Set-Indexed Processes with Independent Increments

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Abstract

Set-indexed process with independent increments are described by 'convolution systems'; the construction of such a process is given using Kolmogorov's extension theorem. When we specialize to the case of Lévy processes, we obtain a double characterization in terms of their characteristic functions and in terms of their generators.

Keywords: set-indexed process; independent increments; convolution system; flow; Lévy process.

1 Introduction

On the real line, processes with independent increments constitute a fundamental class of stochastic processes, Brownian motion, compound Poisson processes, and stable processes being typical examples. Historically, their systematic study began with the characterization of the infinitely divisible distributions given by Lévy's (1934) impressive paper. It turned out that any such process can be written as the sum of a non-random function, a discrete process, and a stochastically continuous process. Moreover, any *Lévy process* (i.e. a stochastically continuous process with independent increments) has a cadlag version, which can be constructed as the sum of a continuous Gaussian process and an independent jump process, obtained as a limit of a sequence of independent compound Poisson processes.

On the other hand, processes with independent increments are prototypes of important classes of processes, like Markov processes and semimartingales.

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An immense amount of literature has been dedicated to them, including two excellent (and quite recent) monographs Bertoin (1996), Sato (1999); a good all-time reference remains chapter IV of Gihman and Skorohod (1975).

Multiparameter processes with independent increments have also been in the literature for many years, their definition being a straightforward generalization of the classical one: they have to assume independent values over disjoint hyper-parallelpipeds. In 1983, Adler et al. proved in a remarkable (and self-contained) paper that most of the classical properties of processes with independent increments on the real line can be transferred to multiparameter processes. However, it was not so straightforward to identify the appropriate type of Markov property satisfied by these processes; a complete answer to this question came almost ten years later in the form of a thorough paper by Dalang and Walsh (1992).

The next level of generality was to consider the class of processes with independent increments, indexed by a collection $\mathcal{A}$ of closed subsets of the d-dimensional unit cube $[0,1]^d$. These processes, called 'set-indexed' processes with independent increments, have to be finitely additive, and to assume independent values over disjoint sets. Adler and Feigin (1984) have introduced the (set-indexed) *Lévy processes* as 'stochastically continuous' processes with independent increments, the set convergence being defined using the metric $d_\lambda(A,B) := \lambda(A\Delta B)$, where λ is the Lebesgue measure; it is proved that the distribution of such a process is completely characterized by a system of infinitely divisible distributions.

In the past decade, processes indexed by sets began increasingly to become the subject of research, and the modern theory of set-indexed processes does not require that the indexing sets be subsets of the Euclidean cube. This theory has been initiated and developed by Ivanoff and Merzbach and there exists already a very good monograph (Ivanoff and Merzbach, 2000) dedicated to the martingale case. In Balan and Ivanoff (2002), a natural Markov property (called the 'set-Markov property') has been introduced in the set-indexed framework, which allowed a complete parallelism with the classical case, and avoided many of the difficulties encountered when working with other Markov properties, even in the multiparameter case.

In the present paper, we will take the first step in the study of set-indexed processes with independent increments (which are automatically set-Markov) and we will show that these processes can also be described by means of convolution systems of distributions on the real line.

The novelty of our approach is that *we do not need to impose a metric on the underlying space*, since our Lévy processes are defined using the monotone convergence of sets. We will show that the distribution of a Lévy process is completely determined by a system of infinitely divisible distributions. Using the set-Markov property, we will provide a second characterization of a Lévy process in terms of *its generator*, a characterization which does not exist in the literature for any other multiparameter (or set-indexed) Lévy process. The value of this result is that it shows that the technical consistency condition given by Assumption 5, of Balan and Ivanoff (2002) (which has to be imposed on a family of generators so that it becomes 'the generator' of a set-Markov process) is automatically satisfied in the case of a Lévy process.

We begin now to introduce the set-indexed framework. Our indexing collection $\mathcal{A}$ will be a *semilattice* of closed subsets of a compact Hausdorff topological space T (i.e. $\mathcal{A}$ is closed under arbitrary intersections), which contains $\emptyset$ and T, but does not contain disjoint (non-empty) sets. In addition, we assume that the collection $\mathcal{A}$ is *separable from above*, in the sense that any set $A \in \mathcal{A}$ can be approximated from above as

$$A = \cap_n g_n(A); \quad g_{n+1}(A) \subseteq g_n(A), A \subseteq g_n(A)^0 \ \forall n$$

