A Markov Property For Set-Indexed Processes

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Abstract

We consider a type of Markov property for set-indexed processes which is satisfied by all processes with independent increments and which allows us to introduce a transition system theory leading to the construction of the process. A set-indexed generator is defined such that it completely characterizes the distribution of the process.

Keywords: set-indexed process; Markov property; transition system; generator.

1 Introduction

The Markov property is without doubt one of the most appealing notions that exists in the classical theory of stochastic processes and many processes modelling physical phenomena enjoy it. Attempts have been made to generalize this concept to processes where the index set is not totally ordered.

However, to define a Markov property for processes indexed by an uncountable partially ordered set is not a straightforward task for someone who has as declared goals to prove that: (i) all processes with independent increments possess this property; (ii) there exists a systematic procedure which allows us to construct a general process which enjoys this property; and (iii) we can define a generator which completely characterizes the finite dimensional distributions of the process. (The case of Markov processes indexed by discrete partially ordered sets was considered by various authors [6], [13] for solving stochastic optimal

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control problems; as our framework is different, we will not discuss this case here.) In the present paper we will define a new type of Markov property for processes indexed by a semilattice of sets, which will attain these three goals.

This paper represents a successful manner of approaching a problem to which much effort has been dedicated in the literature to date. In fact it will be seen that our definition is completely analogous to the classical definition of the Markov property on $\mathbf{R}$, and our approach avoids many of the technical problems which arise with other definitions of Markov properties. We begin now to describe the two levels of generality of this problem.

The first level deals with the case when the index set is the Euclidean space $[0,1]^2$ (or, more generally $[0,1]^d$) and therefore it inherits the extra structure introduced by the total ordering of the coordinate axes. In this case, there are at least three types of Markov properties which can be considered: the sharp Markov property, the germ Markov property, and the 'simultaneously vertical and horizontal' Markov property.

The basic idea behind the sharp Markov property (first introduced in 1948 by Lévy [14]) was to consider as the history of the process at location z in the plane or space, all the information that we have about the values of the process inside the rectangle $[0,z]$; all the information about the values of the process outside the rectangle was regarded as 'future'; and the past and the future should be independent given the values of the process on the boundary of the rectangle. In 1984, Russo [19] proved that all processes with independent increments are sharp Markov with respect to all finite unions of rectangles. The next step was to see if we can replace the rectangles with more general sets; in other words a two-parameter process $(X_z)_{z\in[0,1]^2}$ is said to have **the sharp Markov property** with respect to a set A if the σ- fields $\mathcal{F}_A$ and $\mathcal{F}_{A^c}$ are conditionally independent given $\mathcal{F}_{\partial A}$ where $\mathcal{F}_D = \sigma(X_z; z \in D)$ for any set $D \subseteq [0,1]^2$. The advantage of this definition is that it does not rely on the partial order of the space. Unfortunately, it turned out that processes having this property are difficult to handle. To attain goal (i) the following question had to be answered: what is the largest class of sets for which all processes with independent increments are sharp Markov? (To answer this question the Gaussian and the jump parts have been treated separately.) Goals (ii) and (iii) do not seem to be even specified anywhere in the literature.

(In 1976 Walsh [20] showed that the Brownian sheet fails to have the sharp Markov property with respect to a very simple set, the triangle with vertices $(0,0),(0,1)$ and $(1,0)$. This led to the conclusion that, instead of the sharp σ-field $\mathcal{F}_{\partial A}$ one has to consider a larger one, called the germ σ-field, which is defined as $\mathcal{G}_{\partial A} = \cap_G \mathcal{F}_G$ where the intersection is taken over all open sets G containing ∂A. The new Markov property, for which $\mathcal{F}_A$ and $\mathcal{F}_{A^c}$ are conditionally independent given $\mathcal{G}_{\partial A}$, was called **the germ Markov property** and it was first introduced by McKean [16]. In 1980 Nualart [17] showed that the Brownian sheet satisfies the germ Markov property with respect to all open sets.)

The complete answer was given in 1992 in the form of two thorough papers written by Dalang and Walsh: in [8] it is shown that the Brownian sheet satisfies the sharp Markov property with respect to all domains whose boundaries are singular curves of bounded variation; on the other hand, the main result of [7] states that for jump processes with independent increments which have only positive jumps, the sharp Markov property holds with respect to all domains.

The third Markov property defined for two-parameter processes was introduced in 1979 by Nualart and Sanz [18] and it was first studied in the Gaussian case. Korezlioglu, Lefort and Mazziotto [12] generalized it and proved that any process satisfying this property is sharp Markov with respect to any finite union of rectangles and germ Markov with respect to any relatively convex domain. The basic idea in this third Markov property is the separation of parameters, that is, the simultaneous definition of a horizontal and vertical Markov property. It is an easy exercise to show that any process with independent increments satisfies this property. The general construction of a process satisfying this property, which corresponds to a certain transition semigroup, was made by Mazziotto [15] (the trajectories of this process have also nice regularity properties).

