Q-Markov Random Probability Measures and Their Posterior Distributions

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Abstract

In this paper, we use the Markov property introduced in Balan and Ivanoff (2002) for set-indexed processes and we prove that a Markov prior distribution leads to a Markov posterior distribution. In particular, by proving that a neutral to the right prior distribution leads to a neutral to the right posterior distribution, we extend a fundamental result of Doksum (1974) to arbitrary sample spaces.

Keywords: random probability measure; Q-Markov process; transition system; Dirichlet process; neutral to the right process

1 Introduction

Bayesian non-parametric statistics is a field that has been introduced by Ferguson in 1973 and has become increasingly popular among the theoretical statisticians in the past few decades. The philosophy behind this field is to assume that the common (unknown) distribution $P$ of a given sample $X = (X_1, \ldots, X_n)$ is also governed by randomness, and therefore can be regarded as a stochastic process (indexed by sets). The best way for a Bayesian statistician to guess the “shape” of the prior distribution $P$ is to identify the posterior distribution of $P$ given $X$ and to prove that it satisfies the same properties as the prior.

Formalizing these ideas, we can say that a typical problem in Bayesian non-parametric statistics is to identify a class $\Sigma$ of “random distributions” $P$ such

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that if \( X \) is a sample of \( n \) observations drawn according to \( P \), then the posterior distribution of \( P \) given \( X \) remains in the class \( \Sigma \). The purpose of this paper is to introduce a new class \( \Sigma \) for which this property is preserved. This is the class of \( Q \)-Markov processes (or distributions), which contains the extensively studied class of neutral to the right processes.

There are two major contributions in the literature in this field. The first one is Ferguson’s (1973) fundamental paper where it is shown that the posterior distribution of a Dirichlet process is also Dirichlet. (By definition, a Dirichlet process with parameter measure \( \alpha \) has a Dirichlet finite dimensional distribution with parameters \( \alpha(A_1), \ldots, \alpha(A_k) \alpha((\bigcup_{i=1}^{k} A_i)^c) \) over any disjoint sets \( A_1, \ldots, A_k \in \mathcal{B} \).) The second one is Doksum’s (1974) fundamental paper where it is proved that if \( X = \mathbb{R} \), then the posterior distribution of a neutral to the right process is also neutral to the right. (A random probability distribution function \( F := (F_t)_{t \in \mathbb{R}} \) is neutral to the right if \( F_{t_1}, (F_{t_2} - F_{t_1})/(1 - F_{t_1}), \ldots, (F_{t_k} - F_{t_{k-1}})/(1 - F_{t_{k-1}}) \) are independent \( \forall t_1 < \ldots < t_k \), or equivalently, \( Y_t := -\ln(1 - F_t), t \in \mathbb{R} \) is a process with independent increments.) A quick review of the literature to date (Ferguson, 1974; Ferguson and Phadia, 1979; Dykstra and Laud, 1981; Hjort, 1990; Walker and Muliere, 1997; Walker and Muliere, 1999) reveals that neutral to the right processes have received considerably attention in the past three decades, especially because of their appealing representation using Lévy processes and because of their applications in survival analysis, reliability theory, life history data.

In the present paper we extend Doksum’s result to the class of \( Q \)-Markov processes introduced in Balan and Ivanoff (2002), which are characterized by Markov-type finite dimensional distributions. Unlike Doksum’s paper (and unlike most of the statistical papers generated by it) our results are valid for arbitrary sample spaces \( \mathcal{X} \), which can be endowed with a certain topological structure (in particular for \( \mathcal{X} = \mathbb{R}^d \)). Our main result (Theorem 3.4) proves that if \( P := (P_A)_{A \in \mathcal{B}} \) is a set-Markov random probability measure and \( X_1, \ldots, X_n \) is a sample from \( P \), then the conditional distribution of \( P \) given \( X_1, \ldots, X_n \) is also set-Markov. This result is new even in the case \( \mathcal{X} = \mathbb{R} \), when the set-Markov property coincides with the classical Markov property.

The paper is organized as follows:

In Section 2 we describe the structure that has to be imposed on the sample space \( \mathcal{X} \) (which will be assumed for the entire paper); under this structure we identify the necessary ingredients for the construction of set-Markov (respectively \( Q \)-Markov) random probability measure.

In Section 3 we introduce the Bayesian nonparametric framework and we prove that a set-Markov prior distribution leads to a set-Markov posterior distribution. The essence of all calculations is an integral form of Bayes’ formula.

In Section 4 we define neutral to the right processes and using their \( Q \)-Markov property we prove that a neutral to the right prior distribution leads to a neutral to the right posterior distribution.

The paper also includes two appendices: Appendix A contains two elemen-
tary results which are used for the proof of Theorem 3.4; Appendix B contains a Bayes property of a classical Markov chain, which is interesting by itself and which has motivated this paper.

2 Q-Markov random probability measures

Let \((X, B)\) be an arbitrary measurable space (the sample space).

**Definition 2.1** A collection \(P := (P_A)_{A \in B}\) of \([0, 1]\)-valued random variables is called a random probability measure if

(i) it is finitely additive in distribution, i.e., for every disjoint sets \((A_j)_{j=1}^k\) and for every \(1 \leq i_1 < \ldots < i_m \leq k\), the distribution of \((P_{\bigcup_{j=i_1}^{i_m} A_j})\) coincides with the distribution of \((\sum_{j=1}^{i_1} P_{A_j}, \ldots, \sum_{j=i_m}^k P_{A_j})\);

(ii) \(P_X = 1\) a.s.; and

(iii) it is countably additive in distribution, i.e., for every decreasing sequence \((A_n)_{n \in \mathbb{N}}\) with \(\bigcap_n A_n = \emptyset\) we have \(\lim_n P_{A_n} = 0\) a.s.

Note that the almost sure convergence of (iii) (in the above definition) is equivalent to the convergence in distribution and the convergence in mean.

In order to construct a random probability measure \(P\) on \(B\) it is enough to specify its finite dimensional distributions \(\mu_{A_1, \ldots, A_k}\) over all un-ordered collections \(\{A_1, \ldots, A_k\}\) of disjoint sets in \(B\). Some conditions need to be imposed.

**Condition C1.** If \(\{A_1, \ldots, A_k\}\) is an un-ordered collection of disjoint sets and we let \(A'_l := \bigcup_{l=1}^{i_l} A_j; l = 1, \ldots, m\) for \(1 \leq i_1 < \ldots < i_m \leq k\), then \(\mu_{A'_1, \ldots, A'_m} = \mu_{A_1 \ldots A_k} \circ \alpha^{-1}\), where \(\alpha(x_1, \ldots, x_k) = (\sum_{j=1}^{i_1} x_j, \ldots, \sum_{j=i_m-1}^{i_m} x_j)\).

**Condition C2.** For every \((A_n)_{n \in \mathbb{N}}\) with \(A_{n+1} \subseteq A_n\) for all \(n\) and \(\cap_n A_n = \emptyset\), we have \(\lim_n \mu_{A_n} = \delta_0\).

In this paper we will assume that the sample space \(X\) has an additional underlying structure which we begin now to explain.

Let \(X\) be a (Hausdorff) topological space and \(B\) its Borel \(\sigma\)-field. We will assume that there exists a collection \(A\) of closed subsets of \(X\) which generates \(B\) (i.e. \(B = \sigma(A)\)) and which has the following properties:

1. \(\emptyset, X \in A\);

2. \(A\) is a semilattice i.e., \(A\) is closed under arbitrary intersections;

3. \(\forall A, B \in A; A, B \neq \emptyset \Rightarrow A \cap B \neq \emptyset\);

4. There exists a sequence \((A_n)_{n \in \mathbb{N}}\) of finite sub-semilattices of \(A\) such that \(\forall A \in A\), there exist \(A_n \in A_n(u), \forall n\) with \(A = \cap_n A_n\) and \(A \subseteq A_0, \forall n\).

(Here \(A_n(u)\) denotes the class of all finite unions of sets in \(A_n\).)
Let \( \mathcal{A} \) be an indexing collection. By properties 2 and 3, the collection \( \mathcal{A} \) has the finite intersection property, and hence its minimal set \( \emptyset' := \cap_{A \in \mathcal{A} \setminus \{\emptyset\}} A \) is non-empty.

The typical example of a sample space \( \mathcal{X} \) which can be endowed with an indexing collection is \( \mathbb{R}^d \); in this case \( \mathcal{A} = \{ [0, z]; z \in \mathbb{R}^d \} \cup \{ \emptyset, \mathbb{R}^d \} \) and the approximation sets \( A_n \) have vertices with dyadic coordinates.

We denote with \( \mathcal{A}(u) \) the class of all finite unions of sets in \( \mathcal{A} \), with \( \mathcal{C} \) the semialgebra of the sets \( C = A \setminus B \) with \( A \in \mathcal{A}, B \in \mathcal{A}(u) \) and with \( \mathcal{C}(u) \) the algebra of sets generated by \( \mathcal{C} \). Note that \( \mathcal{B} = \sigma(\mathcal{C}(u)) \).

We introduce now the definition of the \( \mathcal{Q} \)-Markov property. This definition has been originally considered in Balan and Ivanoff (2002) for finitely additive real-valued processes indexed by the algebra \( \mathcal{C}(u) \). In this paper, we will restrict our attention to random probability measures.

**Definition 2.2 (a)** For each \( B_1, B_2 \in \mathcal{A}(u) \) with \( B_1 \subseteq B_2 \), let \( Q_{B_1,B_2} \) be a transition probability on \([0, 1]\). The family \( \mathcal{Q} := (Q_{B_1,B_2})_{B_1 \subseteq B_2} \) is called a transition system if \( \forall B_1 \subseteq B_2 \subseteq B_3 \text{ in } \mathcal{A}(u), \forall z_1 \in [0,1], \forall \Gamma_3 \in \mathcal{B}([0,1]) \)

\[
Q_{B_1,B_2}(z_1;\Gamma_3) = \int_{[0,1]} Q_{B_2,B_3}(z_2;\Gamma_3)Q_{B_1,B_2}(z_1;dz_2)
\]

(b) Given a transition system \( \mathcal{Q} := (Q_{B_1,B_2})_{B_1 \subseteq B_2} \), a random probability measure \( P := (P_A)_{A \in \mathcal{B}} \), defined on a probability space \((\Omega, \mathcal{F}, P)\), is called \( \mathcal{Q} \)-Markov if \( \forall B_1 \subseteq B_2 \text{ in } \mathcal{A}(u), \forall \Gamma_2 \in \mathcal{B}([0,1]) \)

\[
P[P_{B_2} \in \Gamma_2|\mathcal{F}_{B_1}] = Q_{B_1,B_2}(P_{B_2};\Gamma_2) \text{ a.s.}
\]

where \( \mathcal{F}_{B_1} := \sigma(\{P_A; A \in \mathcal{A}, A \subseteq B_1\}) \).

A \( \mathcal{Q} \)-Markov random probability measure can be constructed using the following additional consistency condition.