where the approximation set $g_n(A)$ can be written as a finite union of sets that lie in a finite sub-semilattice $\mathcal{A}_n$ of $\mathcal{A}$; moreover, $\mathcal{A}_n \subseteq \mathcal{A}_{n+1} \ \forall n$ and g_n preserves arbitrary intersections and finite unions i.e. $g_n(\cap_{\alpha \in \Lambda} A_\alpha) = \cap_{\alpha \in \Lambda} g_n(A_\alpha), \ \forall A_\alpha \in \mathcal{A}$; and $\cup_{i=1}^k A_i = \cup_{j=1}^m A'_j \Rightarrow \cup_{i=1}^k g_n(A_i) = \cup_{j=1}^m g_n(A'_j), \ \forall A_i, A'_j \in \mathcal{A}$. By convention, $g_n(\emptyset) = \emptyset$. We will also suppose that, if $(A_n)_n$ is an increasing sequence of sets in $\mathcal{A}$, then $\overline{\cup_n A_n} \in \mathcal{A}$.

There are many examples of classes of sets which have these properties, from which one can recognize easily the case of processes indexed by the lower sets of the d-dimensional space $[0,1]^d$ (A is a 'lower set' if $z \in A$ implies $[0,z] \subseteq A$), and in particular the case of processes indexed by the 'rectangles' $[0,z], z \in [0,1]^d$ (or equivalently, the case of multiparameter processes).

The collection $\mathcal{A}$ induces a semialgebra $\mathcal{C}$ of sets defined by

$$\mathcal{C} := \{C = A \backslash B; A \in \mathcal{A}, B \in \mathcal{A}(u)\}$$

where $\mathcal{A}(u)$ denotes the class of all finite unions of sets in $\mathcal{A}$. We denote with $\mathcal{C}(u)$ the algebra of sets generated by $\mathcal{C}$ and with $\emptyset' := \cap_{A \in \mathcal{A} \backslash \{\emptyset\}} A$ the minimal set in $\mathcal{A}$ (note that $\emptyset' \neq \emptyset$).

Note that any set $B \in \mathcal{A}(u)$ admits at least one *extremal representation* of the form $B = \cup_{i=1}^n A_i, A_i \in \mathcal{A}$ with $A_i \not\subseteq \cup_{j \neq i} A_j \ \forall i$. For a set $C \in \mathcal{C}$ we will say that the representation $C = A \backslash B$ is extremal if the representation of B is extremal.

Let $\mathcal{A}'$ be a finite sub-semilattice of $\mathcal{A}$ (i.e $\mathcal{A}'$ is closed under intersections). An ordering $\{A_0, \ldots, A_n\}$ of $\mathcal{A}'$ is called *consistent* if $A_i \subset A_j \Rightarrow i < j$, where $\subset$ denotes strict inclusion. Without loss of generality we will always assume that $A_0 := \emptyset'$. The set $C_i := A_i \backslash (\cup_{A \in \mathcal{A}', A_i \not\subseteq A} A) = A_i \backslash (\cup_{j=1}^{i-1} A_j) \in \mathcal{C}$ is called *the left neighbourhood* of the set A_i, $i = 1, \ldots, n$; we set $C_0 := \emptyset'$.

An increasing function $f : [0,a] \to \mathcal{A}(u)$ is called a *flow*; we will say that a flow f is *continuous* if $f(t) = \cap_n f(t_n)$ whenever $t_n \searrow t$, and $f(t) = \overline{\cup_n f(t_n)}$ whenever $t_n \nearrow t$.

Lemma 1.1 *(Lemma 5.1.7, Ivanoff and Merzbach, 2000) For any consistently ordered sub-semilattice $\mathcal{A}' = \{A_0 = \emptyset', A_1, \ldots, A_n\}$ there exists a continuous* *flow f such that $f(t) = \cup_{j=1}^{i} A_j \cup f_i(t)$, $f_i(t) \in \mathcal{A}$, $t \in [t_i, t_{i+1}]$. (Any continuous flow with this property is called 'simple').*

A *set-indexed process* is a collection $X := (X_A)_{A \in \mathcal{A}}$ of random variables which has an almost surely unique additive extension to the algebra $\mathcal{C}(u)$.

Definition 1.2 **(a)** *A set-indexed process $X := (X_A)_{A\in\mathcal{A}}$ is said to have* **independent increments** *if $X_\emptyset = X_{\emptyset'} = 0$ a.s. and $X_{C_1}, \dots, X_{C_n}$ are independent whenever $C_1, \dots, C_n \in \mathcal{C}$ are pairwise disjoint.*

(b) *A set-indexed process $X := (X_A)_{A\in\mathcal{A}}$ is called* **set-Markov** *if $\forall A \in \mathcal{A}, \forall B \in \mathcal{A}(u)$ the σ-field $\mathcal{F}_B := \sigma(\{X_{A'}; A' \in \mathcal{A}, A' \subseteq B\})$ is conditionally independent of $X_{A \setminus B}$ given X_B.*

Note: Any process with independent increments is automatically set-Markov.

