A note on intermittency for the fractional heat equation

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Abstract
The goal of the present note is to study intermittency properties for the solution to the fractional heat equation

$$\frac{\partial u}{\partial t}(t,x) = -(-\Delta)^{\beta/2}u(t,x) + u(t,x)\dot{W}(t,x), \quad t > 0, x \in \mathbb{R}^d$$

with initial condition bounded above and below, where $\beta \in (0, 2]$ and the noise $W$ behaves in time like a fractional Brownian motion of index $H > 1/2$, and has a spatial covariance given by the Riesz kernel of index $\alpha \in (0, d)$. As a by-product, we obtain that the necessary and sufficient condition for the existence of the solution is $\alpha < \beta$.

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1 Introduction

In this article we consider the fractional heat equation

$$\begin{cases}
\frac{\partial u}{\partial t}(t,x) = -(-\Delta)^{\beta/2}u(t,x) + u(t,x)\dot{W}(t,x), \quad t > 0, x \in \mathbb{R}^d \\
u(0,x) = u_0(x), \quad x \in \mathbb{R}^d.
\end{cases} \tag{1}$$

where $\beta \in (0, 2], (-\Delta)^{\beta/2}$ denotes the fractional power of the Laplacian, and $u_0$ is a deterministic function such that

$$a \leq u_0(x) \leq b \quad \text{for all} \quad x \in \mathbb{R}^d \quad \text{(2)}$$

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for some constants \(b \geq a > 0\). We let \(W = \{W(\varphi); \varphi \in \mathcal{H}\}\) be a zero-mean Gaussian process with covariance

\[ E(W(\varphi)W(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}. \]

Here \(\mathcal{H}\) is a Hilbert space defined as the completion of the space \(C_0^{\infty}(\mathbb{R}_+ \times \mathbb{R}^d)\) of infinitely differentiable functions with compact support on \(\mathbb{R}_+ \times \mathbb{R}^d\), with respect to the inner product \(\langle \cdot, \cdot \rangle_{\mathcal{H}}\) defined by:

\[
\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_{[\mathbb{R}_+ \times \mathbb{R}^d]^2} \varphi(t, x)\psi(s, y)|t - s|^{2H-2}|x - y|^{-\alpha} dt ds dy,
\]

where \(\alpha_H = H(2H - 1)\), \(H \in (1/2, 1)\) and \(\alpha \in (0, d)\). We denote by \(\dot{W}\) the formal derivative of \(W\). The noise \(W\) is spatially homogeneous with spatial covariance given by the Riesz kernel \(f(x) = |x|^{-\alpha}\) and behaves in time like a fractional Brownian motion of index \(H\). We refer to [2, 3, 5] for more details.

Let \(G(t, x)\) be the fundamental solution of \(\frac{\partial u}{\partial t} + (-\Delta)^{\beta/2} u = 0\) and

\[ w(t, x) = \int_{\mathbb{R}^d} u_0(y)G(t, x - y)dy \]

be the solution of the equation \(\frac{\partial u}{\partial t} + (-\Delta)^{\beta/2} u = 0\) with initial condition \(u(0, x) = u_0(x)\). Note that

\[ G(t, \cdot) \text{ is the density of } X_t \]

where \(X = (X_t)_{t \geq 0}\) is a symmetric Lévy process with values in \(\mathbb{R}^d\). If \(\beta = 2\), then \(X\) coincides with a Brownian motion \(B = (B_t)_{t \geq 0}\) in \(\mathbb{R}^d\) with variance 2. If \(\beta < 2\), then \(X\) is a \(\beta\)-stable Lévy process given by \(X_t = B_{S_t}\), where \((S_t)_{t \geq 0}\) is a \((\beta/2)\)-stable subordinator with Lévy measure

\[ \nu(dx) = \frac{\beta/2}{\Gamma(1 - \beta/2)}x^{-\beta/2-1}1_{\{x > 0\}}dx. \]

(See for instance the explanation on page 62 of [19] on how to construct a new Lévy process from a Wiener process using subordination, and in particular Example 4.38 of [19]). Due to (2) and (4), it follows that for all \(t > 0\) and \(x \in \mathbb{R}^d\),

\[ a \leq w(t, x) \leq b. \]

There is a rich literature dedicated to the case \(H = 1/2\), when the noise \(W\) is white in time. We refer to [10, 13] for some general properties, and to [12, 9, 7] for intermittency properties of the solution to the heat equation with this type of noise. Different methods have to be used for \(H > 1/2\), since in this case the noise is not a semi-martingale in time. The stochastic heat equation driven by a fractional noise in time with index \(H \in (\frac{1}{2}, \frac{3}{2})\) was studied in [15], assuming that the noise has a \(\gamma\)-continuous spatial covariance function.