**Condition C3.** If \((Y_1, \ldots, Y_k)\) is a vector with distribution \( \mu_{C_1 \ldots C_k} \) where \( C_1 = B_1; C_i = B_i \setminus B_{i-1}; i = 2, \ldots, k \) and \( B_1 \subseteq \ldots \subseteq B_k \) are sets in \( \mathcal{A}(u) \), then for every \( i = 2, \ldots, k \), the distribution of \( Y_i \) given \( Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1} \) depends only on \( y := \sum_{j=1}^{i-1} y_j \) and is equal to \( Q_{B_{i-1},B_i}(y; y+\cdot) \).

The next result follows immediately by Kolmogorov’s extension theorem.

**Theorem 2.3** Let \( \mathcal{Q} := (Q_{B_1,B_2})_{B_1 \subseteq B_2} \) be a transition system. For each unordered collection \( \{A_1, \ldots, A_k\} \) of disjoint sets in \( \mathcal{B} \) let \( \mu_{A_1 \ldots A_k} \) be a probability measure on \([0,1]^k, \mathcal{B}([0,1]^k)\) such that C1-C3 hold; let \( \mu_0 = \delta_0, \mu_{X'} = \delta_1 \). Then there exists a probability measure \( \mathcal{P}^{1} \) on \([0,1]^\mathcal{B}, \mathcal{B}([0,1]^\mathcal{B})\) under which the coordinate-variable process \( P := (P_A)_{A \in \mathcal{B}} \) is a \( \mathcal{Q} \)-Markov random probability measure whose finite dimensional distributions are the measures \( \mu_{A_1 \ldots A_k} \).
Examples:

1. Let $P$ be the Dirichlet process with parameter measure $\alpha$. For any disjoint sets $A_1, \ldots, A_k$ in $\mathcal{B}$, $(P_{A_1}, \ldots, P_{A_k})$ has a Dirichlet distribution with parameters $\alpha(A_1), \ldots, \alpha(A_k), \alpha((\cup_{j=1}^k A_j)^c)$. The ratio $P_{A_i}/(1-\sum_{j=1}^{i-1} P_{A_j})$ is independent of $P_{A_1}, \ldots, P_{A_{i-1}}$ and has a Beta distribution with parameters $\alpha(A_i), \alpha((\cup_{j=1}^i A_j)^c)$: hence the distribution of $P_{A_i}$ given $P_{A_1}, \ldots, P_{A_{i-1}}$ depends only on $\sum_{j=1}^{i-1} P_{A_j}$. The process $P$ is Q-Markov with $Q_{B_1B_2}(z_1; \Gamma_2)$ given $Q_{B_1B_2}(z_1; \Gamma_1)$ equal to the value at $(\Gamma_2 - z_1)/(1-z_1)$ of the Beta distribution with parameters $\alpha(B_2 \setminus B_1), \alpha(B_1^c)$. 

2. Let $P := (1/N)\sum_{j=1}^N \delta_{Z_j}$ be the empirical measure of a sample $Z_1, \ldots, Z_N$ from a non-random distribution $P_0$ on $\mathcal{X}$. For any disjoint sets $A_1, \ldots, A_k$ in $\mathcal{B}$, $(NP_{A_1}, \ldots, NP_{A_k})$ has a multinomial distribution with $N$ trials and $P_0(A_1), \ldots, P_0(A_k)$ probabilities of success; hence the distribution of $NP_{A_i}$ given $NP_{A_1}, \ldots, NP_{A_{i-1}}$ depends only on $\sum_{j=1}^{i-1} P_{A_j}$ (it is a binomial distribution with $N(1-\sum_{j=1}^{i-1} P_{A_j})$ trials and $P_0(A_i)/(1-\sum_{j=1}^{i-1} P_{A_j})$ probability of success). The process $P$ is Q-Markov with

$$Q_{B_1B_2}\left(m_1/N, \left\{m_2/N\right\}\right) = \left(\begin{array}{c} N-m_1 \cr m_2-m_1 \end{array}\right) \frac{P_0(C)^{m_2-m_1}P_0(B_1^{c})^{N-m_2}}{P_0(B_1^{c})^{N-m_1}}$$

where $\left(\begin{array}{c} a \cr b \end{array}\right) = a!/b!(a-b)!$ is the binomial coefficient and $C = B_2 \setminus B_1$.

3. Let $P := (1/N)\sum_{j=1}^N \delta_{W_j}$ be the empirical measure of a sample $W_1, \ldots, W_N$ from a Dirichlet process with parameter measure $\alpha$. For any disjoint sets $A_1, \ldots, A_k$ in $\mathcal{B}$, $(NP_{A_1}, \ldots, NP_{A_k})$ has a Pólya distribution with $N$ trials and parameters $\alpha(A_1), \ldots, \alpha(A_k), \alpha((\cup_{j=1}^k A_j)^c)$; hence the distribution of $NP_{A_i}$ given $NP_{A_1}, \ldots, NP_{A_{i-1}}$ depends only on $\sum_{j=1}^{i-1} P_{A_j}$ (it is a Pólya distribution with $N(1-\sum_{j=1}^{i-1} P_{A_j})$ trials and parameters $\alpha(A_i), \alpha((\cup_{j=1}^i A_j)^c)$). The process $P$ is Q-Markov with

$$Q_{B_1B_2}\left(m_1/N, \left\{m_2/N\right\}\right) = \left(\begin{array}{c} N-m_1 \cr m_2-m_1 \end{array}\right) \frac{\alpha(C)^{m_2-m_1}\alpha(B_2^{c})^{N-m_2}}{\alpha(B_1^{c})^{N-m_1}}$$

where $\alpha^{[x]} = \alpha(\alpha + 1) \ldots (\alpha + x - 1)$ and $C = B_2 \setminus B_1$.

3 The posterior distribution of a Q-Markov random probability measure

We begin to introduce the Bayesian nonparametric framework.
Let \( P := (P_A)_{A \in B} \) be a \( Q \)-Markov random probability measure defined on a probability space \((\Omega, \mathcal{F}, \mathcal{P})\) and \( X_i: \Omega \to \mathcal{X}, i = 1, \ldots, n \) some \( \mathcal{F}/\mathcal{B} \)-measurable functions such that \( \forall A_1, \ldots, A_n \in \mathcal{B} \)

\[
\mathcal{P}[X_1 \in A_1, \ldots, X_n \in A_n|P] = \prod_{i=1}^n P_{A_i} \quad \text{a.s.}
\]

We say that \( \tilde{X} := (X_1, \ldots, X_n) \) is a sample from \( P \). The distribution of \( P \) is called prior, while the distribution of \( P \) given \( \tilde{X} \) is called posterior. Note that \((P_A)_{A \in B} \) and \( X_1, \ldots, X_n \) can be constructed as coordinate-variables on the space \(([0, 1]^B \times \mathcal{X}^n, \mathcal{B}(0, 1])^B \times \mathcal{B}^n \) under the probability measure \( \mathcal{P} \) defined by

\[
\mathcal{P}(D \times \prod_{i=1}^n A_i) := \int_D \prod_{i=1}^n \omega_{A_i} \mathcal{P}^1(d\omega), \quad D \in \mathcal{B}(0, 1]^B, A_i \in \mathcal{B}
\]

where \( \mathcal{P}^1 \) is the probability measure given by Theorem 2.3.

The goal of this section is to prove that the posterior distribution of \( P \) given \( \tilde{X} = \tilde{x} \) is \( Q(\tilde{x}) \)-Markov (for some “posterior” transition system \( Q(\tilde{x}) \)).

Let \( \alpha_n \) be the law of \( X \) under \( \mathcal{P} \) and \( \mu_{A_1, \ldots, A_k} \) be the law of \((P_{A_1}, \ldots, P_{A_k}) \) under \( \mathcal{P} \), for every \( A_1, \ldots, A_k \in \mathcal{B} \). Note that \( \alpha_n(\prod_{i=1}^n A_i) = E[\prod_{i=1}^n P_{A_i}] \), where \( E \) denotes the expectation with respect to \( \mathcal{P} \).

For each set \( B_1 \in \mathcal{A}(u) \), let \( \nu_{B_1} \) be the law of \((X_1, \ldots, X_n, P_{B_1}) \) under \( \mathcal{P} \). Note that \( \nu_{B_1}(\prod_{i=1}^n A_i \times \Gamma_1) = E[\prod_{i=1}^n P_{A_i} \cdot I_{\Gamma_1}(P_{B_1})] \) and

\[
\nu_{B_1}(\tilde{A} \times \Gamma_1) = \int_{\tilde{A}} \mu_{B_1}^{(\tilde{x})}(\Gamma_1) \alpha_n(d\tilde{x}) = \int_{\Gamma_1} \tilde{Q}_{B_1}(z_1; \tilde{A}) \mu_{B_1}(dz_1) \quad (1)
\]

where \( \mu_{B_1}^{(\tilde{x})}(\Gamma_1) := \mathcal{P}[P_{B_1} \in \Gamma_1|X = \tilde{x}] \) and \( \tilde{Q}_{B_1}(z_1; \tilde{A}) := \mathcal{P}[X \in \tilde{A}|P_{B_1} = z_1] \).

For each sets \( B_1, B_2 \in \mathcal{A}(u); B_1 \subseteq B_2 \), let \( \nu_{B_1, B_2} \) be the law of \((X_1, \ldots, X_n, P_{B_1}, P_{B_2}) \) under \( \mathcal{P} \). Note that \( \nu_{B_1, B_2}(\prod_{i=1}^n A_i \times \Gamma_1 \times \Gamma_2) = E[\prod_{i=1}^n P_{A_i} \cdot I_{\Gamma_1}(P_{B_1}) \cdot I_{\Gamma_2}(P_{B_2})] \) and

\[
\nu_{B_1, B_2}(\tilde{A} \times \Gamma_1 \times \Gamma_2) = \int_{\tilde{A}} \int_{\Gamma_1} \int_{\Gamma_2} Q_{B_1, B_2}^{(\tilde{x})}(z_1; \tilde{A}) \mu_{B_1}^{(\tilde{x})}(dz_1) \alpha_n(d\tilde{x}) = \int_{\Gamma_1 \times \Gamma_2} \tilde{Q}_{B_1, B_2}(z_1, z_2; \tilde{A}) \mu_{B_1, B_2}(dz_1 \times dz_2) \quad (2)
\]

\[
\text{where} \quad Q_{B_1, B_2}^{(\tilde{x})}(z_1; \Gamma_2) := \mathcal{P}[P_{B_2} \in \Gamma_2|X = \tilde{x}, P_{B_1} = z_1] \quad (4)
\]

and \( \tilde{Q}_{B_1, B_2}(z_1, z_2; \tilde{A}) := \mathcal{P}[X \in \tilde{A}|P_{B_1} = z_1, P_{B_2} = z_2] \). (For the first equality we used the first integral in the decomposition (1) of \( \nu_{B_1} \).)
Using the second integral in the decomposition (1) of \( \nu_{B_1} \) and the \( Q \)-Markov property for representing \( \mu_{B_1B_2} \) we get: (for \( \mu_{B_1} \)-almost all \( z_1 \))

\[
\int_{A} Q_{B_1B_2}^{(2)}(z_1; \Gamma_2)Q_{B_1}(z_1; dz_2) = \int_{\Gamma_2} \tilde{Q}_{B_1}(z_1, z_2; \tilde{A})Q_{B_1B_2}(z_1; dz_2). \tag{5}
\]

This very important equation is the key for determining the posterior transition probabilities \( Q_{B_1B_2}^{(2)} \) from the prior transition probabilities \( Q_{B_1B_2} \), providing that \( \tilde{Q}_{B_1}(z_1; \prod_{i=1}^{n} A_i) = \mathcal{E}[\prod_{i=1}^{n} P_{A_i} | P_{B_1} = z_1] \) and \( Q_{B_1B_2}(z_1, z_2; \prod_{i=1}^{n} A_i) = \mathcal{E}[\prod_{i=1}^{n} P_{A_i} | P_{B_1} = z_1, P_{B_2} = z_2] \) are easily computable.