If $X := (X_A)_{A\in\mathcal{A}}$ is a process with independent increments and F_C is the distribution of $X_C, C \in \mathcal{C}$, then the family $(F_C)_{C\in\mathcal{C}}$ is a *convolution system*, i.e. $F_\emptyset = F_{\emptyset'} = \delta_0$ and $F_{C_1} * \dots * F_{C_n} = F_{C'_1} * \dots * F_{C'_m}$ for any sets $C_i, C'_j \in \mathcal{C}$ with $\dot\cup_{i=1}^{n} C_i = \dot\cup_{j=1}^{m} C'_j$ (here $*$ denotes the convolution and $\dot\cup$ the disjoint union).

Comment 1.3 In order to show that a family $(F_C)_{C\in\mathcal{C}}$ is a convolution system it is enough to verify that for any finite sub-semilattices $\mathcal{A}'' \subseteq \mathcal{A}'$, and for any consistent orderings ord'$=\{A'_0 = \emptyset', A'_1, \dots, A'_m\}$, ord$=\{A_0 = \emptyset', A_1, \dots, A_n\}$ of $\mathcal{A}'', \mathcal{A}'$ which are chosen such that, if $A'_l = A_{i_l}; l = 1, \dots, m$ then $\cup_{s=1}^{l} A_{i_s} = \cup_{j=1}^{i_l} A_j$ for any $l = 1, \dots, m$ we have

$$F_{C'_l} = *_{j=i_{l-1}+1}^{i_l} F_{C_j}, \ \forall l = 1, \dots, m$$

where C'_l, respectively C_j are the left neighbourhoods of A'_l, A_j in $\mathcal{A}'', \mathcal{A}'$.

Examples 1.4

1. Let $\Lambda(dz)$ be a finite positive measure on $(T, \sigma(\mathcal{A}))$. A process $X := (X_A)_{A\in\mathcal{A}}$ with independent increments, for which the distribution F_C of each increment $X_C, C \in \mathcal{C}$ is normal with mean 0 and variance Λ_C, is called a *Brownian motion* (with variance measure Λ).

2. Let $\Pi(dz; dx)$ be a finite positive measure on $(T \times \mathbf{R}, \sigma(\mathcal{A}) \times \mathcal{B}(\mathbf{R}))$ and let $\Pi_A(\Gamma) := \Pi(A \times \Gamma), A \in \sigma(\mathcal{A}), \Gamma \in \mathcal{B}(\mathbf{R})$. A process $X := (X_A)_{A\in\mathcal{A}}$ with independent increments, for which the distribution F_C of each increment $X_C, C \in \mathcal{C}$ has the log-characteristic function

$$\psi_C(u) = \int_{\mathbf{R}} (e^{iux} - 1) \Pi_C(dx)$$

is called a *compound Poisson* process (corresponding to the bimeasure Π).

Here is the main result of this section.

Theorem 1.5 *Given any convolution system $(F_C)_{C\in\mathcal{C}}$, there exists a process $X := (X_A)_{A\in\mathcal{A}}$ with independent increments such that the distribution of X_C is F_C for every $C \in \mathcal{C}$.*

Proof: Let $(\mathbf{R}^{\mathcal{C}}, \mathcal{R}^{\mathcal{C}}) := \prod_{C\in\mathcal{C}}(R_C, \mathcal{R}_C)$ where $(R_C, \mathcal{R}_C), C \in \mathcal{C}$ represent $\mathcal{C}$ copies of the real space $\mathbf{R}$ with its Borel subsets. For each k-tuple $(C_1, \ldots, C_k)$ of distinct sets in $\mathcal{C}$ we will define a probability measure $\mu_{C_1\ldots C_k}$ on $(\mathbf{R}^k, \mathcal{R}^k)$ in the following manner: let $C_i = A_i \backslash \cup_{j=1}^{n_i} A_{ij}; i = 1, \ldots, k$ be extremal representations of these sets, $\mathcal{A}'$ the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets A_i, A_{ij}, $\{B_0 = \emptyset', B_1, \ldots, B_n\}$ a consistent ordering of $\mathcal{A}'$, and D_j the left neighbourhood of B_j for $j = 1, \ldots, n$; if $C_i = \dot\cup_{j\in J_i} D_j$ for some $J_i \subseteq \{1, \ldots, n\}; i = 1, \ldots, k$, then define

$$\mu_{C_1\ldots C_k} := (F_{D_1} \times \ldots \times F_{D_n}) \circ \alpha^{-1}$$

where $\alpha(x_1, \ldots, x_n) := (\sum_{j\in J_1} x_j, \ldots, \sum_{j\in J_k} x_j)$.