In the present article, we follow the approach of [16, 5] for defining the concept of solution. We say that a process \(u = \{u(t, x); t \geq 0, x \in \mathbb{R}^d\}\) defined
on a probability space \((\Omega, F, P)\) is a mild solution of (1) if it is square-integrable, adapted with respect to the filtration induced by \(W\), and satisfies:

\[
u(t, x) = w(t, x) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)\nu(s, y)W(\delta s, \delta y),
\]

where the stochastic integral is interpreted as the divergence operator of \(W\) (see (18)). Using Malliavin calculus techniques, it can be shown that the mild solution (if it exists) is unique and has the Wiener chaos decomposition:

\[
u(t, x) = \sum_{n \geq 0} I_n(f_n(\cdot, t, x))
\]

where \(I_n\) denotes the multiple Wiener integral (with respect to \(W\)) of order \(n\), and the kernel \(f_n(\cdot, t, x)\) is given by:

\[
f_n(t_1, x_1, \ldots, t_n, x_n, t, x) = G(t - t_n, x - x_n) \ldots G(t_2 - t_1, x_2 - x_1)w(t_1, x_1)1_{\{0 < t_1 < \ldots < t_n < t\}}
\]

(see page 303 of [16]). By convention, \(f_0(\cdot, t, x) = w(t, x)\) and \(I_0\) is the identity map on \(\mathbb{R}\).

The necessary and sufficient condition for the existence of the mild solution is that the series in (6) converges in \(L^2(\Omega)\), i.e.

\[
S(t, x) := \sum_{n \geq 0} \frac{1}{n!} \alpha_n(t, x) < \infty,
\]

where

\[
\alpha_n(t, x) = n!E|I_n(f_n(\cdot, t, x))|^2 = (n!)^2 \| \tilde{f}_n(\cdot, t, x) \|_{H^\otimes n}^2
\]

and \(\tilde{f}_n(\cdot, t, x)\) is the symmetrization of \(f_n(\cdot, t, x)\) in the \(n\) variables \((t_1, x_1), \ldots, (t_n, x_n)\). If the solution \(\nu\) exists, then \(E|\nu(t, x)|^2 = S(t, x)\). We refer to Section 4.1 of [16] and Section 2 of [5] for the details. Note that if \(\nu_0(x) = u_0\) for all \(x \in \mathbb{R}^d\), then the law of \(\nu(t, x)\) does not depend on \(x\), and hence \(\alpha_n(t, x) = \alpha_n(t)\).

The goal of the present work is to give an upper bound for the \(p\)-th moment of the solution of (1) (for \(p \geq 2\)), and a lower bound for its second moment. In particular, this will show that, if \(\nu_0(x)\) does not depend on \(x\), then the solution \(\nu\) of (1) is weakly \(\rho\)-intermittent, in a sense which has been recently introduced in [4], i.e. \(\gamma_{\rho}(2) > 0\) and \(\gamma_{\rho}(p) < \infty\) for all \(p \geq 2\), where

\[
\gamma_{\rho}(p) = \limsup_{t \to \infty} \frac{1}{t^p} \log E|\nu(t, x)|^p
\]

is a modified Lyapunov exponent (which does not depend on \(x\)), and

\[
\rho = \frac{2H\beta - \alpha}{\beta - \alpha}.
\]
As a by-product, we obtain that the necessary and sufficient condition for the existence of the solution is \( \alpha < \beta \). Note that this condition is equivalent to

\begin{equation}
I_\beta(\mu) := \int_{\mathbb{R}^d} \left( \frac{1}{1 + |\xi|^2} \right)^{\beta/2} \mu(d\xi) < \infty \tag{9}
\end{equation}

with \( \mu(d\xi) = \mu_{\alpha,d}[|\xi|^{-d+\alpha}d\xi] \), which is encountered in the study of equations with white noise in time. When \( \beta = 2 \), (9) is called Dalang’s condition (see [10]). (If \( \beta/2 = k \) was a positive integer, (9) would coincide with condition (3.3) of [11].)

In the case \( \beta = 2 \), a lower bound for the \( p \)-the moment of the solution has been obtained in the recent preprint [14], for the equation interpreted in the Skorohod sense (as in the present paper), and also in the Stratonovich sense. The method of [14] is based on a Feynman-Kac (FK) type representation for the moments of the solution. A similar approach may work in the case \( \beta < 2 \), as this type of FK representations might still hold for the solution of the fractional heat equation, under some additional constraints on the parameters \( H \) and \( \alpha \) of the noise. (This problem was considered in [8] for a noise with spatial covariance \( f(x) = \prod_{r=1}^n |x_i|^{2H_r-2} \) with \( H_i \in (\frac{1}{2},1) \).) We do not investigate this problem here.

### 2 The result

The goal of the present article is to prove the following result.

**Theorem 2.1.** The necessary and sufficient condition for equation (1) to have a mild solution is \( \alpha < \beta \). If the solution \( u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\} \) exists, then for any \( p \geq 2 \), for any \( x \in \mathbb{R}^d \) and for any \( t > 0 \) such that \( pt^{2H-\alpha/\beta} > t_1 \)

\[
E|u(t,x)|^p \leq b^p \exp(C_1(p^{(2\beta-\alpha)/(\beta-\alpha)}t^\rho))
\]

and for any \( x \in \mathbb{R}^d \) and for any \( t > t_2 \),

\[
E|u(t,x)|^2 \geq a^2 \exp(C_2t^\rho),
\]

where \( \rho \) is given by (8), \( a, b \) are the constants given by (2), and \( t_1, t_2, C_1, C_2 \) are some positive constants depending on \( d, \alpha, \beta \) and \( H \).