We note that each \( Q_{B_1B_2}^{(2)}(z_1; \cdot) \) is well-defined only for \( \nu_{B_1} \)-almost all \((z, z_1)\). Moreover, as we will see in the proof of Theorem 3.4 and it was correctly pointed out by an anonymous referee, \( Q^{(2)} \) may not be a genuine transition system as introduced by Definition 2.2.(a). To avoid any confusion we introduce the following terminology.

**Definition 3.1** The family \( Q^{(2)} := (Q_{B_1B_2}^{(2)})_{B_1 \subseteq B_2} \) defined by (4) is called a posterior transition system (corresponding to \( P \) and \( \mathcal{X} \)) if \( \forall B_1 \subseteq B_2 \subseteq B_3 \) in \( \mathcal{A}(u) \), \( \forall \Gamma_3 \in \mathcal{B}([0,1]) \) and for \( \nu_{B_1} \)-almost all \((z, z_1)\)

\[
Q_{B_1B_2}^{(2)}(z_1; \Gamma_3) = \int_{[0,1]} Q_{B_2B_3}^{(2)}(z_2; \Gamma_3)Q_{B_1B_2}^{(2)}(z_1; dz_2)
\]

In this case, we will say that the conditional distribution of \( P \) given \( \mathcal{X} = z \) is \( Q^{(2)} \)-Markov if \( \forall B_1 \subseteq B_2 \) in \( \mathcal{A}(u) \), \( \forall \Gamma_2 \in \mathcal{B}([0,1]) \)

\[
\mathcal{P}[P_{B_2} \in \Gamma_2 | \mathcal{F}_{B_1}, \mathcal{X}] = Q_{B_1B_2}^{(2)}(P_{B_1}; \Gamma_2) \ a.s.
\]

We proceed now to the proof of the main theorem. Two preliminary lemmas are needed.

Let \( B_1 \subseteq B_2 \) be some arbitrary sets in \( \mathcal{A}(u) \), \( C := B_2 \setminus B_1 \) and \( 0 \leq l \leq r \leq n \). The next lemma shows us what happens intuitively with the probability that the first \( l \) observations fall in \( B_1 \), the next \( r - l \) observations fall in \( C \) and the remaining \( n - r \) observations fall in \( B_2 \), given \( P_{B_1} \) and \( P_{B_2} \).

**Lemma 3.2** For each \( B_1 \subseteq B_2 \) in \( \mathcal{A}(u) \) and \( A_1, \ldots, A_n \in \mathcal{B} \), let

\[
\tilde{A} := \prod_{i=l}^{l}(A_i \cap B_1) \times \prod_{i=l+1}^{r}(A_i \cap C) \times \prod_{i=r+1}^{n}(A_i \cap B_2) \tag{6}
\]

where \( C := B_2 \setminus B_1 \) and \( 0 \leq l \leq r \leq n \). Let \( \tilde{A}_1 := \prod_{i=l}^{l}(A_i \cap B_1) \times X^{n-l} \), \( \tilde{A}_2 := \prod_{i=l+1}^{r}(A_i \cap C) \times X^{r-l} \), \( \tilde{A}_3 := \prod_{i=r+1}^{n}(A_i \cap B_2) \times X^{n-r} \), \( \tilde{A}_3 := \tilde{A}_2 \cup \tilde{A}_3 \).

(a) For \( \mu_{B_1} \)-almost all \( z_1 \), \( \tilde{Q}_{B_1}(z_1; \tilde{A}) = \tilde{Q}_{B_1}(z_1; \tilde{A}_1) \cdot \tilde{Q}_{B_1B_2}(z_1, z_2; \tilde{A}_2) \cdot \tilde{Q}_{B_2}(z_2; \tilde{A}_3) \).

(b) For \( \mu_{B_1B_2} \)-almost all \( (z_1, z_2) \),

\[
\tilde{Q}_{B_1B_2}(z_1, z_2; \tilde{A}) = \tilde{Q}_{B_1}(z_1; \tilde{A}_1) \cdot \tilde{Q}_{B_1B_2}(z_1, z_2; \tilde{A}_2) \cdot \tilde{Q}_{B_2}(z_2; \tilde{A}_3).
\]
Proof: We will prove only (b) since part (a) follows by a similar argument. Note that the sets $\tilde{A}$ form a $\pi$-system generating the $\sigma$-field $\mathcal{B}^n$ on $B_1 \times C^{r-1} \times (B_2)^{n-r}$.

Since $\sigma(A) = B$ and $A$ is a $\pi$-system, using a Dynkin system argument, it is enough to consider the case $A_1, \ldots, A_n \in A$. Note that

$$E[\prod_{i=r+1}^{n} P_{A_i \cap B_2} | \mathcal{F}_{B_2}] = E[\prod_{i=r+1}^{n} P_{A_i \cap B_2} | P_{B_2}] = \tilde{Q}_{B_2}(P_{B_2}; \tilde{A}_3).$$

By double conditioning with respect to $\mathcal{F}_{B_2}$, we have

$$\tilde{Q}_{B_1B_2}(z_1, z_2; \tilde{A}) = E[\prod_{i=1}^{l} P_{A_i \cap B_1} \prod_{i=l+1}^{r} P_{A_i \cap B_2} \prod_{i=r+1}^{n} P_{A_i \cap B_2} | P_{B_1} = z_1, P_{B_2} = z_2] =$$

$$\tilde{Q}_{B_2}(z_2; \tilde{A}_3) \cdot E[\prod_{i=1}^{l} P_{A_i \cap B_1} \cdot \prod_{i=l+1}^{r} P_{A_i \cap B_2} | P_{B_1} = z_1, P_{B_2} = z_2].$$

For the second term we have

$$E[\prod_{i=1}^{l} P_{A_i \cap B_1} \prod_{i=l+1}^{r} P_{A_i \cap B_2} | P_{B_1}, P_{B_2}] =$$

$$E[\prod_{i=1}^{l} P_{A_i \cap B_1} E[\prod_{i=l+1}^{r} P_{A_i \cap B_2} | (P_{A_i \cap B_1})_{i \leq l}, P_{B_1}, P_{B_2}] | P_{B_1}, P_{B_2}].$$

Since $P_{A_i \cap C} = P_{B_1 \cup (A_i \cap B_2)} - P_{B_1}$, using Lemma A.1 (Appendix A)

$$E[\prod_{i=l+1}^{r} P_{A_i \cap C} | (P_{A_i \cap B_1})_{i \leq l}, P_{B_1}, P_{B_2}] = \tilde{Q}_{B_2}(P_{B_1}, P_{B_2}; \tilde{A}_2).$$

(In order to use Lemma A.1, we need $A_{l+1} \subseteq A_{l+2} \subseteq \ldots \subseteq A_r$. Note that this is not a restriction since if we can consider the minimal semilattice $\{A_{1}', \ldots, A_{m}'\}$ determined by the sets $A_{l+1}, \ldots, A_r$, which is ordered such that $A_{i}' \nsubseteq \cup_{i \neq j} A_{j}' \forall j$, and we let $B_{j}' = \cup_{s=1}^{l} A_{s}'$ and $C_{j}' = B_{j}' \setminus B_{j-1}'$, then each $A_i = \cup_{j \in J_i} C_{j}'$ for some $J_i \subseteq \{1, \ldots, m\}$. We have $A_i \cap C = \cup_{j \in J_i} (B_{j}' \cap C \setminus (B_{j-1}' \cap C))$ and $\prod_{i=l+1}^{r} P_{A_i \cap C} = h(P_{B_1 \cap C}, \ldots, P_{B_m \cap C})$ for some function $h$.) Finally, since $\mathcal{F}_{B_1}$ is conditionally independent of $P_{B_2}$ given $P_{B_1}$ and $P_{A_i \cap B_1}, i \leq l$ are $\mathcal{F}_{B_2}$-measurable, we have $E[\prod_{i=1}^{l} P_{A_i \cap B_1} | P_{B_1}, P_{B_2}] = E[\prod_{i=1}^{l} P_{A_i \cap B_1} | P_{B_1}] = \tilde{Q}_{B_1}(P_{B_1}; \tilde{A}_1)$, which concludes the proof. \square

Note: Let $\tilde{A}_{12} := \tilde{A}_1 \cap \tilde{A}_2$. By a similar argument one can show that

$$\tilde{Q}_{B_1B_2}(z_1, z_2; \tilde{A}_{12}) = \tilde{Q}_{B_1}(z_1; \tilde{A}_1) \cdot \tilde{Q}_{B_2}(z_1; \tilde{A}_2)$$

(7)
The next lemma tells us that if \( B_1 \subseteq B_2 \) are “nicely-shaped” regions and we want to predict the value of \( P_{B_2} \) given the value of \( P_{B_1} \), and a sample \( X \) from \( P \), then we can forget all about those values \( X_i \) which fall inside the region \( B_1 \). The reason for this phenomenon is the very essence of the Markov property given by Definition 2.2(b), which says that for predicting the value of \( P_{B_2} \) it suffices to know the value of \( P_{B_1} \), i.e. all the information about the values of \( P \) inside the region \( B_1 \) can be discarded.

**Lemma 3.3** For every \( B_1, B_2 \in \mathcal{A}(u) \) with \( B_1 \subseteq B_2 \), for every \( \Gamma_2 \in \mathcal{B}([0, 1]) \) and for \( \nu_{B_1} \)-almost all \( (z_1, z_2) \), \( \tilde{Q}_{B_1, B_2}^{(z)}(z_1; \Gamma_2) \) does not depend on those \( x_i \)'s that fall in \( B_1 \); in particular, for \( \nu_{B_1} \)-almost all \( (z_1, z_2) \) in \( B_1^\circ \times [0, 1] \), \( \tilde{Q}_{B_1, B_2}^{(z)}(z_1; \Gamma_2) = Q_{B_1, B_2}(z_1; \Gamma_2) \).