It is clear that $\mu_{C_1\ldots C_k}$ does not depend on the ordering of the semilattice $\mathcal{A}'$; on the other hand, Lemma 5 of Balan and Ivanoff (2002) (with $Q_{BB'}(x; \Gamma) := F_{B'\backslash B}(\Gamma - x)$) shows that $\mu_{C_1\ldots C_k}$ does not depend on the extremal representations of the sets C_i.

The fact that the family of all probability measures $\mu_{C_1\ldots C_k}$ satisfies the two consistency conditions of Kolmogorov's extension theorem, follows exactly as in proof of Theorem 1 of Balan and Ivanoff (2002). Therefore, there exists a probability measure P on $(\mathbf{R}^{\mathcal{C}}, \mathcal{R}^{\mathcal{C}})$ such that the coordinate-variable process $X := (X_C)_{C\in\mathcal{C}}$ defined by $X_C(x) := x_C$ has the measures $\mu_{C_1\ldots C_k}$ as its finite dimensional distributions. Moreover, this process has an almost surely unique additive extension to $\mathcal{C}(u)$, since its finite dimensional distributions are additive. Finally, it is not hard to see that the process X has independent increments. $\Box$

We will conclude this section by giving an equivalent definition for a set-indexed process with independent increments, and the construction of such a process which is suggested by this property.

Proposition 1.6 *A set-indexed process $X := (X_A)_{A\in\mathcal{A}(u)}$ has independent increments if and only if for for every (simple) flow f the process $X^f := (X_{f(t)})_t$ has independent increments.*

Proof: Clearly $X_t^f - X_s^f$ is independent of $\sigma(X_{f(u)}; u \le s)$ for any flow f.
To prove the converse, let $C_1, \ldots, C_k$ be some pairwise disjoint sets in $\mathcal{C}$, and say $C_i = A_i \backslash \cup_{j=1}^{n_i} A_{ij}; i = 1, \ldots, k$ are some extremal representations. Let $\mathcal{A}'$ be the minimal finite sub-semilattice of $\mathcal{A}$ determined by the sets A_i, A_{ij}, $\mathrm{ord} = \{B_0 = \emptyset', B_1, \ldots, B_n\}$ a consistent ordering of $\mathcal{A}'$, and D_j the left neighbourhood of the set B_j; let f be a simple flow which connects the sets of the semilattice $\mathcal{A}'$ in the sense of the ordering ord, and say $D_j = f(t_j)\backslash f(t_{j-1})$ for some $0 = t_1 < t_2 < \ldots < t_n$. Clearly each set C_i can be expressed uniquely as $C_i = \dot\cup_{j\in J_i} D_j$

for some pairwise disjoint subsets $J_i \subseteq \{1, \ldots, n\}$, and therefore the variables $X_{C_i} = \sum_{j\in J_i} X_{D_j} = \sum_{j\in J_i}(X^f_{t_j} - X^f_{t_{j-1}}), i = 1, \ldots, k$ are independent. $\square$

A family $(F_{st})_{s,t\in[0,a];s<t}$ of probability measures on $\mathbf{R}$ is called a 'one-dimensional' convolution system if $F_{tt} = \delta_0\ \forall t$ and $F_{st} * F_{tu} = F_{su}, \forall s < t < u$.

For each finite sub-semilattice $\mathcal{A}'$ of $\mathcal{A}$ and for each consistent ordering ord$=\{A_0 = \emptyset', A_1, \ldots, A_n\}$ of $\mathcal{A}'$ pick one simple flow $f := f_{\mathcal{A}',\mathrm{ord}}$ which connects the sets of $\mathcal{A}'$ in the sense of the ordering ord (as in Lemma 1.1). Let $\mathcal{S}$ be the collection of all the simple flows $f_{\mathcal{A}',\mathrm{ord}}$.

Assumption 1.7 *If $ord1 = \{A_0 = \emptyset', A_1, \ldots, A_n\}$ and $ord2 = \{A'_0 = \emptyset', A'_1, \ldots, A'_m\}$ are two consistent orderings of some finite semilattices $\mathcal{A}', \mathcal{A}''$ such that $(\cup_{j=1}^n A_j)\backslash(\cup_{j=1}^k A_j) = (\cup_{j=1}^m A'_j)\backslash(\cup_{j=1}^l A'_j)$ for some $k < n, l < m$, and we denote $f := f_{\mathcal{A}',ord1}, g := f_{\mathcal{A}'',ord2}$ with $f(t_i) = \cup_{j=1}^i A_j, g(u_i) = \cup_{j=1}^i A'_j$, then*

$$F^f_{t_k t_n} = F^g_{u_l u_m}$$

The following result is immediate.