We suspect that the inequalities given by Theorem 2.1 cannot be improved, except for possibly different constants \( C_1 \) and \( C_2 \). This problem is not investigated in the present article.

Before giving the proof, we recall from [5] that

\[
\alpha_n(t,x) = \alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} \psi_n(t,s)dtds \tag{10}
\]

where

\[
\psi_n(t,s) = \int_{\mathbb{R}^{2d}} \prod_{j=1}^n |x_j - y_j|^{-\alpha} \tilde{f}_n(t_1, x_1, \ldots, t_n, x_n) \tilde{f}_n(s_1, y_1, \ldots, s_n, y_n) dxdy
\]
and we denote $t = (t_1, \ldots, t_n)$, $s = (s_1, \ldots, s_n)$ with $t_i, s_i \in [0, t]$ and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ with $x_i, y_i \in \mathbb{R}^d$.

Note that the Fourier transform of $G(t, \cdot)$ is given by:

$$FG(t, \cdot)(\xi) := \int_{\mathbb{R}^d} e^{-i\xi \cdot x} G(t, x) dx = \exp(-t|\xi|^{\beta}), \quad \xi \in \mathbb{R}^d$$  \hspace{1cm} (11)

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. Recall that for any $\varphi, \psi \in L^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x)\psi(y)|x-y|^{-\alpha} dxdy = c_{\alpha,d} \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\mathcal{F}\psi(\xi)|\xi|^{-\alpha + \delta} d\xi$$  \hspace{1cm} (12)

where $\mathcal{F}\varphi$ is the Fourier transform of $\varphi$, $c_{\alpha,d} = (2\pi)^{-d}C_{\alpha, d}$ and $C_{\alpha, d}$ is the constant given by (21) (see Appendix A). This identity can be extended to functions $\varphi, \psi \in L^1(\mathbb{R}^{nd})$:

$$\int_{\mathbb{R}^{nd}} \int_{\mathbb{R}^{nd}} \varphi(x)\psi(y) \prod_{j=1}^{n} |x_j - y_j|^{-\alpha} dxdy = \int_{\mathbb{R}^{nd}} \mathcal{F}\varphi(\xi_1, \ldots, \xi_n)\mathcal{F}\psi(\xi_1, \ldots, \xi_n) \prod_{j=1}^{n} |\xi_j|^{-\alpha + \delta} d\xi_1 \ldots \xi_n.$$  \hspace{1cm} (13)

We will use the following elementary inequality.

**Lemma 2.2.** For any $t > 0$ and $\eta \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} e^{-t|\xi|^{\beta}} |\xi - \eta|^{-\alpha + \delta} d\xi \leq K_{d, \alpha, \beta} t^{-\alpha / \beta}$$

where

$$K_{d, \alpha, \beta} := \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{1 + |\xi - \eta|^{\beta}} |\xi|^{-\alpha + \delta} d\xi.$$

**Proof:** Using the change of variable $z = t^{1/\beta}(\eta - \xi)$, we have:

$$\int_{\mathbb{R}^d} e^{-t|\xi|^{\beta}} |\xi - \eta|^{-\alpha + \delta} d\xi = t^{-\alpha / \beta} \int_{\mathbb{R}^d} e^{-z^{1/\beta}|\eta|^{\beta}} |z|^{-\alpha + \delta} dz.$$  \hspace{1cm} (14)

The result follows using the inequality $e^{-x} \leq 1/(1 + x)$ for $x > 0$. \hspace{1cm} \(\Box\)

**Proof of Theorem 2.1:** Step 1. (Sufficiency and upper bound for the second moment) Suppose that $\alpha < \beta$. We will prove that the series (7) converges, by providing upper bounds for $\psi_n(t, s)$ and $\alpha_n(t, x)$.

By the Cauchy-Schwarz inequality, $\psi_n(t, s) \leq \psi_n(t, t)^{1/2} \psi_n(s, s)^{1/2}$. So it is enough to consider the case $t = s$. Let $u_j = t_{\sigma(j+1)} - t_{\sigma(j)}$ where $\sigma$ is a permutation of $\{1, \ldots, n\}$ such that $t_{\sigma(1)} < \ldots < t_{\sigma(n)}$ and $t_{\sigma(n+1)} = t$. Using (5), (11) and (13), and arguing as in the proof of Lemma 3.2 of [3], we obtain:

$$\psi_n(t, t) \leq b^2 c_{\alpha,d} \int_{\mathbb{R}^d} d\gamma_1 \exp(-u_1|\gamma_1|^{\beta}) |\gamma_1|^{-\alpha + \delta} \int_{\mathbb{R}^d} d\gamma_2 \exp(-u_2|\gamma_2|^{\beta}) |\gamma_2 - \gamma_1|^{-\alpha + \delta} \hspace{1cm} (15)$$
\[ \int_{\mathbb{R}^d} d\eta_n \exp(-u_n |\eta_n|^\beta) |\eta_n - \eta_{n-1}|^{-d+\alpha}. \]