**Proof:** Let \( A_1, \ldots, A_n \in \mathcal{B} \) and \( \tilde{A} \) defined by (6). Using (5) and Lemma 3.2,(b) combined with (8) we have

\[
\int_{A} \tilde{Q}_{B_1, B_2}^{(z)}(z_1; \Gamma_2) \tilde{Q}_{B_1}(z_1; d\tilde{z}) = \int_{\Gamma_2} \tilde{Q}_{B_1, B_2}(z_1, z_2; \tilde{A}) \tilde{Q}_{B_1}(z_1; d\tilde{z}) = \int_{\Gamma_2} \tilde{Q}_{B_1, B_2}(z_1, z_2; \tilde{A}_2) \tilde{Q}_{B_1}(z_1; d\tilde{z}) = \int_{\tilde{A}_2} \tilde{Q}_{B_1}(z_1; \Gamma_2) \tilde{Q}_{B_1}(z_1; d\tilde{z}).
\]

The result follows by Lemma A.2 (Appendix A) since on the set \( B_1^\circ \times C_{\nu-1} \times (B_2^\circ)^{n-l} \), \( \tilde{Q}_{B_1}(z_1; \cdot) \) is the product measure between its marginal with respect to the first \( l \) components restricted to \( B_1^\circ \) and its marginal with respect to the remaining \( n-l \) components restricted to \( C_{\nu-1} \times (B_2^\circ)^{n-l} \) (by Lemma 3.2,(a)). \( \square \)

Here is the main result of the paper.

**Theorem 3.4** If \( P := (P_A)_{A \in \mathcal{B}} \) is a \( Q \)-Markov random probability measure and \( X := (X_1, \ldots, X_n) \) is a sample from \( P \), then the family \( \mathcal{Q}^{(z)} = \{Q_{B_1, B_2}^{(z)}\}_{B_1 \subseteq B_2} \) defined by (4) is a posterior transition system and the conditional distribution of \( P \) given \( X = z \) is \( \mathcal{Q}^{(z)} \)-Markov.

**Proof:** By Proposition 5 of Balan and Ivanoff (2002), it is enough to show that \( \forall B_1 \subseteq B_2 \subseteq \ldots \subseteq B_k \in \mathcal{A}(u), \forall \Gamma \in \mathcal{B}([0, 1])^k \) and for \( \alpha_n \)-almost all \( z \)

\[
\mathcal{P}(P_{B_1}, \ldots, P_{B_k}) \in \tilde{\Gamma} | X = z = \int_{\Gamma} Q_{B_k, B_{k-1}}^{(z)}(z_{k-1}; dz_{k-1}) \ldots Q_{B_1, B_2}^{(z)}(z_1; dz_2) \mu_{B_1}^{(z)}(dz_1)
\]
Using (10) we can conclude that for 

$$P(X \in \hat{A}, (P_{B_j})_j \in \hat{\Gamma}) = \int_{\hat{A}} \int_{\hat{\Gamma}} Q_{B_{k-1}B_k}(z_{k-1}; dz_k) \cdots \mu_{B_1}(dz_1) \alpha_n(dx).$$ (9)$$

Note also that (9) will imply that \(Q^{(z)}\) is a posterior transition system.

For the proof of (9) we will use an induction argument on \(k \geq 2\). The statement for \(k = 2\) is exactly (2). Assume that the statement is true for \(k - 1\). For each \(B_1 \subseteq B_2 \subseteq \ldots \subseteq B_k\) in \(\mathcal{A}(\mu)\) we let \(\mu_{B_1\ldots B_k}\) be the law of \((X_1, \ldots, X_n, P_{B_1}, \ldots, P_{B_k})\) under \(P\). Note that \(\forall A_1, \ldots, A_n \in \mathcal{B}, \forall \Gamma_1, \ldots, \Gamma_k \in \mathcal{B}(0, 1), \mu_{B_1\ldots B_k}(\prod_{i=1}^n A_i \times \prod_{j=1}^k \Gamma_j) = \mathcal{E}[\prod_{i=1}^n P_{A_i} \cdot \prod_{j=1}^k \Gamma_j(P_{B_j})].\) On the other hand, \(\mu_{B_1\ldots B_k}(\hat{A} \times \prod_{j=1}^k \Gamma_j)\) is also equal to

$$\int_{\hat{A} \times \prod_{j=1}^{k-1} \Gamma_j} Q^{(z)}_{B_1\ldots B_k}(z_1, \ldots, z_{k-1}; \Gamma_k) \nu_{B_1\ldots B_{k-1}}(dz_1 \times \ldots \times dz_{k-1}) = (10)$$

where \(Q^{(z)}_{B_1\ldots B_k}(z_1, \ldots, z_{k-1}; \Gamma_k) := P[P_{B_k} \in \Gamma_k|X = z, P_{B_j} = z_j, j < k]\) and \(\hat{Q}_{B_1\ldots B_k}(z_1, \ldots, z_k; \prod_{i=1}^n A_i) := P[X_1 \in A_1, \ldots, X_n \in A_n|P_{B_j} = z_j, j \leq k] = \mathcal{E}[\prod_{i=1}^n P_{A_i}|P_{B_j} = z_j, j \leq k].\)

Using the induction hypothesis, the measure \(\nu_{B_1\ldots B_{k-1}}\) disintegrates as

$$Q^{(z)}_{B_{k-2}B_{k-1}}(z_{k-2}; dz_{k-1}) \cdots Q^{(z)}_{B_1B_2}(z_1; dz_2) \mu^{(z)}_{B_1}(dz_1) \alpha_n(dx).$$

Therefore, it is enough to prove that for every \(\Gamma_k \in \mathcal{B}(0, 1)\) and for \(\nu_{B_1\ldots B_{k-1}}\)-almost all \((z, z_1, \ldots, z_{k-1})\)

$$Q^{(z)}_{B_1\ldots B_k}(z_1, \ldots, z_{k-1}; \Gamma_k) = Q^{(z)}_{B_{k-1}B_k}(z_{k-1}; \Gamma_k).$$ (11)

On the other hand, the measure \(\nu_{B_1\ldots B_{k-1}}\) disintegrates also as

$$\hat{Q}_{B_1\ldots B_{k-1}}(z_1, \ldots, z_{k-1}; dz) \mu_{B_1\ldots B_{k-1}}(dz_1 \times \ldots \times dz_{k-1})$$

with respect to its marginal \(\mu_{B_1\ldots B_{k-1}}\) with respect to the last \(k-1\) components. By the Q-Markov property, the measure \(\mu_{B_1\ldots B_k}\) disintegrates as

$$Q_{B_{k-1}B_k}(z_{k-1}; dz_k) \mu_{B_1\ldots B_{k-1}}(dz_1 \times \ldots \times dz_{k-1}).$$

Using (10) we can conclude that for \(\mu_{B_1\ldots B_{k-1}}\)-almost all \((z_1, \ldots, z_{k-1})\)

$$\int_{\hat{A}} Q^{(z)}_{B_1\ldots B_k}(z_1, \ldots, z_{k-1}; \Gamma_k) \hat{Q}_{B_1\ldots B_{k-1}}(z_1, \ldots, z_{k-1}; dz) = (12)$$
\[ \int \Gamma_n \hat{Q}_{B_1 \ldots B_k}(z_1, \ldots, z_k; \hat{A})Q_{B_{k-1}B_k}(z_{k-1}; dz_k). \]

Let \( C_1 = B_1; C_j = B_j \setminus B_{j-1}, j = 2, \ldots, k; C_{k+1} = B_k^c \). Note that each \( C_j \in \mathcal{C}(u) \) and \( (C_1, \ldots, C_{k+1}) \) is a partition of \( \mathcal{X} \); hence each point \( x_i \) falls into exactly one set of this partition.

We proceed to the proof of (11) and we will suppose that for some \( 0 \leq l \leq r \leq n \), the points \( x_1, \ldots, x_l \) fall into \( B_{k-1} \) (more precisely, each \( x_i \) falls into some \( C_{j_i} \) with \( 1 \leq j_1 < \ldots < j_l \leq k - 1 \)), the points \( x_{l+1}, \ldots, x_r \) fall into \( C_k \) and the points \( x_{r+1}, \ldots, x_n \) fall into \( C_{k+1} \).

The main tool will be (12) where we will consider a set \( \hat{A} \) of the form

\[ \hat{A} := \prod_{i=1}^l (A_i \cap C_{j_i}) \times \prod_{i=l+1}^r (A_i \cap C_k) \times \prod_{i=r+1}^n (A_i \cap C_{k+1}), \quad A_i \in \mathcal{B}. \]

Let \( \hat{A}_2 := \prod_{i=l+1}^r (A_i \cap C_k) \times \mathcal{A}^{n-r+l}, \hat{A}_3 := \prod_{i=r+1}^n (A_i \cap C_{k+1}) \times \mathcal{A}^r \) and \( \hat{A}_{23} := \hat{A}_2 \cap \hat{A}_3 \). We will prove that

\[ \hat{Q}_{B_1 \ldots B_k}(z_1, \ldots, z_k; \hat{A}) = M \cdot \hat{Q}_{B_{k-1}B_k}(z_{k-1}, z_k; \hat{A}_{23}) \quad (13) \]

\[ \hat{Q}_{B_1 \ldots B_k}(z_1, \ldots, z_{k-1}; \hat{A}) = M \cdot \hat{Q}_{B_{k-1}B_k}(z_{k-1}; \hat{A}_{23}) \quad (14) \]

where \( M := \prod_{i=1}^l \hat{Q}_{B_{j_i-1}B_{j_i}}(z_{j_i-1}, z_{j_i}; (A_i \cap C_{j_i}) \times \mathcal{A}^{n-1}) \). Then we will have

\[ \int_{\hat{A}} Q^{(2)}_{B_1 \ldots B_k}(z_1, \ldots, z_{k-1}; \Gamma_k)\hat{Q}_{B_1 \ldots B_{k-1}}(z_1, \ldots, z_{k-1}; d\hat{x}) = \]

\[ M \cdot \int_{\hat{A}_2} \hat{Q}_{B_{k-1}B_k}(z_{k-1}, z_k; \hat{A}_{23})Q_{B_{k-1}B_k}(z_{k-1}; dz_k) = \]

\[ M \cdot \int_{\hat{A}_{23}} Q^{(2)}_{B_{k-1}B_k}(z_{k-1}; \Gamma_k)\hat{Q}_{B_{k-1}B_k}(z_{k-1}; d\hat{x}) = \]

\[ \int_{\hat{A}} Q^{(2)}_{B_1 \ldots B_k}(z_{k-1}; \Gamma_k)\hat{Q}_{B_1 \ldots B_{k-1}}(z_1, \ldots, z_{k-1}; d\hat{x}) \]

where we used (12) and (13) for the first equality, (5) for the second equality and (14) for the third equality (taking in account that \( Q^{(2)}_{B_{k-1}B_k}(z_{k-1}; \Gamma_k) \) does not depend on \( x_1, \ldots, x_l \)). Relation (11) will follow immediately.

