typical 1-form looks like

$$f_1(x_1, x_2, \ldots, x_n)dx_1 + f_2(x_1, x_2, \ldots, x_n)dx_2 + \\ f_n(x_1, x_2, \ldots, x_n)dx_n.$$

- So $\Omega^1_R \cong \bigoplus_1^n R$, with $\Omega^p = \wedge^p \Omega^1$, if $p \leq n$.

Theorem

Let R be a commutative k-algebra. Then

$$H_1(R,R) \cong \Omega^1_R$$

Hochschild Homology IV: Antisymmetrization

Theorem (See Loday, *Cyclic Homology*)

For any commutative k-algebra R, there is a canonical **antisymmetrization** *map*

$$\varepsilon_n \colon \Omega^n_R \longrightarrow H_n(R, R)$$

It is the extension of the previous isomorphism $H_1(R, R) \cong \Omega^1_R$, but in general is not an isomorphism.

However, for $R = k[x_1, x_2, \ldots, x_n]$, we do get an isomorphism at all levels. When does this hold in general?

When considering $R = k[x_1, x_2, \ldots, x_n]/I$, the map ε is an isomorphism if and only if (intuitively) the corresponding solution set has no singularities. This leads to the definition of *smoothness* of an algebra. (The formal definition is technical.)

Hochschild Homology V: Hochschild-Kostant-Rosenberg Theorem

This leads to the definition of *smoothness* of an algebra. (The formal definition is technical.) Here's a special case:

Definition

An algebra of the form $A = k[x_1, x_2, \ldots, x_n]/I$ is *smooth* if the object Ω_A^1 is a projective module.

Theorem (HKR)

If A is a finitely generated, smooth commutative k-algebra, then the map

$$\varepsilon_n \colon \Omega_A^n \longrightarrow H_n(A, A)$$

is an isomorphism for all n.

Hochschild Homology VI: More on HKR

There is an extension of this result to algebras of the form $C^\infty(M)$, smooth maps from M, a smooth manifold, to $\mathbb{R}$, due to Connes. It involves de Rham currents.

Ongoing Project

- There is a theorem of Cockett that given an algebra modality T, every T-algebra (in the sense of monads) $(C, \rho : TC \to C)$ has a canonical commutative algebra structure.
- Can we calculate the Kähler n-forms and the Hochschild homology, based on our calculation in the free case?
- What is the abstract version of smoothness? Previous abstract versions of smoothness did not have a good notion of Kähler forms.

Smooth Differential Forms I: Discussion

- We have a notion of differential form associated to algebras in the opposite of a differential category.
- We want a more direct notion, which applies directly in differential category, and can be lifted to manifolds.
- We should be guided by convenient vector spaces.
- Here are key criteria:
  - We need a differential with $d^2 = 0$.
  - We should be able to pull back forms along arbitrary smooth maps.
  - This pull-back should be functorial.
  - Pulling back should commute with the differential.

All of these are true in classical differential geometry.

But we have a problem:

Smooth Differential Forms II: More discussion

Theorem (Kriegl,Michor)

All hell breaks loose.

The passage to infinite-dimensional structures reveals a great deal of new structure, which is both interesting and confusing.

- There are two notions of tangent vector, *operational* and *kinematic* tangent vectors. In this case, it's clear that the kinematic definition is better. In particular, the kinematic tangent bundle functor is product preserving. The other isn't.
- There are (at least) 12 distinct possible definitions of differential form on a convenient manifold.

Smooth Differential Forms III: Proposed Formula:

It turns out, quite nicely, that

- the formula we are proposing agrees with the one that Kriegl and Michor eventually settle on,
- the points of our forms correspond to the forms Geoff defined,
- our formula for smooth forms is the linear dual of the formula for Kähler forms,
- most, and hopefully all, of the above desired properties hold.

The proposed formula for forms is $\Omega^n_{DR}(V) = [!V \otimes \wedge^n V]^*$. (Note this is the linear dual.)

This is obviously functorial on linear maps, but we want it functorial on smooth maps as well.

Smooth Differential Forms IV: Pulling back along smooth maps

Given a smooth map $f\colon !V \to W$, we wish to define
$f^*\colon \Omega^n_{DR}(W) \to \Omega^n_{DR}(V)$.
Our formula is the linear dual of the following map:

$$!V \otimes \wedge^n V \underbrace{\to}_{1} !V \otimes (\otimes^n V) \underbrace{\to}_{2} !V^{n+1} \otimes (\otimes^n V) \underbrace{\to}_{3} !V \otimes [!V \otimes V]^n$$

1. The splitting of the coequalizer, as before.
2. Using comultiplication on $!V$ to create $n+1$ copies of $!V$.
3. Symmetry.

Smooth Differential Forms V: Pulling back continued

$$\underbrace{\rightarrow}_{1} !W \otimes (\otimes^n W) \underbrace{\rightarrow}_{2} !W \otimes (\wedge^n W)$$

1. Uses $\delta; !f : !V \rightarrow !W$ and $D[f] : !V \otimes V \rightarrow W$.
2. Quotienting to get to the wedge product.

We can show:

Lemma

Given $f : !V \rightarrow W$ and $g : !W \rightarrow Z$, we have $(f;g)^* = g^*; f^*$.

But we haven't shown that $d; f^* = f^*; d$, as yet.

Future work

- Finish present work.
- Extend constructions to manifolds (Cockett, Cruttwell), calculate de Rham cohomology etc in these settings.
- There is already a theory of convenient manifolds. Do our constructions agree? As of now, Yes.
- Finiteness manifolds?