Corollary 1.8 *If $\{(F^f_{st})_{s,t\in[0,a];s<t}; f : [0,a] \to \mathcal{A}(u), f \in \mathcal{S}\}$ is a collection of one-dimensional convolution systems satisfying Assumption 1.7, then there exists a process $X := (X_A)_{A\in\mathcal{A}}$ with independent increments such that for every flow $f \in \mathcal{S}$, the distribution of the increment $X^f_t - X^f_s$ is F^f_{st}.*

Proof: Let C be an arbitrary set in $\mathcal{C}$ and $C = A\backslash B, A \in \mathcal{A}, B = \cup_{j=1}^m A'_j \in \mathcal{A}(u)$ one of its extremal representations. Let $\mathcal{A}'$ be the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets $A, A'_1, \ldots, A'_m$, and ord $= \{A_0 = \emptyset', A_1, \ldots, A_n\}$ a consistent ordering of $\mathcal{A}'$ such that $B = \cup_{j=0}^k A_j, A \cup B = \cup_{j=0}^n A_j$ for some $k < n$; let $f := f_{\mathcal{A}',\mathrm{ord}}$ with $f(t_i) = \cup_{j=1}^n A_j$. Define

$$F_C := F^f_{t_k t_n}$$

Note that this definition does not depend on the extremal representation of the set C, because of Assumption 1.7. Using Comment 1.3, we show that the family $(F_C)_{C\in\mathcal{C}}$ is a convolution family: denote $f := f_{\mathcal{A}',\mathrm{ord}}, g := f_{\mathcal{A}'',\mathrm{ord}'}$ with $f(t_i) = \cup_{j=1}^i A_j; i = 1, \ldots, n$ and $g(u_l) = \cup_{s=1}^l A'_s; l = 1, \ldots, m$; then

$$F_{C'_l} = F^g_{u_{l-1}u_l} = F^f_{t_{i_{l-1}}t_{i_l}} = *_{j=i_{l-1}+1}^{i_l} F^f_{t_{j-1}t_j} = *_{j=i_{l-1}+1}^{i_l} F_{C_j}$$

$\square$

2 Lévy processes

We begin with the definition of a set-indexed Lévy process.

Definition 2.1 *We say that* $X := (X_A)_{A \in \mathcal{A}}$ *is a* **Lévy process** *if it has independent increments and* $\forall (A_n)_n \subseteq \mathcal{A}, \forall A \in \mathcal{A}$ *such that either* $A_n \supseteq A_{n+1}\ \forall n; A = \cap_n A_n$*, or* $A_n \subseteq A_{n+1}\ \forall n; A = \overline{\cup_n A_n} \in \mathcal{A}$*, we have* $X_{A_n} \xrightarrow{\mathrm{P}} X_A$*.*

The following result gives the expected correspondence via flows.

Proposition 2.2 *A process* $X := (X_A)_{A \in \mathcal{A}}$ *is Lévy if and only if for every simple flow* f *the process* $X^f := (X_{f(t)})_t$ *is Lévy.*

Proof: Suppose first that X is a Lévy process and let $f : [0, a] \to \mathcal{A}(u)$ be a simple flow of the form $f(t) = \cup_{j=0}^{i} f_j(a_j) \cup f_i(t), t \in [a_i, a_{i+1}]$ where $0 = a_0 < a_1 < \ldots a_m = a$ and $f_i : [a_i, a_{i+1}] \to \mathcal{A}$ are continuous flows.

Let $t, t_n \in [0, a]$ be such that $t_n \to t$. Suppose that $t \in [a_i, a_{i+1}]$ for some i; then $f(t) = B \cup A$ for $A := f_i(t) \in \mathcal{A}$ and some $B \in \mathcal{A}(u)$. For n large enough, $t_n \in [a_i, a_{i+1}]$ and $f(t_n) = B \cup A_n$ for $A_n := f_i(t_n) \in \mathcal{A}$.

If the sequence $(t_n)_n$ is decreasing, then $(A_n)_n$ is a decreasing sequence of sets in $\mathcal{A}$ with intersection A, by the right-continuity of the flow f_i. Using the stochastic continuity of X on $\mathcal{A}$, we can conclude that $X^f_{t_n} = X_{B \cup A_n} \xrightarrow{\mathrm{P}} X_{B \cup A} = X^f_t$. If the sequence $(t_n)_n$ is increasing we get the same conclusion using the left-continuity of the flow f_i.