It remains to prove (13) and (14). Using Lemma 3 of Balan and Ivanoff (2002) we have (for \( A_i \in \mathcal{A} \)):

\[ \mathcal{E}[\prod_{i=r+1}^n P_{A_i \cap C_{k+1}}|\mathcal{F}_{B_k}] = \mathcal{E}[\prod_{i=r+1}^n P_{A_i \cap C_{k+1}}|\mathcal{F}_{B_k}] = \hat{Q}_{B_k}(P_{B_k}; \hat{A}_3) \]

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and therefore, by double conditioning with respect to $\mathcal{F}_{B_k}$

$$
\tilde{Q}_{B_1 \ldots B_k}(\{z_j\}_{j \leq k}; \tilde{A}) = \mathcal{E}\left[ \prod_{i=1}^{l} P_{A_i \cap C_{i_j}} \prod_{i=l+1}^{r} P_{A_i \cap C_k} \prod_{i=r+1}^{n} P_{A_i \cap C_{k+1}} | P_{B_j} = z_j, j \leq k \right]
$$

$$
= \tilde{Q}_{B_k}(z_k; \tilde{A}_3) \cdot \mathcal{E}\left[ \prod_{i=1}^{l} P_{A_i \cap C_{i_j}} \prod_{i=l+1}^{r} P_{A_i \cap C_k} | P_{B_j} = z_j, j \leq k \right].
$$

For the second term we have

$$
\mathcal{E}\left[ \prod_{i=1}^{l} P_{A_i \cap C_{i_j}} \prod_{i=l+1}^{r} P_{A_i \cap C_k} | P_{B_j}, j \leq k \right] = \mathcal{E}\left[ \prod_{i=1}^{l} P_{A_i \cap C_{i_j}} \cdot \mathcal{E}\left[ \prod_{i=l+1}^{r} P_{A_i \cap C_k} | P_{B_j}, j \leq \tilde{k} \right] \right].
$$

$$
\mathcal{E}\left[ \prod_{i=1}^{l} P_{A_i \cap C_{i_j}} \cdot \mathcal{E}\left[ \prod_{i=l+1}^{r} P_{A_i \cap C_k} | P_{B_j}, j \leq \tilde{k} \right] \right] = \tilde{Q}_{B_{k-1}, P_{B_k}}(P_{B_{k-1}}, P_B; \tilde{A}_2).
$$

(In order to use Lemma A.1 we need $A_{l+1} \subseteq A_{l+2} \subseteq \ldots \subseteq A_r$, but this is not a restriction as we have seen in the proof of Lemma 3.2.)

Note that by (8), $\tilde{Q}_{B_{k-1}, B_k}(z_{k-1}, z_k; \tilde{A}_2) \cdot \tilde{Q}_{B_k}(z_k; \tilde{A}_3) = \tilde{Q}_{B_{k-1}, B_k}(z_{k-1}, z_k; \tilde{A}_3)$. Hence the proof of (13) will be complete once we show that

$$
\mathcal{E}\left[ \prod_{i=1}^{l} P_{A_i \cap C_{i_j}} | P_{B_j}, j \leq \tilde{k} \right] = \prod_{i=1}^{l} \tilde{Q}_{B_{j_1-1}, B_{j_1}}(z_{j_1-1}, z_{j_1}; (A_i \cap C_{j_1}) \times \mathcal{X}^{n-1}). \quad (15)
$$

But this follows by induction on $l$, using Lemma A.1 (Appendix A).

We turn now to the proof of (14). Using Lemma 3 of Balan and Ivanoff (2002) we have:

$$
\mathcal{E}\left[ \prod_{i=l+1}^{r} P_{A_i \cap C_k} \prod_{i=r+1}^{n} P_{A_i \cap C_{k+1}} | \mathcal{F}_{B_k-1} \right] = \tilde{Q}_{B_{k-1}, A_{23}}
$$

and therefore, by double conditioning with respect to $\mathcal{F}_{B_k-1}$ we obtain the following expression for $\tilde{Q}_{B_1 \ldots B_{k-1}}(z_1, \ldots, z_{k-1}; \tilde{A}_{23})$:

$$
\mathcal{E}\left[ \prod_{i=1}^{l} P_{A_i \cap C_{i_j}} \prod_{i=l+1}^{r} P_{A_i \cap C_k} \prod_{i=r+1}^{n} P_{A_i \cap C_{k+1}} | P_{B_j} = z_j, j \leq k-1 \right] = \ldots
$$
\[ \hat{Q}_{B_{k-1}}(z_{k-1}; \hat{A}_{23}) \cdot \mathcal{E}[\prod_{i=1}^{l} P_{A_{i} \cap C_{j_{i}}} \mid P_{B_{j}} = z_{j}, j \leq k - 1] \]

and (14) follows, using (15). The proof of the theorem is complete. \( \square \)

The posterior distribution of a Dirichlet process is also Dirichlet. In the case of an empirical measure which corresponds to a sample either from a non-random distribution or from a Dirichlet process, the calculations for the posterior transition probabilities \( Q_{B_{1}B_{2}}^{(2)} \) are not straightforward for samples of size greater than 1; however, in the case of a sample of size 1 we have the following result.

**Proposition 3.5** If \( P := (P_{A})_{A \in \mathcal{B}} \) is the empirical measure of a sample of size \( N \) from a non-random distribution \( P_{0} \) (respectively from a Dirichlet process with parameter measure \( \alpha \)) and \( X \) is a sample of size 1 from \( P \), then the conditional distribution of \( P \) given \( X = x \) is \( Q^{(x)} \text{-Markov} \) with

\[ Q^{(x)}_{B_{1}B_{2}} \left( \frac{m_{1}}{N}; \left\{ \frac{m_{2}}{N} \right\} \right) = Q^{(1)}_{B_{1}B_{2}} \left( \frac{m_{1} - \delta x(B_{1})}{N - 1}; \left\{ \frac{m_{2} - \delta x(B_{2})}{N - 1} \right\} \right) \]

where \( Q^{(1)} \) is the transition system of the empirical measure of a sample of size \( N - 1 \) from \( P_{0} \) (respectively from a Dirichlet process with parameter measure \( \alpha \)).

**Proof:** Let \( P \) be the empirical measure of a sample from a non-random distribution \( P_{0} \). Note that \( \alpha_{1}(A) = \mathcal{E}[P_{A}] = P_{0}(A), \forall A \in \mathcal{B} \). We have

\[ Q^{(x)}_{B_{1}B_{2}} \left( \frac{m_{1}}{N}; \left\{ \frac{m_{2}}{N} \right\} \right) = Q_{B_{1}B_{2}} \left( \frac{m_{1}}{N}; \left\{ \frac{m_{2}}{N} \right\} \right) = Q^{(1)}_{B_{1}B_{2}} \left( \frac{m_{1} - 1}{N - 1}; \left\{ \frac{m_{2} - 1}{N - 1} \right\} \right) \]

for \( \alpha_{1} \)-almost all \( x \in B_{1} \). The fact that

\[ Q^{(x)}_{B_{1}B_{2}} \left( \frac{m_{1}}{N}; \left\{ \frac{m_{2}}{N} \right\} \right) = Q^{(1)}_{B_{1}B_{2}} \left( \frac{m_{1} - 1}{N - 1}; \left\{ \frac{m_{2} - 1}{N - 1} \right\} \right) \]

for \( \alpha_{1} \)-almost all \( x \in C \) follows from (5), since for every \( A \in \mathcal{B} \)

\[ \hat{Q}_{B_{1}} \left( \frac{m_{1}}{N}; A \cap C \right) = \mathcal{E}[P_{A \cap C} \mid P_{B_{1}} = \frac{m_{1}}{N}] = \frac{N - m_{1}}{N} \cdot \frac{P_{0}(A \cap C)}{P_{0}(B_{1})} \]
\[ \hat{Q}_{B_{1}B_{2}} \left( \frac{m_{1}}{N}, \frac{m_{2}}{N}; A \cap C \right) = \mathcal{E}[P_{A \cap C} \mid P_{B_{1}} = \frac{m_{1}}{N}, P_{B_{2}} = \frac{m_{2}}{N}] = \frac{m_{2} - m_{1}}{N} \cdot \frac{P_{0}(A \cap C)}{P_{0}(C)} \]

Similarly one can show that

\[ \hat{Q}_{B_{1}} \left( \frac{m_{1}}{N}; A \cap B_{2} \right) = \frac{N - m_{1}}{N} \cdot \frac{P_{0}(A \cap B_{2})}{P_{0}(B_{2})} \]
\[ \hat{Q}_{B_{1}B_{2}} \left( \frac{m_{1}}{N}, \frac{m_{2}}{N}; A \cap B_{2} \right) = \frac{N - m_{2}}{N} \cdot \frac{P_{0}(A \cap B_{2})}{P_{0}(B_{2})} \]
and hence
\[ Q_{B_1B_2}^{(x)} \left( \frac{m_1}{N}; \left\{ \frac{m_2}{N} \right\} \right) = Q_{B_1B_2}^{(1)} \left( \frac{m_1}{N-1}; \left\{ \frac{m_2}{N-1} \right\} \right) \]
for \( \alpha_1 \)-almost all \( x \) in \( B_2^* \).

If \( P \) is the empirical measure of a sample from a Dirichlet process with parameter measure \( \alpha \), then \( \alpha_1(A) = \alpha(A)/\alpha(X) \) and a similar argument can be used. \( \square \)

4 Neutral to the right random probability measures

Let \( P := (P_A)_{A \in B} \) be a random probability measure on \( X \). For every sets \( B_1, B_2 \in \mathcal{A}(u) \) with \( B_1 \subseteq B_2 \), we define \( V_{B_1B_2} \) to be equal to \( (P_{B_2} - P_{B_1})/(1 - P_{B_1}) \) on the set \( \{ P_{B_1} < 1 \} \) and 1 elsewhere; let \( F_{B_1B_2} \) be the distribution of \( V_{B_1B_2} \). The next definition generalizes the definition of Doksum (1974).

**Definition 4.1** A random probability measure \( P := (P_A)_{A \in B} \) is called **neutral to the right** if for every sets \( B_1 \subseteq \ldots \subseteq B_k \) in \( \mathcal{A}(u) \), \( P_{B_1}, V_{B_1B_2}, \ldots, V_{B_{k-1}B_k} \) are independent.

*Comments:* 1. A random probability measure \( P := (P_A)_{A \in B} \) is neutral to the right if and only if \( \forall B_1, B_2 \in \mathcal{A}(u), B_1 \subseteq B_2, V_{B_1B_2} \) is independent of \( F_{B_1} \).

2. The Dirichlet process with parameter measure \( \alpha \) is neutral to the right with \( F_{B_1B_2} \) equal to the Beta distribution with parameters \( \alpha(B_2 \setminus B_1), \alpha(B_2^*) \).

3. If we denote \( C_1 = B_1; C_i = B_i \setminus B_{i-1}; i = 2, \ldots, k \), then \( (P_{C_1}, \ldots, P_{C_k}) \) has a ‘completely neutral’ distribution (see Definition B.2); this distribution was formally introduced by Connor and Mosimann (1969), although the concept itself goes back to Halmos (1944). Note that the Dirichlet process is the only non-trivial process which has completely neutral distributions over any disjoint sets \( \{ A_1, \ldots, A_k \} \) in \( B \) (according to Ferguson 1974, p. 622).