(Since Lemma 3.2 of [3] refers to the wave equation, the argument has to be adjusted by replacing the Fourier transform \( \mathcal{F}G_w(t,\cdot)(\xi) = \sin(t |\xi|)/|\xi| \) of the fundamental solution \( G_w \) of the wave equation by (11).) By Lemma 2.2, it follows that:

\[ \psi_n(t, t) \leq b_1^2 C_{n,d,\alpha,\beta}(u_1 \ldots u_n)^{-\alpha/\beta}. \]

By inequality (26) (Appendix A), \( K_{d,\alpha,\beta} \leq I_{d,\alpha,\beta} \), where

\[ I_{d,\alpha,\beta} := \int_{\mathbb{R}^d} \left( \frac{1}{1 + |\xi|^2} \right)^{\beta/2} |\xi|^{-d+\alpha} d\xi = \frac{(2\pi)^d c_d \Gamma((\beta - \alpha)/2) \Gamma(\alpha/2)}{2 \Gamma(\beta/2)} \]

(see relation (24) and Remark A.3, Appendix A). Hence,

\[ \psi_n(t, s) \leq b_1^2 C_{n,d,\alpha,\beta} \beta(t)^{\beta(s)} - \alpha/(2\beta) \]

where \( \beta(t) = u_1 \ldots u_n \), \( \beta(s) \) is defined similarly, and \( C_{d,\alpha,\beta} > 0 \) is a constant depending on \( d, \alpha, \beta \). Similarly to the proof of Proposition 3.6 of [5], we have:

\[ \alpha_n(t, x) \leq b_2^2 C_{n,d,\alpha,\beta,H}(n!)^{\alpha/\beta} t^{n(2H-\alpha/\beta)}, \]

where \( C_{d,\alpha,\beta,H} > 0 \) is a constant depending on \( d, \alpha, \beta, H \). (The only change compared to the proof mentioned above is the fact that \( 2\beta \) was equal to 4 in [5]. These authors worked with a different parametrization: their \( d - \alpha \) is denoted here by \( \alpha \).)

Since \( \alpha < \beta \), it follows that the series (7) converges and

\[ E|u(t, x)|^2 = \sum_{n \geq 0} \frac{1}{n!} \alpha_n(t, x) \leq b_2^2 \sum_{n \geq 0} C_{n,d,\alpha,\beta,H}(n!)^{\alpha/\beta} t^{n(2H-\alpha/\beta)} \leq b_2^2 \exp(C_0 t^\rho), \]

for all \( t > t_0 \), where \( C_0 > 0 \) and \( t_0 > 0 \) are constants depending in \( d, \alpha, \beta, H \). We used the fact that for any \( a > 0 \) and \( x > 0 \),

\[ \sum_{n \geq 0} \frac{x^n}{(n!)^2} \leq \exp(c_0 x^{1/a}) \quad \text{for all} \quad x > x_0, \]

where \( x_0 > 0 \) and \( c_0 > 0 \) are some constants depending on \( a \) (see e.g. Lemma A.1 of [4]).

**Step 2. (Upper bound for the p-th moment)** Note that \( u(t, x) = \sum_{n \geq 0} J_n(t, x) \) in \( L^2(\Omega) \), where \( J_n(t, x) \) lies in the \( n \)-th order Wiener chaos \( \mathcal{H}_n \) associated to the Gaussian process \( W \) (see [18]). Hence,

\[ E|u(t, x)|^p = \sum_{n \geq 0} E|J_n(t, x)|^p = \sum_{n \geq 0} \frac{1}{n!} \alpha_n(t, x). \]
We denote by \( \| \cdot \|_p \) the \( L^p(\Omega) \)-norm. We use the fact that for a fixed Wiener chaos \( \mathcal{H}_n \), the \( \| \cdot \|_p \) are equivalent, for all \( p \geq 2 \) (see the last line of page 62 of [18] with \( q = p \) and \( p = 2 \)). Hence,

\[
\|J_n(t, x)\|_p \leq (p - 1)^{n/2}\|J_n(t, x)\|_2 = (p - 1)^{n/2} \left( \frac{1}{n!} \alpha_n(t, x) \right)^{1/2} \leq b(p - 1)C_{d, \alpha, \beta, H}^{n/2} \frac{1}{(n!)^{(\beta - \alpha)/(2\beta)}} t^{n(2H\beta - \alpha)/(2\beta)}
\]

using (14) for the last inequality. Using Minkowski’s inequality for integrals (see Appendix A.1 of [20]) and inequality (15), we obtain that:

\[
\|u(t, x)\|_p \leq \sum_{n \geq 0} \|J_n(t, x)\|_p \leq b \exp(C_1(p - 1)^{\beta/(\beta - \alpha)}) t^p
\]

if \( pt^{2H - \alpha/\beta} > t_1 \), where the constants \( C_1 > 0 \) and \( t_1 > 0 \) depend on \( d, \alpha, \beta, H \).