Suppose now that each of the projected processes X^f is continuous in probability. Let $(A_n)_n$ be a decreasing sequence in $\mathcal{A}$ with intersection A and $A_0 := T$. For each $A_{n-1} \subseteq A_n \subseteq A_{n+1}$ there exist two continuous $\mathcal{A}$-valued flows f_{n-1}, f'_n such that $f_{n-1}(t_{n-1}) = A_{n-1}, f_{n-1}(t_n) = A_n$ and $f'_n(t'_n) = A_n, f'_n(t'_{n+1}) = A_{n+1}$ for some $t_{n-1} \le t_n$ and $t'_n \le t'_{n+1}$ (using Lemma 5.1.6 of Ivanoff and Merzbach, 2000). We can certainly find a linear rescaling f_n of the flow f'_n such that $f_n(t_n) = A_n$, that is the set A_n is the image of the same point t_n under the two different flows. The sequence $(t_n)_n$ is decreasing and let $t_* := \lim_n t_n$. Let $f : [0, t_0] \to \mathcal{A}$ be defined by

$$f(t) := \begin{cases} f_n(t) & \text{if } t \in [t_n, t_{n+1}] \\ g(t) & \text{if } t \in [0, t_*] \end{cases}$$

where $g : [0, t_*] \to \mathcal{A}$ is a continuous flow with $g(0) = \emptyset', g(t^*) = A$. Clearly f is a continuous $\mathcal{A}$-valued flow. Using the stochastic right-continuity of X^f, we have $X_{A_n} = X^f_{t_n} \xrightarrow{\mathrm{P}} X^f_{t_*} = X_A$.

If the sequence $(A_n)_n$ is increasing and $\overline{\cup_n A_n} := A \in \mathcal{A}$ then we start with $A_0 := \emptyset'$ and construct a continuous $\mathcal{A}$-valued flow f such that $f(t_n) = A_n\ \forall n$ for an increasing sequence $(t_n)_n$ with $t^* = \lim_n t_n$. We can set $f(t^*) := A$. The stochastic left-continuity of the process X^f implies that $X_{A_n} \xrightarrow{\mathrm{P}} X_A$. $\Box$

A set-function $x : \mathcal{A} \to \mathbf{R}$ is monotone outer-(respectively monotone inner-) continuous if $\lim_n x(A_n) = x(A)$ for any decreasing (respectively increasing) sequence $(A_n)_n \subseteq \mathcal{A}$ with $A = \cap_n A_n$ (respecively $A = \overline{\cup_n A_n} \in \mathcal{A}$).

Note that $x : \mathcal{A} \to \mathbf{R}$ is monotone outer- and monotone inner- continuous if and only if for every simple flow f, the function $x^f := x \circ f$ is continuous.

Recall that if F is an infinitely divisible distribution, then its log-characteristic function $\psi(u) := \log \int e^{iuy} F(dy)$ can be written uniquely as

$$\psi(u) = i\gamma u - \frac{1}{2}\Lambda u^2 + \int_{\{|y|>1\}} (e^{iuy} - 1)\Pi(dy) + \int_{\{0<|y|\leq 1\}} (e^{iuy} - 1 - iuy)\Pi(dy)$$

where Π is a Lévy measure on $\mathbf{R}$; (γ, Λ, Π) is called the generating triplet of F.

Theorem 2.3 *(Characterization) (i) If $X := (X_A)_{A\in\mathcal{A}}$ is a Lévy process, then the distribution of each $X_C, C \in \mathcal{C}(u)$ is infinitely divisible with generating triplet $(\gamma_C, \Lambda_C, \Pi_C)$ satisfying the following properties:*

(a) $\gamma_{\emptyset'} = 0$*; the function $C \mapsto \gamma_C$ is additive on $\mathcal{C}(u)$, monotone outer- and monotone inner-continuous on $\mathcal{A}$;*

(b) $\Lambda_{\emptyset'} = 0$*; the function $C \mapsto \Lambda_C$ has a unique additive extension to a finite positive measure on $\sigma(\mathcal{A})$;*

(c) $\Pi_{\emptyset'} = 0$*; for any Borel set $\Gamma \subseteq \{y; |y| > \epsilon\}, \epsilon > 0$, the function $C \mapsto \Pi_C(\Gamma)$ has a unique extension to a finite positive measure on $\sigma(\mathcal{A})$.*

(ii) If $F_C, C \in \mathcal{C}$ are infinitely divisible distributions with generating triplets $(\gamma_C, \Lambda_C, \Pi_C)$ satisfying conditions (a), (b), (c), then there exists a Lévy process $X := (X_A)_{A\in\mathcal{A}}$ such that F_C is the distribution of $X_C, \forall C \in \mathcal{C}$.