4. In general, the process \( Y_A := -\ln(1 - P_A), A \in B \) is not additive and hence it does not have independent increments, even if \( Y_{B_1}, Y_{B_2} - Y_{B_1}, \ldots, Y_{B_k} - Y_{B_{k-1}} \) are independent for any sets \( B_1 \subseteq B_2 \subseteq \ldots \subseteq B_k \) in \( \mathcal{A}(u) \) (the increment \( Y_{B_2 \setminus B_1} \) is not equal to \( Y_{B_2} - Y_{B_1} \)); therefore, the theory of processes with independent increments cannot be used in higher dimensions.

**Proposition 4.2** A neutral to the right random probability measure is \( Q \)-Markov with

\[ Q_{B_1B_2}(z_1; \Gamma_2) := \begin{cases} F_{B_1B_2} \left( \frac{\Gamma_2 - z_1}{1 - z_1} \right) & \text{if } z_1 < 1 \\ \delta_1(\Gamma_2) & \text{if } z_1 = 1 \end{cases} \quad (16) \]
Proof: For any sets $B_1 \subseteq \ldots \subseteq B_k$ in $\mathcal{A}(u)$, $P_{B_1}, \ldots, P_{B_k}$ is a Markov chain:

$$P[B_j \in \Gamma|P_{B_1} = z_1, \ldots, P_{B_{j-1}} = z_{j-1}] =
$$

$$P[V_{B_{j-1}B_j} \in \frac{\Gamma_j - z_{j-1}}{1 - z_{j-1}}|P_{B_1} = z_1, V_{B_1B_2} = v_2, \ldots, V_{B_{j-2}B_{j-1}} = v_{j-1}] =
$$

$$P[V_{B_{j-1}B_j} \in \frac{\Gamma_j - z_{j-1}}{1 - z_{j-1}}] = P[V_{B_{j-1}B_j} \in \frac{\Gamma_j - z_{j-1}}{1 - z_{j-1}}|P_{B_{j-1}} = z_{j-1}] =
$$

$$P[P_{B_j} \in \Gamma|P_{B_{j-1}} = z_{j-1}]$$

where $v_i := (z_i - z_{i-1})/(1 - z_{i-1})$, $i = 2, \ldots, j - 1$ and assuming $z_i < 1, \forall i$. □

For any sets $B_1 \subseteq B_2 \subseteq B_3$ in $\mathcal{A}(u)$, $V_{B_1B_2} = V_{B_1B_2} + V_{B_2B_3} - V_{B_1B_3}$.

This leads us to the following definition.

**Definition 4.3** For each $B_1, B_2 \in \mathcal{A}(u)$ with $B_1 \subseteq B_2$, let $F_{B_1B_2}$ be a probability measure on $[0, 1]$. The family $(F_{B_1B_2})_{B_1 \subseteq B_2}$ is called a neutral to the right system if for any sets $B_1 \subseteq B_2 \subseteq B_3$ in $\mathcal{A}(u)$

$$F_{B_1B_2}(\Gamma) = \int_{[0,1]^2} I_\Gamma(y + z - yz)F_{B_2B_3}(dz)F_{B_1B_2}(dy).$$

**Comments:**

1. If we let $U_{B_1B_2} := -\ln(1 - V_{B_1B_2})$ and $G_{B_1B_2}$ be the distribution of $U_{B_1B_2}$, then for every $B_1 \subseteq B_2 \subseteq B_3$ in $\mathcal{A}(u)$, $U_{B_1B_3} = U_{B_1B_2} + U_{B_2B_3}$ and $G_{B_1B_3} = G_{B_1B_2} \ast G_{B_2B_3}$.

2. Let $Q_{B_1B_2}(z_1; \Gamma_2) := F_{B_1B_2}(\Gamma_2 - z_1)/(1 - z_1)$ for $z_1 < 1$ and $Q_{B_1B_2}(1; \cdot) = \delta_1$; then $(F_{B_1B_2})_{B_1 \subseteq B_2}$ is a neutral to the right system if and only if $(Q_{B_1B_2})_{B_1 \subseteq B_2}$ is a transition system.

The following result is the converse of Proposition 4.2.

**Proposition 4.4** If $P := (P_A)_{A \in \mathcal{B}}$ is a $\mathcal{Q}$-Markov random probability measure with a transition system $\mathcal{Q}$ given by (16) for a neutral to the right system $(F_{B_1B_2})_{B_1 \subseteq B_2}$, then $P$ is neutral to the right.

**Proof:** We want to prove that for every $B_1, B_2 \in \mathcal{A}(u)$ with $B_1 \subseteq B_2$ and for every $A_1, \ldots, A_k \in \mathcal{A}$, $A_i \subseteq B$, $A_k = B_1$, $V_{B_1B_2}$ is independent of $(P_{A_1}, \ldots, P_{A_k})$.

Using the $\mathcal{Q}$-Markov property we have: $P[V_{B_1B_2} \in \Gamma|P_{A_i} = z_i; i = 1, \ldots, k] = P[P_{B_2} \in z_k + (1 - z_k)\Gamma|P_{B_1} = z_k] = Q_{B_1B_2}(z_k; z_k + (1 - z_k)\Gamma) = F_{B_1B_2}(\Gamma) = P(V_{B_1B_2} \in \Gamma)$. Since this holds for any Borel set $\Gamma$ in $[0, 1]$, the proof is complete. □

In what follows we will prove that the posterior distribution of a neutral to the right random probability measure is also neutral to the right, by showing that the posterior transition probabilities $Q_{B_1B_2}^{(x)}$ are of the form (16) for a “posterior” neutral to the right system $(F_{B_1B_2}^{(x)})_{B_1 \subseteq B_2}$. This extends Doksum’s (1974) result to an arbitrary space $\mathcal{X}$, which can be endowed with an indexing collection $\mathcal{A}$. 15
Let \( P := (P_A)_{A \in B} \) be a neutral to the right process and \( \mathbf{X} := (X_1, \ldots, X_n) \) a sample from \( P \). In order to define the probability measures \( F^{(x)}_{B_1B_2} \), we will use the same Bayesian technique as in Section 3.

For each sets \( B_1, B_2 \in \mathcal{A}(u); B_1 \subseteq B_2 \), let \( \phi_{B_1B_2} \) be the law of \( X_1, \ldots, X_n \), \( V_{B_1B_2} \) under \( \mathcal{P} \). Note that \( \phi_{B_1B_2}(\prod_{i=1}^n A_i \times \Gamma) = \mathcal{E}[\prod_{i=1}^n P_{A_i} \cdot I_{\Gamma}(V_{B_1B_2})] \). On the other hand, we have

\[
\phi_{B_1B_2}(\tilde{A} \times \Gamma) = \int_{\tilde{A}} F^{(x)}_{B_1B_2}(\Gamma) \alpha_n(dx) = \int_{\Gamma} \tilde{T}_{B_1B_2}(z; \tilde{A}) F^{(x)}_{B_1B_2}(dz) \tag{17}
\]

where

\[
F^{(x)}_{B_1B_2}(\Gamma) := \mathcal{P}[V_{B_1B_2} \in \Gamma| \mathbf{X} = x] \tag{18}
\]

and \( \tilde{T}_{B_1B_2}(z; \tilde{A}) := \mathcal{P}[X \in \tilde{A}|V_{B_1B_2} = z] \).

In the proof of Theorem 4.8 we will see that \( (F^{(x)}_{B_1B_2})_{B_1 \subseteq B_2} \) may not be a genuine neutral to the right system as introduced by Definition 4.3. Therefore we need to introduce the following terminology.

**Definition 4.5** The family \( (F^{(x)}_{B_1B_2})_{B_1 \subseteq B_2} \) defined by (18) is called a posterior neutral to the right system (corresponding to \( P \) and \( \mathbf{X} \)) if \( \forall B_1 \subseteq B_2 \subseteq B_3 \) in \( \mathcal{A}(u), \forall \Gamma \in \mathcal{B}([0,1]) \) and for \( \alpha_n \)-almost all \( x \)

\[
F^{(x)}_{B_1B_3}(\Gamma) = \int_{[0,1]^2} I_\Gamma(y + z - yz) F^{(x)}_{B_2B_3}(dz) F^{(x)}_{B_1B_2}(dy).
\]

The conditional distribution of \( P \) given \( \mathbf{X} = x \) is called neutral to the right if \( \forall B_1 \subseteq B_2 \) in \( \mathcal{A}(u), V_{B_1B_2} \) is conditionally independent of \( \mathcal{F}_{B_1} \) given \( \mathbf{X} \).

Let \( C := B_2 \setminus B_1 \). For fixed \( 0 \leq l \leq r \leq n \) we will consider sets of the form \( \tilde{A}_{23} := \prod_{i=l+1}^{r}(A_i \cap C) \times \prod_{i=r+1}^{n}(A_i \cap B_2^c) \), where \( A_i \in \mathcal{B} \).

**Lemma 4.6** (a) For \( \mu_{B_1} \)-almost all \( z_1 \),

\[
\tilde{Q}_{B_1}(z_1; \tilde{A}_{23}) = \frac{(1 - z_1)^{n-l}}{\alpha_n((B_1^c)^{n-l} \times \mathcal{X}^l)} \cdot \alpha_n(\tilde{A}_{23}).
\]

(b) For \( \mu_{B_1B_2} \)-almost all \( (z_1, z_2) \),

\[
\tilde{Q}_{B_1B_2}(z_1, z_2; \tilde{A}_{23}) = \frac{(1 - z_1)^{n-l}}{\alpha_n((B_1^c)^{n-l} \times \mathcal{X}^l)} \cdot \tilde{T}_{B_1B_2} \left( \frac{z_2 - z_1}{1 - z_1}; \tilde{A}_{23} \right).
\]

**Proof:** Without loss of generality we will assume that \( A_i \in \mathcal{A}, \forall i \). We have

\[
\prod_{i=l+1}^{r} P_{A_i \cap C} \cdot \prod_{i=r+1}^{n} P_{A_i \cap B_2^c} = (1 - P_{B_1})^{n-l} \cdot \prod_{i=l+1}^{r} \frac{P_{A_i \cap C}}{1 - P_{B_1}} \cdot \prod_{i=r+1}^{n} \frac{P_{A_i \cap B_2^c}}{1 - P_{B_1}}. \tag{19}
\]
Note that $P_{A_i \cap C}/(1 - P_{B_i}) = V_{B_i,(A_i \cap B_2) \cup B_1}, P_{A_i \cap B_2}/(1 - P_{B_i}) = V_{B_i,A_i \cup B_2} - V_{B_i,B_2}$ and $P_{B_i}$ is independent of $V_{B_i,(A_i \cap B_2) \cup B_1}, i = l + 1, \ldots, r, V_{B_i,B_2}$ and $V_{B_i,A_i \cup B_2}, i = r + 1, \ldots, n$.