**Step 3. (Necessity and lower bound for the second moment)** Suppose that equation (1) has a mild solution \( u \), i.e. the series (7) converges. In particular,

\[
\infty > \alpha_1(t, x) \geq a^2 \alpha_H \int_{[0, t]^2} \int_{\mathbb{R}^{2d}} |r - s|^{2H - s} |y - z|^{-\alpha} G(s, y) G(r, z) dy dz dr ds
\]

\[
= a^2 \alpha_H c_{d, \alpha} \int_{\mathbb{R}^d} \left( \int_0^t \int_0^t |r - s|^{2H - 2} e^{-(r + s)|\xi|^\beta} dr ds \right) |\xi|^{-d - \alpha} d\xi
\]

\[
\geq a^2 \alpha_H c_{d, \alpha} \int_{\mathbb{R}^d} \left( \frac{1}{1/t + |\xi|^\beta} \right)^{2H} |\xi|^{-d + \alpha} d\xi,
\]

where we used (12) for the equality and Theorem 3.1 of [2] for the last inequality. From here, we infer that

\[
\alpha < 2H \beta.
\]

(In particular, this implies that \( \alpha < 2\beta \) since \( H < 1 \).)

Note that one can replace \( \psi_n(t, s) \) by \( \psi_n(te - t, te - s) \) in the definition (10) of \( \alpha_n(t, x) \), where \( e = (1, \ldots, 1) \in \mathbb{R}^n \). By Lemma 2.2 of [1], we have:

\[
\psi_n(te - t, te - s) = E \left[ w(t - t^*, x + X_{t^*}) w(t - s^*, x + X_{s^*}) \prod_{j=1}^n |X_{t_j} - X_{s_j}|^{-\alpha} \right],
\]

where \( t^* = \max\{t_1, \ldots, t_n\} \), \( s^* = \max\{s_1, \ldots, s_n\} \) and \( X^1, X^2 \) are two independent copies of the Lévy process \( X = (X_t)_{t \geq 0} \) mentioned in the Introduction. (Lemma 2.2 of [1] was proved for \( \beta = 2 \). The same proof is valid for \( \beta < 2 \), the only change required for this case being to replace the fundamental solution \( p_t(x) = (2\pi t)^{-d/2} \exp(-|x|^2/2t) \) of the heat equation by \( G(t, x) \) given by (4).)

Due to (5), it follows that

\[
a^2 M_n(t) \leq \alpha_n(t, x) \leq b^2 M_n(t)
\]

(17)
where
\[
M_n(t) := E \left[ \alpha_n^2 \int_{0,t}^{2\alpha} \prod_{j=1}^{n} |t_j - s_j|^{2H-2} \prod_{j=1}^{n} |X_{t_j}^1 - X_{s_j}^2|^{-\alpha} dt ds \right] = E(L(t)^n)
\]
and \(L(t)\) is a random variable defined by:
\[
L(t) := \alpha_H \int_0^t \int_0^t |r - s|^{2H-2} |X_r^1 - X_s^2|^{-\alpha/\beta} dr ds.
\]

To prove that \(L(t)\) is finite a.s., we show that its mean is finite. Note that
\[
X_r^1 - X_s^2 \overset{d}{=} X_{r+s} \overset{d}{=} (r + s)^{1/\beta} X_1,
\]
and hence
\[
E[L(t)] = \alpha_H C_{d,\alpha,\beta} \int_0^t \int_0^t |r - s|^{2H-2} (r + s)^{-\alpha/\beta} dr ds,
\]
where
\[
C_{d,\alpha,\beta} := E|X_1|^{-\alpha} = \frac{c_d C_{d,\alpha}}{\beta} \Gamma(\alpha/\beta).
\]
(see (28), Appendix A). Due to (16), it follows that \(E[L(t)] < \infty\).

By (17), we have:
\[
a^2 E(e^{L(t)}) \leq E|u(t,x)|^2 = \sum_{n \geq 0} \frac{1}{n!} \alpha_n(t,x) \leq b^2 E(e^{L(t)}). \tag{18}
\]

We consider also the random variable
\[
\zeta(t) := \int_0^t \int_0^t |X_r^1 - X_s^2|^{-\alpha} dr ds.
\]
Since \(|r - s|^{2H-2} \geq (2t)^{2H-2}\) for any \(r,s \in [0,t]\), \(L(t) \geq \beta_H t^{2H-2} \zeta(t)\), where \(\beta_H = \alpha_H t^{2H-2}\). Hence \(\zeta(t)\) is finite a.s.

By the self-similarity (of index \(1/\beta\)) of the processes \(X^1\) and \(X^2\), it follows that for any \(t > 0\) and \(c > 0\),
\[
\zeta(t) \overset{d}{=} t^{(2\beta-\alpha)/\beta} \zeta(t/c).
\]
In particular, for \(c = t^{-(2H-2)\beta/(2\beta-\alpha)}\), we obtain that
\[
t^{2H-2} \zeta(t) \overset{d}{=} \zeta(t^\delta), \quad \text{with} \quad \delta = \frac{2H\beta - \alpha}{2\beta - \alpha}
\]
and for \(c = t\), we obtain that \(\zeta(t) \overset{d}{=} t^{(2\beta-\alpha)/\beta} \zeta(1)\). Hence,
\[
E(e^{L(t)}) \geq E(e^{\beta_H t^{2H-2} \zeta(t)}) = E(e^{\beta_H \zeta(t^\delta)}). \tag{19}
\]
The asymptotic behavior of the moments of \( \zeta(t) \) was investigated in [6], under the condition \( \alpha < 2\beta \). More precisely, under this condition, by relation (2.3) of [6], we know that:

\[
\lim_{n \to \infty} \frac{1}{n} \log \left\{ \frac{1}{(n!)^{\alpha/\beta}} E[\zeta(1)^n] \right\} = \log \left( \frac{2\beta}{2\beta - \alpha} \right)^{\frac{(2\beta - \alpha)/\beta}{\beta - \alpha}} + \log \gamma,
\]

where \( \gamma > 0 \) is a constant depending on \( d, \alpha, \beta \). Hence, there exists some \( n_1 \geq 1 \) such that for all \( n \geq n_1 \), \( E[\zeta(1)^n] \geq c^n (n!)^{\alpha/\beta} \), where \( c > 0 \) is a constant depending on \( d, \alpha, \beta \). Consequently, for any \( t > 0 \),

\[
E[\zeta(t)^n] \geq c^n t^{n(2\beta - \alpha)/\beta} (n!)^{\alpha/\beta} \quad \text{for all } n \geq n_1.
\]

Hence, for any \( \theta > 0 \),

\[
E(e^{\theta \zeta(t)}) = \sum_{n=0}^{\infty} \frac{1}{n!} \theta^n E[\zeta(t)^n] \geq \sum_{n \geq n_1} \frac{1}{(n!)^{\alpha/\beta}} \theta^n c^n t^{n(2\beta - \alpha)/\beta}.
\]

Using (18), (19) and (20), we obtain that:

\[
\infty > E |u(t, x)|^2 \geq a^2 E(e^{\beta H \zeta(t \delta)}) \geq a^2 \sum_{n \geq n_1} \beta_H^n c^n t^{n(2H \beta - \alpha)/\beta} (n!)^{1 - \alpha/\beta}.
\]

This implies that \( \alpha < \beta \). For any \( \theta > 0 \) and \( h \in (0, 1) \), we note that

\[
E_h(x) := \sum_{n \geq 0} \frac{x^n}{(n!)^h} \geq \left( \sum_{n \geq 0} \frac{(x^{1/h})^n}{n!} \right)^h = \exp(hx^{1/h}).
\]

We denote \( x_t = \theta ct^{(2\beta - \alpha)/\beta} \) and \( h = 1 - \alpha/\beta \). Writing the last sum in (20) as the sum for all terms \( n \geq 0 \), minus the sum \( S_t \) with terms \( n \leq n_1 \), we see that for all \( \theta > 0 \), and for all \( t \geq t_0 \),

\[
E(e^{\theta \zeta(t)}) \geq E_h(x_t) - S_t \geq \exp(hx_t^{1/h}) - S_t \geq \frac{1}{2} \exp(hx_t^{1/h})
\]
\[
\geq \exp(c_0 \theta^{\beta/(\beta - \alpha)} t^{(2\beta - \alpha)/(\beta - \alpha)}),
\]

where \( c_0 = hc^{1/h} \) and \( t_0 > 0 \) is a constant depending on \( \theta, \alpha, \beta \). Using this last inequality with \( \theta = \beta_H \) and \( t^\delta \) instead of \( t \), we obtain that:

\[
E |u(t, x)|^2 \geq a^2 E(e^{\beta_H \zeta(t^\delta)}) \geq a^2 \exp(C_2 t^\delta),
\]

where \( C_2 = c_0 \beta_H^{\beta/(\beta - \alpha)} \) depends on \( d, \alpha, \beta, H \). □
A Some useful identities

In this section, we give a result which was used in the proof of Theorem 2.1 for finding an upper bound for $\psi_n(t,t)$. This result may be known, but we were not able to find a reference. We state it in a general context.

Following Definition 5.1 of [17], we say that a function $f : \mathbb{R}^d \to [0,\infty]$ is a kernel of positive type if it is locally integrable and its Fourier transform in $S'(\mathbb{R}^d)$ is a function $g$ which is non-negative almost everywhere. Here we denote by $S'(\mathbb{R}^d)$ the dual of the space $S(\mathbb{R}^d)$ of rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^d$.

The Riesz kernel defined by $f(x) = |x|^{-\alpha}$ for $x \in \mathbb{R}^d \setminus \{0\}$ and $f(0) = \infty$ (with $\alpha \in (0,d)$), is a kernel of positive type. Its Fourier transform in $S'(\mathbb{R}^d)$ is given by $g(\xi) = C_{\alpha,d}|\xi|^{-(d-\alpha)}$ where

$$C_{\alpha,d} = \pi^{-d/2}2^{-\alpha}\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}$$

(see Lemma 1, page 117 of [20]).