Proof: (i) Let $C \in \mathcal{C}$ be arbitrary, say $C = A\backslash B$ with $A \in \mathcal{A}, B \in \mathcal{A}(u)$. We know that we can find a *simple* flow f such that $f(s) = B, f(t) = A \cup B$ for some $s < t$; the process $X^f := (X_{f(t)})_t$ is a classical Lévy process, and hence the distribution of $X_C = X^f_t - X^f_s$ is infinitely divisible (Theorem 2.9.1 of Sato, 1999). If $C \in \mathcal{C}(u)$, the distribution of X_C is also infinitely divisible.

Clearly, the functions $A \mapsto \gamma_A, A \mapsto \Lambda_A, A \mapsto \Pi_A(\Gamma)$ (for any Borel set $\Gamma \subseteq \mathbf{R}$) are additive on $\mathcal{C}(u)$. Moreover, these functions are continuous over any simple flow f, i.e. the functions $t \mapsto \gamma^f_t := \gamma_{f(t)}, t \mapsto \Lambda^f_t := \Lambda_{f(t)}, t \mapsto \Pi^f_t(\Gamma) := \Pi_{f(t)}(\Gamma)$ are continuous (using Theorem 2.9.8 of Sato, 1999; here $\Gamma \subseteq \{y; |y| > \epsilon\}$ for some $\epsilon > 0$). Therefore these functions are monotone outer- and monotone inner-continuous.

The functions Λ, Π are also increasing, i.e. $\Lambda_C \geq 0, \Pi_C(\Gamma) \geq 0, \ \forall C \in \mathcal{C}$; using Proposition 1.4.8 of Ivanoff and Merzbach (2000) it follows that they are countably additive on the algebra $\mathcal{C}(u)$, and hence by Carathéodory extension theorem, they can be extended to (unique) finite positive measures on $\sigma(\mathcal{A})$.

(ii) By the additivity of the functions γ, Λ, Π, it follows that the family $(F_C)_{C\in\mathcal{C}}$ is a convolution system. Using Theorem 1.5, there exists a process $X := (X_A)_{A\in\mathcal{A}}$ with independent increments, such that F_C is the distribution of X_C, $\forall C \in \mathcal{C}$.

This process will automatically be a Lévy process: let $(A_n)_n$ be a decreasing sequence of sets in $\mathcal{A}$ with intersection A; then $(A_n \backslash A)_n$ is a decreasing sequence of sets in $\mathcal{C}$ with empty intersection and by the monotone outer-continuity of the functions γ, Λ, Π it follows that $X_{A_n \backslash A} \xrightarrow{\mathcal{D}} 0$ i.e. $X_{A_n} \xrightarrow{P} X_A$. A similar argument may be used for an increasing sequence $(A_n)_n$. $\Box$

Note: Fundamental examples of Lévy processes are the Brownian motion and the compound Poisson process.

Since a set-indexed Lévy process is a classical Lévy process over any simple flow (and hence a classical Markov process over any simple flow), the generator of such a process over a simple flow has a known form. The next result says that conversely, given a family of classical Lévy generators, indexed by a suitable class of simple flows, it is possible to reconstruct a set-indexed Lévy process, providing certain consistency conditions hold.

Let $B(\mathbf{R})$ be the Banach space of all bounded measurable functions $h : \mathbf{R} \to \mathbf{R}$ (with the supremum norm) and $C_b^2(\mathbf{R})$ the subspace of all twice continuously differentiable functions h which have the property that h, h', h'' are bounded.

We recall that the generator at time s of a weakly differentiable Lévy process with generating triplet $(\gamma_t, \Lambda_t, \Pi_t)$ is given by

$$(\mathcal{G}_s h)(x) = \gamma_s' h'(x) + \frac{1}{2}\Lambda_s' h''(x) + \int_{\{|y|>1\}} (h(x+y) - h(x))\Pi_s'(dy) +$$

$$\int_{\{0<|y|\le 1\}} (h(x+y) - h(x) - yh'(x))\Pi_s'(dy)$$

where $\gamma_s', \Lambda_s', \Pi_s'(\Gamma)$ are the derivatives at s of the functions $(\gamma_t)_t (\Lambda_t)_t, (\Pi_t(\Gamma))_t$; the domain of $\mathcal{G}_s$ contains $C_b^2(\mathbf{R})$.