(a) Take $\mathcal{E}[\cdot], \mathcal{E}[\cdot | P_{B_i} = z_1]$ in (19); we get

$$\mathcal{E}\left[ \prod_{i=l+1}^{r} \frac{P_{A_i \cap C}}{1 - P_{B_i}} \cdot \prod_{i=r+1}^{n} \frac{P_{A_i \cap B_2}}{1 - P_{B_i}} \right] = \frac{\alpha_n(\tilde{A}_{23})}{\alpha_n((B_1^*)^{n-l} \times X^l)}$$

(b) Take $\mathcal{E}[\cdot | V_{B_i,B_2} = z], \mathcal{E}[\cdot | P_{B_i} = z_1, P_{B_i} = z_2]$ in (19); we get

$$\tilde{Q}_{B_i,B_2}(z_1, z_2; \tilde{A}_{23}) = (1 - z_1)^{n-l} \cdot \mathcal{E}[\prod_{i=l+1}^{r} \frac{P_{A_i \cap C}}{1 - P_{B_i}} \prod_{i=r+1}^{n} \frac{P_{A_i \cap B_2}}{1 - P_{B_i}} | V_{B_i,B_2} = z] = \frac{\tilde{T}_{B_i,B_2}(z; \tilde{A}_{23})}{\alpha_n((B_1^*)^{n-l} \times X^l)}$$

which concludes the proof. \(\square\)

**Lemma 4.7** For every $B_1, B_2 \in \mathcal{A}(u)$ with $B_1 \subseteq B_2$, for every $\Gamma \in \mathcal{B}([0,1])$ and for $\alpha_n$-almost all $\bar{x}$, $F^{(2)}_{B_1,B_2}(\Gamma)$ does not depend on those $x_i$’s that fall in $B_1$; in particular, for $\alpha_n$-almost all $\bar{x}$ in $B_1^n$, $F^{(2)}_{B_1,B_2}(\Gamma) = F_{B_1,B_2}(\Gamma)$.

**Proof:** For arbitrary $A_1, \ldots, A_n \in \mathcal{A}$ we write

$$\prod_{i=1}^{l} P_{A_i \cap B_1} \prod_{i=l+1}^{r} P_{A_i \cap C} \prod_{i=r+1}^{n} P_{A_i \cap B_2} = (1 - P_{B_i})^{n-l} \prod_{i=1}^{l} P_{A_i \cap B_1} \prod_{i=l+1}^{r} \frac{P_{A_i \cap C}}{1 - P_{B_i}} \prod_{i=r+1}^{n} \frac{P_{A_i \cap B_2}}{1 - P_{B_i}}.$$

Taking $\mathcal{E}[\cdot], \mathcal{E}[\cdot | V_{B_i,B_2} = z]$ and using (20), respectively (21) we get

$$\alpha_n(\tilde{A}) = \frac{\alpha_n(\prod_{i=1}^{l} (A_i \cap B_1) \times (B_1^*)^{n-l})}{\alpha_n((B_1^*)^{n-l} \times X^l)} \cdot \alpha_n(\tilde{A}_{23})$$

$$\tilde{T}_{B_i,B_2}(z; \tilde{A}) = \frac{\alpha_n(\prod_{i=1}^{l} (A_i \cap B_1) \times (B_1^*)^{n-l})}{\alpha_n((B_1^*)^{n-l} \times X^l)} \cdot \tilde{T}_{B_i,B_2}(z; \tilde{A}_{23}).$$
Using (17) we get

\[ \int_{\bar{A}} F^{(x)}_{B_1 B_2}(\Gamma) \alpha_n(d\bar{x}) = \int_{\Gamma} \tilde{T}_{B_1 B_2}(z; \tilde{A}) F_{B_1 B_2}(d\bar{z}) = \]

\[ \frac{\alpha_n(\prod_{i=1}^l (A_i \cap B_1) \times (B_i')^{n-i})}{\alpha_n((B_1')^{n-l} \times \mathcal{X}^l)} \cdot \int_{\Gamma} \tilde{T}_{B_1 B_2}(z; \tilde{A}_{23}) F_{B_1 B_2}(d\bar{z}) = \]

\[ \frac{\alpha_n(\prod_{i=1}^l (A_i \cap B_1) \times (B_i')^{n-i})}{\alpha_n((B_1')^{n-l} \times \mathcal{X}^l)} \cdot \int_{\bar{A}_{23}} F^{(x)}_{B_1 B_2}(\Gamma) \alpha_n(d\bar{x}). \]

The result follows by Lemma A.2 (Appendix A). \( \square \)

Here is the main result of this section.

**Theorem 4.8** If \( P := (P_A)_{A \in B} \) is a neutral to the right random probability measure and \( \bar{X} := (X_1, \ldots, X_n) \) is a sample from \( P \), then the conditional distribution of \( P \) given \( \bar{X} = \bar{x} \) is also neutral to the right.

**Proof:** Since \( P \) is \( \mathcal{Q} \)-Markov, by Theorem 3.4 the conditional distribution of \( P \) given \( \bar{X} = \bar{x} \) is \( \mathcal{Q}^{(\bar{x})} \)-Markov. Using Lemma 4.6, the key equation (5) becomes

\[ \int_{A_{23}} Q^{(\bar{x})}_{B_1 B_2}(z_1; \Gamma_2) \alpha_n(d\bar{x}) = \int_{\bar{A}_{23}} \tilde{T}_{B_1 B_2} \left( z_2 - z_1 \over 1 - z_1 ; \tilde{A}_{23} \right) Q_{B_1 B_2}(z_1; d\bar{z}_2). \]

Using Proposition 4.2 and relation (17), the right-hand side becomes (for \( z_1 < 1 \))

\[ \int_{z_2 - z_1 \over 1 - z_1} \tilde{T}_{B_1 B_2}(z; \tilde{A}_{23}) F_{B_1 B_2}(d\bar{z}) = \int_{\bar{A}_{23}} F^{(x)}_{B_1 B_2} \left( \Gamma_2 - z_1 \over 1 - z_1 \right) \alpha_n(d\bar{x}). \]

This proves that \( \forall z_1 \in [0, 1), \forall \Gamma_2 \in B([0, 1]) \) and for \( \alpha_n \)-almost all \( \bar{x} \)

\[ Q^{(\bar{x})}_{B_1 B_2}(z_1; \Gamma_2) = F^{(x)}_{B_1 B_2} \left( \Gamma_2 - z_1 \over 1 - z_1 \right). \]

Since \( \mathcal{Q}^{(\bar{x})} \) is a posterior transition system, it follows that \( (F^{(x)}_{B_1 B_2})_{B_1 \subseteq B_2} \) is a posterior neutral to the right system. By Proposition 4.4, the distribution of \( P \) given \( \bar{X} \) is neutral to the right. \( \square \)

The next result gives some simple formulas for calculating the posterior distribution of \( P_{B_1} \), when all the observations fall outside \( B_1 \), and the posterior distribution of \( V_{B_1 B_2} \) when all the observations fall outside \( B_2 \setminus B_1 \).

**Proposition 4.9** (a) For \( \alpha_n \)-almost all \( \bar{x} \) with \( x_i \in B_1^c \forall i \)

\[ \mu^{(x)}_{B_1}(\Gamma) = P[P_{B_1} \in \Gamma | \bar{X} = \bar{x}] = \frac{\mathcal{E}[I_{\Gamma}(P_{B_1})(1 - P_{B_1})^{n}]}{\mathcal{E}[(1 - P_{B_1})^{n}]} . \]
(b) For \( \alpha_n \)-almost all \( x \) with \( x_i \in (B_2 \setminus B_1)^c \) \( \forall i \)

\[
F_{B_1B_2}^{(z)}(\Gamma) = \mathcal{P}[V_{B_1B_2} \in \Gamma | X = x] = \frac{\mathcal{E}[I_\Gamma(V_{B_1B_2})(1 - P_{B_2})^m]}{\mathcal{E}[(1 - P_{B_2})^m]}
\]

where \( m \) denotes the number of \( x_i \)'s that fall outside \( B_2 \).

**Proof:** Note that (a) is a particular case of (b) since \( \mu_{B_1}^{(z)} = F_{\emptyset B_1}^{(z)} \). We proceed to the proof of (b). For fixed \( 0 \leq l \leq n \), let \( \tilde{A} := \prod_{i=1}^{l} (A_i \cap B_1) \times \prod_{i=l+1}^{n} (A_i \cap B_2^c) \), where \( A_i \in \mathcal{B} \). We claim that

\[
\tilde{T}_{B_1B_2}(z; \tilde{A}) = (1 - z)^{n-l} \cdot \frac{\alpha_n((B_1^c)^{n-l} \times X^l)}{\alpha_n((B_2^c)^{n-l} \times X^l)} \cdot \alpha_n(\tilde{A})
\]

(22)

Using (17), it follows that

\[
\int_{\tilde{A}} F_{B_1B_2}^{(z)}(\Gamma) \alpha_n(dx) = \frac{\alpha_n((B_1^c)^{n-l} \times X^l)}{\alpha_n((B_2^c)^{n-l} \times X^l)} \cdot \alpha_n(\tilde{A}) \cdot \int_{\Gamma} (1 - z)^{n-l} F_{B_1B_2}(dz)
\]

and hence for \( \alpha_n \)-almost all \( x \) with \( x_i \in (B_2 \setminus B_1)^c \), \( \forall i \)

\[
F_{B_1B_2}^{(z)}(\Gamma) = \frac{\alpha_n((B_1^c)^{n-l} \times X^l)}{\alpha_n((B_2^c)^{n-l} \times X^l)} \cdot \int_{\Gamma} (1 - z)^{n-l} F_{B_1B_2}(dz) = \frac{\mathcal{E}[(1 - P_{B_2})^{n-l}]}{\mathcal{E}[(1 - P_{B_1})^n]} \cdot \mathcal{E}[I_\Gamma(V_{B_1B_2})(1 - V_{B_1B_2})^{n-l}] = \frac{\mathcal{E}[I_\Gamma(V_{B_1B_2})(1 - P_{B_2})^{n-l}]}{\mathcal{E}[(1 - P_{B_1})^n]}
\]

since \( P_{B_1} \) is independent of \( V_{B_1B_2} \).