Let $f$ be a continuous symmetric kernel of positive type such that $f(x) < \infty$ if and only if $x \neq 0$. By Lemma 5.6 of [17], for any Borel probability measures $\mu$ and $\nu$ on $\mathbb{R}^d$, we have:

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y)\mu(dx)\nu(dy) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}\mu(\xi)\mathcal{F}\nu(\xi)g(\xi)d\xi,$$

where $\mathcal{F}\mu,\mathcal{F}\nu$ denote the Fourier transforms of $\mu,\nu$. In particular, if $\mu(dx) = \varphi(x)dx$ and $\nu(dy) = \psi(y)dy$ for some density functions $\varphi,\psi$ in $\mathbb{R}^d$, then

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y)\varphi(x)\psi(y)dxdy = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\mathcal{F}\psi(\xi)g(\xi)d\xi.$$  

(22)

This relation holds for arbitrary non-negative functions $\varphi,\psi \in L^1(\mathbb{R}^d)$. (To see this, we consider the normalized functions $\varphi/\|\varphi\|_1$ and $\psi/\|\psi\|_1$, where $\|\cdot\|_1$ denotes the $L^1(\mathbb{R}^d)$-norm.) Using the decomposition $\varphi = \varphi^+ - \varphi^-$ with non-negative functions $\varphi^+,\varphi^-$, we see that (22) holds for any functions $\varphi,\psi \in L^1(\mathbb{R}^d)$. In fact, (22) holds for any functions $\varphi,\psi \in L^1_{\mathbb{C}}(\mathbb{R}^d)$, replacing $\psi(y)$ by its conjugate $\psi(y)$ on the left-hand side. (To see this, we write $\varphi = \varphi_1 + i\varphi_2$ where $\varphi_1,\varphi_2$ are the real and imaginary parts of $\varphi$.)

We consider the Bessel kernel (in $\mathbb{R}^d$) of order $\beta > 0$:

$$G_{d,\beta}(x) = \frac{1}{\Gamma(\beta/2)} \int_0^\infty u^{\beta/2-1}e^{-u} \frac{1}{(4\pi u)^d/2} e^{-|x|^2/(4u)}du.$$  

Note that $G_{d,\beta}$ is a density function (see Remark A.3 below) and

$$\mathcal{F}G_{d,\beta}(\xi) = \left(\frac{1}{1+|\xi|^2}\right)^{\beta/2}, \quad \xi \in \mathbb{R}^d.$$  

(23)

Moreover, $G_{d,\alpha} * G_{d,\beta} = G_{d,\alpha+\beta}$ for any $\alpha, \beta > 0$ (see pages 130-135 of [20]).

The following result is an extension of relations (3.4) and (3.5) of [11] to the case of arbitrary $\beta > 0$. 

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Lemma A.1. Let \( f \) be a continuous symmetric kernel of positive type such that \( f(x) < \infty \) if and only if \( x \neq 0 \). Let \( \mu(d\xi) = (2\pi)^{-d}g(\xi)d\xi \), where \( g \) is the Fourier transform of \( f \) in \( \mathcal{S}'(\mathbb{R}^d) \). Let \( \beta > 0 \) be arbitrary. Then

\[
\int_{\mathbb{R}^d} G_{d,\beta}(x)f(x)dx = \int_{\mathbb{R}^d} \left( \frac{1}{1 + |\xi|^2} \right)^{\beta/2} \mu(d\xi) := I_\beta(\mu). \tag{24}
\]

If \( I_\beta(\mu) < \infty \), then, for any \( a \in \mathbb{R}^d \),

\[
\int_{\mathbb{R}^d} e^{ia \cdot x} G_{d,\beta}(x)f(x)dx = \int_{\mathbb{R}^d} \left( \frac{1}{1 + |\xi - a|^2} \right)^{\beta/2} \mu(d\xi). \tag{25}
\]

Proof: Relation (24) follows from (22) with \( \varphi = \psi = G_{d,\beta/2} \). On the left-hand side (LHS), we use the fact that \( G_{d,\beta/2} * G_{d,\beta/2} = G_{d,\beta} \). On the right-hand side (RHS), we use (23) (with \( \beta/2 \) instead of \( \beta \)).

To prove (25), we apply (22) to the complex-valued functions:

\[
\varphi(x) = \psi(x) = e^{ia \cdot x} G_{d,\beta/2}(x).
\]

The term on the LHS is

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(a-x \cdot y)} G_{d,\beta/2}(x)G_{d,\beta/2}(y)f(x-y)dxdy = \int_{\mathbb{R}^d} e^{ia \cdot x} f(x)G_{d,\beta}(x)dx,
\]

using Fubini’s theorem. The application of Fubini’s theorem is justified since

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |e^{i(a-x \cdot y)} G_{d,\beta/2}(x)G_{d,\beta/2}(y)f(x-y)|dxdy = \int_{\mathbb{R}^d} G_{d,\beta}(x)f(x)dx < \infty.
\]

For the term on the RHS, we use the fact that

\[
\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-i(\xi - a) \cdot x} G_{d,\beta/2}(x)dx = \mathcal{F}G_{d,\beta/2}(\xi - a) = \left( \frac{1}{1 + |\xi - a|^2} \right)^{\beta/4}.
\]

\( \square \)

Corollary A.2. Let \((f, \mu)\) be as in Lemma A.1 and \( \beta > 0 \) be arbitrary. Assume that \( I_\beta(\mu) < \infty \). Then

\[
\sup_{a \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left( \frac{1}{1 + |\xi - a|^2} \right)^{\beta/2} \mu(d\xi) = I_\beta(\mu).
\]