Let $\mathcal{S}$ be a collection of simple flows $f_{\mathcal{A}',\mathrm{ord}}$ as in the previous section, and denote by f_A the simple flow in the class $\mathcal{S}$ which connects the sets of the semilattice $\{A_0 = \emptyset', A_1 = A\}$; say $f_A(t_A) = A$.

Let $\{(\mathcal{G}_s^f)_s \ ; f \in \mathcal{S}\}$ be a collection of families of classical Lévy generators with generating triplets $(\gamma_s^f, \Lambda_s^f, \Pi_s^f)$ and domains $C_b^2(\mathbf{R})$.

Assumption 2.4 *If* $\mathrm{ord1} = \{A_0 = \emptyset', A_1, \ldots, A_n\}$ *and* $\mathrm{ord2} = \{A_0' = \emptyset', A_1', \ldots, A_m'\}$ *are two consistent orderings of some finite semilattices* $\mathcal{A}', \mathcal{A}''$ *such that* $(\cup_{j=1}^n A_j) \backslash (\cup_{j=1}^k A_j) = (\cup_{j=1}^m A_j') \backslash (\cup_{j=1}^l A_j')$ *for some* $k < n, l < m$, *and we denote* $f := f_{\mathcal{A}',\mathrm{ord1}}, g := f_{\mathcal{A}'',\mathrm{ord2}}$ *with* $f(t_i) = \cup_{j=1}^i A_j, g(u_i) = \cup_{j=1}^i A_j'$, *then*

$$\gamma_{t_k t_n}^f = \gamma_{u_l u_m}^g, \ \Lambda_{t_k t_n}^f = \Lambda_{u_l u_m}^g, \ \Pi_{t_k t_n}^f = \Pi_{u_l u_m}^g$$

where $\gamma_{st}^f := \gamma_t^f - \gamma_s^f, \Lambda_{st}^f := \Lambda_t^f - \Lambda_s^f, \Pi_{st}^f := \Pi_t^f - \Pi_s^f$.

Theorem 2.5 *If* $\{(\mathcal{G}_s^f)_s \ ; f \in \mathcal{S}\}$ *is a collection of families of Lévy generators with generating triplets* $(\gamma_s^f, \Lambda_s^f, \Pi_s^f)$ *satisfying Assumption 2.4, then there exists* *a process $X := (X_A)_{A\in\mathcal{A}}$ with independent increments such that $\forall f \in \mathcal{S}$, the generator of $X^f := (X_{f(t)})_t$ at time s is an extension of $\mathcal{G}^f_s$. The process X is Lévy if for any decreasing (respectively increasing) sequence $(A_n)_n \subseteq \mathcal{A}$ with $A := \cap_n A_n$ (respectively $A := \overline{\cup_n A_n} \in \mathcal{A}$) we have*

$$\gamma^{f_{A_n}}_{t_{A_n}} \to \gamma^{f_A}_{t_A},\ \Lambda^{f_{A_n}}_{t_{A_n}} \to \Lambda^{f_A}_{t_A},\ \Pi^{f_{A_n}}_{t_{A_n}}(\Gamma) \to \Pi^{f_A}_{t_A}(\Gamma). \tag{1}$$

Proof: For each flow $f \in \mathcal{S}$, let F^f_{st} be the infinitely divisible distribution with generating triplet $(\gamma^f_{st}, \Lambda^f_{st}, \Pi^f_{st})$; the family $(F^f_{st})_{s<t}$ is a one-dimensional convolution system. The generator at time s of the transition system defined by $Q^f_{st}(x;\Gamma) := F^f_{st}(\Gamma - x),\ x \in \mathbf{R},\ \Gamma \in \mathcal{B}(\mathbf{R})$, is an extension of $\mathcal{G}^f_s$.

The collection $\{(F^f_{st})_{s<t}; f \in \mathcal{S}\}$ satisfies Assumption 1.7; hence, by Corollary 1.8, there exists a process $X := (X_A)_{A\in\mathcal{A}}$ with independent increments such that $\forall f \in \mathcal{S}$ the distribution of $X_{f(t)} - X_{f(s)}$ is F^f_{st}.

By Assumption 2.4, we can define the functions $(\gamma_A)_{A\in\mathcal{A}}, (\Lambda_A)_{A\in\mathcal{A}}, (\Pi_A)_{A\in\mathcal{A}}$ (with unique additive extensions to $\mathcal{C}(u)$) such that the distribution of each $X_C, C \in \mathcal{C}$ is infinitely divisible with generating triplet $(\gamma_C, \Lambda_C, \Pi_C)$. These functions will be monotone outer-continuous and monotone inner-continuous on $\mathcal{A}$ (which is equivalent to saying that the process X is Lévy) if (1) holds. $\square$

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