We turn now to the proof of (22). Without loss of generality we will assume that \( A_i \in \mathcal{A}, \forall i \). Let \( \tilde{A}_2 = \prod_{i=l+1}^{n} (A_i \cap B_2^c) \times X^l \). We have

\[
\prod_{i=1}^{l} P_{A_i \cap B_1} \prod_{i=l+1}^{n} P_{A_i \cap B_2^c} = (1 - P_{B_1})^{n-l} \prod_{i=1}^{l} P_{A_i \cap B_1} (1 - V_{B_1B_2})^{n-l} \prod_{i=l+1}^{n} \frac{P_{A_i \cap B_2^c}}{1 - P_{B_2}}
\]

Note that \( (1 - P_{B_1})^{n-l} \prod_{i=1}^{l} P_{A_i \cap B_1} \) is \( \mathcal{F}_{B_1} \)-measurable and \( \mathcal{F}_{B_2} \) is independent of \( V_{B_1B_2}, V_{B_2, A_i \cap B_2^c}, i = l + 1, \ldots, n \). By taking \( \mathcal{E}[\cdot | V_{B_1B_2} = \tilde{z}] \) we get

\[
\tilde{T}_{B_1B_2}(z; \tilde{A}) = (1 - z)^{n-l} \cdot \mathcal{E}[\prod_{i=1}^{l} P_{A_i \cap B_1} \cdot (1 - P_{B_1})^{n-l}] \cdot \mathcal{E}[\prod_{i=l+1}^{n} \frac{P_{A_i \cap B_2^c}}{1 - P_{B_2}}] = (1 - z)^{n-l} \cdot \frac{\alpha_n(\tilde{A}_2)}{\alpha_n((B_1^c)^{n-l} \times X^l)}
\]

Finally, by taking expectation in

\[
\prod_{i=1}^{l} P_{A_i \cap B_1} \cdot \prod_{i=l+1}^{n} P_{A_i \cap B_2^c} = \prod_{i=1}^{l} P_{A_i \cap B_1} \cdot (1 - P_{B_1})^{n-l} \cdot \prod_{i=l+1}^{n} \frac{P_{A_i \cap B_2^c}}{1 - P_{B_2}}
\]

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we get \( \alpha_n(\tilde{A}) = \alpha_n(\prod_{i=1}^l (A_i \cap B_1) \times (B_1^{n-l}) \cdot \alpha_n(\tilde{A}_2)/\alpha_n((B_1^{n-l}) \times \mathcal{X}^i) \). \qed

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A Some elementary results

Lemma A.1 If \((X_t)_{t \in \mathbb{R}}\) is a Markov process, then for every \(s_1 < \ldots < s_n < s < u_1 < \ldots < u_p < t < t_1 < \ldots < t_m\) and for every bounded measurable function \(h\)
\[
\mathcal{E}[h(X_{u_1}, \ldots, X_{u_p})|X_{s_1}, \ldots, X_{s_n}, X_s, X_t, X_{t_1}, \ldots, X_{t_m}] = \mathcal{E}[h(X_{u_1}, \ldots, X_{u_p})|X_s, X_t],
\]

The proof of the previous lemma is elementary and will be omitted.

Lemma A.2 Let \((X, \mathcal{X}, \mu), (Y, \mathcal{Y}, \nu)\) be probability spaces and \(f : X \times Y \to \mathbb{R}\) a bounded measurable function. If for every \(A \in \mathcal{X}, \forall B \in \mathcal{Y}\)
\[
\int_{A \times B} f(x, y)(\mu \times \nu)(dx \times dy) = \mu(A) \int_X f(x, y)(\mu \times \nu)(dx \times dy)
\]
then \(f(x, y)\) does not depend on \(x\), for \((\mu \times \nu)\)-almost all \((x, y)\).

Proof: Let \(f_B(x) = \int_B f(x, y)\nu(dy), x \in X\) and \(I_B = \int_X f_B(x)\mu(dx)\). We have
\[
\int_A f_B(x)\mu(dx) = \mu(A)I_B = \int_A I_B\mu(dx), \forall A \in \mathcal{X}
\]
and hence \(f_B(x) = I_B, \forall x \in N_0^c\), where \(N_0\) is a \(\mu\)-negligible set. For each \(x \in N_0^c\)
\[
\int_B f(x, y)\nu(dy) = \int_X \int_B f(x, y)\nu(dy)\mu(dx) = \int_B \int_X f(x, y)\mu(dx)\nu(dy), \forall B \in \mathcal{Y}.
\]
Hence \(f(x, y) = \int_X f(x, y)\mu(dx) := g(y)\) for all \(y \in N_0^c\), where \(N_0\) is a \(\nu\)-negligible set. If we take \(N := \{(x, y); x \in N_0^c, y \in N_0^c\}^c\), then \((\mu \times \nu)(N) = 0\) and \(f(x, y) = g(y), \forall (x, y) \in N^c\). \(\square\)

B A Bayes property of a Markov chain

Lemma B.1 Let \((Z_1, \ldots, Z_k)\) be an increasing Markov chain with values in \([0,1]\), with initial distribution \(\mu\) and transition probabilities \((Q_{i-1,i})_{i=2,\ldots,k}\); let \(Z_0 := 0\) and \(Z_{k+1} := 1\). Let \(Y_j = Z_j - Z_{j-1}; j = 1, \ldots, k+1\) and \(X\) be a random variable such that
\[
\mathbb{P}[X = j|Y_1, \ldots, Y_{k+1}] = Y_j, \forall j = 1, \ldots, k+1.
\]
Then for every $j = 1, \ldots, k + 1$, the conditional distribution of $(Z_1, \ldots, Z_k)$ given $X = j$ coincides with the distribution of a Markov chain with some initial distribution $\mu^{(j)}$ and some transition probabilities $(Q_{i-1,i}^{(j)})_{i=2,\ldots,k}$.

**Proof:** Let $\alpha_j := P(X = j) = \mathcal{E}[Y_j]$. We consider first the case $j > 1$. For any sets $\Gamma_1, \ldots, \Gamma_k \in \mathcal{B}([0,1])$ we have

$$
P[Z_1 \in \Gamma_1, \ldots, \Gamma_k \in \Gamma_k | X = j] = \frac{1}{\alpha_j} \int_{\cap_{i=1}^k \{Z_i \in \Gamma_i\}} \mathcal{P}[X = j | Z_1, \ldots, Z_k] d\mathcal{P} =$$

$$
\frac{1}{\alpha_j} \int_{\Gamma_1} \cdots \int_{\Gamma_k} h(z_j)(z_j - z_{j-1})Q_{j-1,j}(z_{j-1}; dz_j) \cdots Q_{12}(z_1; dz_2) \mu(dz_1)
$$

where $h(z_j) = \int_{\Gamma_{j+1}} \cdots \int_{\Gamma_n} Q_{k-1,k}(z_{k-1}; dz_k) \cdots Q_{j,j+1}(z_j; dz_{j+1})$. We denote $\alpha_j^{(j)}(y) := \mathcal{E}[Y_j | Z_{j-1} = y]$ and $\alpha_i^{(j)}(y) := \mathcal{E}[\alpha_{i+1}^{(j)}(Z_i) | Z_{i-1} = y], i < j$; we have

$$
P[Z_i \in \Gamma_i; i \leq k | X = j] = \frac{1}{\alpha_j} \int_{\Gamma_1} \alpha_2^{(j)}(z_1) \cdot \frac{1}{\alpha_2^{(j)}(z_1)} \int_{\Gamma_2} \cdots \alpha_j^{(j)}(z_{j-1}).
$$

$$
\int_{\Gamma_1} \cdots \int_{\Gamma_k} Q_{j-1,j}^{(j)}(z_{j-1}; dz_{j-1}) \cdots Q_{k-1,k}^{(j)}(z_{k-1}; dz_k) \cdots Q_{12}^{(j)}(z_1; dz_2) \mu^{(j)}(dz_1)
$$

where $\mu^{(j)}(\Gamma) := (1/\alpha_j) \int_{\Gamma} \alpha_2^{(j)}(y) \mu(dy)$ and

$$
Q_{i-1,i}^{(j)}(y; \Gamma) := Q_{i-1,i}(y; \Gamma) \text{ if } i > j
$$

$$
Q_{j-1,j}^{(j)}(y; \Gamma) := \frac{1}{\alpha_j^{(j)}(y)} \int_{\Gamma} (z - y)Q_{j-1,j}(y; dz)
$$

$$
Q_{i-1,i}^{(j)}(y; \Gamma) := \frac{1}{\alpha_i^{(j)}(y)} \int_{\Gamma} \alpha_{i+1}^{(j)}(z)Q_{i-1,i}(y; dz) \text{ if } i < j.
$$

We consider next the case $j = 1$. For any sets $\Gamma_1, \ldots, \Gamma_k \in \mathcal{B}([0,1])$ we have

$$
P[Z_1 \in \Gamma_1, \ldots, \Gamma_k \in \Gamma_k | X = 1] = \frac{1}{\alpha_1} \int_{\cap_{i=1}^k \{Z_i \in \Gamma_i\}} \mathcal{P}[X = 1 | Z_1, \ldots, Z_k] d\mathcal{P} =$$

$$
\int_{\Gamma_1} \cdots \int_{\Gamma_k} Q_{k-1,k}^{(1)}(z_{k-1}; dz_k) \cdots Q_{12}^{(1)}(z_1; dz_2) \mu^{(1)}(dz_1)
$$

where $\mu^{(1)}(\Gamma) := (1/\alpha_1) \int_{\Gamma} \mu(dy)$ and $Q_{i-1,i}^{(1)} = Q_{i-1,i}, \forall i = 2, \ldots, k$. □

The following definition is taken from Fang, Kotz and Ng (1990), p.163.
Definition B.2 A random vector \((Y_1, \ldots, Y_k)\) with values in the simplex \(S = \{(y_j); y_j \in [0, 1], \sum_{j=1}^{k} y_j \leq 1\}\) has a completely neutral distribution if there exist some independent random variables \(V_1, \ldots, V_k\) such that \((Y_1, \ldots, Y_k)\) has the same distribution as \((V_1, V_2(1 - V_1), \ldots, V_k \prod_{j=1}^{k-1}(1 - V_j))\).

The following result can be viewed as a complement to Theorem 4 of Asgharian and Wolfson (2001).

Corollary B.3 If \((Y_1, \ldots, Y_k)\) has a completely neutral distribution, \(Y_{k+1} := 1 - \sum_{j=1}^{k} Y_j\) and \(X\) is a random variable such that

\[P[X = j|Y_1, \ldots, Y_{k+1}] = Y_j \quad \forall j = 1, \ldots, k + 1\]

then the conditional distribution of \((Y_1, \ldots, Y_k)\) given \(X = j\) is completely neutral.

Proof: Let \(Z_i = \sum_{j=1}^{i} Y_j\) and \(V_i = Y_i/(1 - Z_{i-1}), i = 2, \ldots, k\). The variables \(V_1, \ldots, V_k\) are independent and \(Z_1, \ldots, Z_k\) is a Markov chain with the transition probabilities \(Q_{i-1,i}(y; \Gamma) = F_i((\Gamma - y)/(1 - y))\), where \(F_i\) is the distribution of \(V_i\). By Lemma B.1, the conditional distribution of \((Z_1, \ldots, Z_k)\) given \(X = j\) coincide with the distribution of a Markov chain with some transition probabilities \(Q_{i-1,i}^{(j)}\). Direct calculations show that \(Q_{i-1,i}^{(j)}(y; \Gamma) = F_i^{(j)}((\Gamma - y)/(1 - y))\) with: \(F_i^{(j)} = F_i\) if \(i > j\),

\[F_j^{(j)}(\Gamma) = \frac{1}{\beta_j} \int_\Gamma v F_j^j(dv), \quad F_i^{(j)}(\Gamma) = \frac{1}{1 - \beta_i} \int_\Gamma (1 - v) F_i(dv) \quad \text{if } i < j,\]

where \(\beta_i = E[V_i]\). The conclusion follows immediately. □

References


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