Consequently,

\[
\sup_{a \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{1 + |\xi - a|^2} \mu(d\xi) \leq I_\beta(\mu). \tag{26}
\]

Proof: The fact that \( I_\beta(\mu) \) is smaller than the supremum is obvious. To prove the other inequality, we take absolute values on both sides of (25) and we use the fact that \( | \cdots | \leq \int | \cdots | \). For the last statement, we use the fact that \( (1 + |\xi - a|^2)^{\beta/2} \leq 1 + |\xi - a|^2 \), since \( \beta/2 \in (0, 1) \) and the following inequality holds: \((a + b)^p \leq a^p + b^p\) for any \( a, b > 0 \) and \( p \in (0, 1] \). \( \square \)
Remark A.3. The Bessel kernel $G_{d,\beta}(x)$ arises in statistics as the density of the random vector $X$ given by the following hierarchical model:

$$X|U = u \sim N_d(0, 2uI) \quad U \sim \text{Gamma}(\beta/2, 1)$$

where $N_d(0, 2uI)$ denotes the $d$-dimensional normal distribution with covariance matrix $2uI$, $I$ being the identity matrix. Hence, the term on the LHS of (24) is

$$\int_{\mathbb{R}^d} G_{d,\beta}(x)f(x)dx = E[f(X)] = \frac{1}{\Gamma(\beta/2)} \int_0^\infty u^{\beta/2-1}e^{-u}E[f(X)|U = u]du.\)

This can be computed explicitly if $f(x) = |x|^{-\alpha}$ with $\alpha \in (0, d)$. First, note that if $Z \sim N_d(0, 2tI)$, then its negative moment of order $-\alpha$ is:

$$E(|Z|^{-\alpha}) = \frac{1}{2}C_{\alpha,d}c_d\Gamma(\alpha/2)t^{-\alpha/2}$$

where $c_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in $\mathbb{R}^d$. To see this, we use the fact that $\mathcal{F}f(\xi) = C_{\alpha,d}|\xi|^{-d+\alpha}d\xi$ in $\mathcal{S}'(\mathbb{R}^d)$. Hence,

$$E(|Z|^{-\alpha}) = \int_{\mathbb{R}^d} |x|^{-\alpha} \frac{1}{(4\pi t)^{d/2}}e^{-|x|^2/(4t)}dx = C_{\alpha,d} \int_{\mathbb{R}^d} |\xi|^{-d+\alpha}e^{-t|\xi|^2}d\xi$$

and (27) follows by passing to the polar coordinates. We obtain that

$$\int_{\mathbb{R}^d} G_{d,\beta}|x|^{-\alpha}dx = \frac{c_{\alpha,d}c_d\Gamma(\alpha/2)}{2\Gamma(\beta/2)} \int_0^\infty u^{(\beta-\alpha)/2-1}e^{-u}du = \frac{C_{\alpha,d}c_d\Gamma((\beta-\alpha)/2)\Gamma(\alpha/2)}{2\Gamma(\beta/2)}.\)

(Note that the integral is finite if and only if $\alpha < \beta$.)

Remark A.4. A relation similar to (27) for stable random variables was used in the proof of Theorem 2.1 (Step 3). More precisely, if $X$ is a $d$-dimensional random variable with a symmetric stable distribution with index $\beta \in (0, 2)$ (i.e. $E(e^{-i\xi \cdot X}) = e^{-|\xi|^\beta}$ for all $\xi \in \mathbb{R}^d$), then

$$E(|X|^{-\alpha}) = \frac{1}{\beta}C_{\alpha,d}c_d\Gamma(\alpha/\beta).$$  \hspace{1cm} (28)

We include the proof of (28) for the sake of completeness. We denote by $f_X$ the density of $X$. Recall that $\mathcal{F}f(\xi) = C_{\alpha,d}|\xi|^{-d+\alpha}d\xi$ in $\mathcal{S}'(\mathbb{R}^d)$ i.e. for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |x|^{-\alpha}\varphi(x)dx = C_{\alpha,d} \int_{\mathbb{R}^d} |\xi|^{-d+\alpha}\mathcal{F}\varphi(\xi)d\xi.$$  

Using a regularization technique, one can show that the previous relation also holds for $\varphi = f_X$, since $\mathcal{F}f_X(\xi) \to 0$ rapidly as $|\xi| \to \infty$ and $f_X$ is bounded and infinitely differentiable (see page 13 of [21]). Hence,

$$E(|X|^{-\alpha}) = \int_{\mathbb{R}^d} |x|^{-(d-\alpha)}f_X(x)dx = C_{\alpha,d} \int_{\mathbb{R}^d} |\xi|^{-d+\alpha}e^{-|\xi|^\beta}d\xi.$$
We now pass to the polar coordinates $\xi = rz$ with $r > 0$ and $z \in S_d$, where $S_d$ is the unit sphere in $\mathbb{R}^d$. Let $c_d$ be the area of $S_d$. We have

$$E(|X|^{-\alpha}) = C_{\alpha,d}c_d \int_0^\infty r^{-d+\alpha}e^{-r^\beta}r^{d-1}dr,$$

and relation (28) follows using the change of variable $s = r^\beta$.

References