The second level of generality deals with the case of processes indexed by a collection $\mathcal{A}$ of closed connected subsets of a compact space. The collection $\mathcal{A}$ is partially ordered by set-inclusion and has the additional structure of a semilattice, being closed under arbitrary intersections. This new approach in the modern theory of stochastic processes indexed by partially ordered sets (which is known in the literature as the theory of set-indexed processes) was initiated and developed by Ivanoff and Merzbach for the martingale case [11]. (Other authors [1],[5] were interested in processes indexed by Borel subsets of $[0,1]^d$, but the approach taken by Ivanoff and Merzbach introduces tools which permit the development of a general theory.) If we identify this class with the class of the rectangles $[0,z], z \in [0,1]^d$ and we denote by X_z the value of the set-indexed process at the rectangle $[0,z]$, then $(X_z)_z$ will be a d-dimensional process; hence, we can view this theory as a generalization of the theory of multiparameter processes.

In [10] both sharp Markov and ('simultaneously vertical and horizontal') Markov properties have been introduced in the set-indexed framework, in such a manner that they generalize the corresponding notions defined in the multiparameter case. In fact, in this new set-up, the Markov property implies the sharp Markov property and there are certain circumstances when they are actually equivalent. Moreover, a highly non-trivial result of [10] shows that all processes with independent increments are Markov (in the specified sense), so that goal (i) is attained; however, the authors of [10] do not consider goals (ii) and (iii) and they do not attempt to construct a general Markov process.

In this paper we will consider another type of Markov property for set-indexed processes, which is less general than the Markov property introduced by Ivanoff and Merzbach, but which has the merit of attaining the three goals (i), (ii) and (iii). This Markov property will be called **the set-Markov property** and we will show that it implies the sharp Markov property. The definition requires that the value $X_{A \setminus B} = X_{A\cup B} - X_B$ of an additive set-indexed process $(X_A)_{A\in\mathcal{A}}$ over the increment $A\setminus B$ be conditionally independent of the history $\mathcal{F}_B := \sigma(\{X_{A'}; A' \in \mathcal{A}, A' \subseteq B\})$ given the present status X_B. We believe that this is a natural definition because:

(a) it captures the essence of the Markov property in terms of the increments, allowing us to have a perfect analogy with the classical case;

(b) all processes with independent increments are (trivially) set-Markov;

(c) a process is set-Markov if and only if it becomes Markov in the classical sense when it is transported by a 'flow'. (A 'flow' is an increasing function mapping a bounded interval of the real line into the collection of all finite unions of sets in $\mathcal{A}$.) This property provides us with the means to define a generator for a set-Markov process.

The paper is organized as follows:

In Section 2 we define the general framework which is used in the theory of set-indexed processes. The main structure that we need is a semilattice of sets which leads us to a semialgebra and an algebra of sets. All the processes that we consider are assumed to be a.s. finitely additive on this algebra.

In Section 3 we define the set-Markov property in terms of the increments of the process. We will prove that this property has two equivalent formulations. The first one (Proposition 1) requires that the 'future' behaviour of the process at a set $B'(\supseteq B)$ be conditionally independent of its history $\mathcal{F}_B$ knowing its present status at the set B. The second formulation (Proposition 2) requires that the (one-parameter) processes X^f obtained as 'traces' of the original set-indexed process X along the paths of all flows f be Markov in the classical sense; this allows us to make use of the rich theory that exists for Markov processes indexed by the real line.

In Section 4 we turn to the question of existence of a set-Markov process and we prove that we can construct such a process if we specify the laws $Q_{BB'}$ that characterize the transition from a set B to a set $B'(\supseteq B)$. A set-Markov process with a given transition system $\mathcal{Q} := (Q_{BB'})_{B\subseteq B'}$ is called '$\mathcal{Q}$-Markov' and can be constructed as soon as we specify its finite dimensional distributions (by Kolmogorov's existence theorem). However, the formula that we obtain for the finite dimensional distribution over some k-tuple $A_1, \ldots, A_k$ of sets turns out to be dependent on the ordering of the sets; hence we have to impose a natural consistency condition (Assumption 1) which makes the finite dimensional distribution 'invariant' under permutations. An important consequence of this condition is that the finite dimensional distributions of a $\mathcal{Q}$-Markov process are also additive (Lemma 5).

In Section 5 we define the generator of a $\mathcal{Q}$-Markov process X as the class $\{\mathcal{G}^f; f \in \mathcal{S}\}$, where $\mathcal{G}^f$ is the generator of the (one-parameter) Markov process X^f and $\mathcal{S}$ is a large enough (uncountable) collection of flows. In other words, at any set B there are infinitely many generators depending on which 'direction' we approach this set along the path of a flow f in $\mathcal{S}$ (note that in the classical case, the class $\mathcal{S}$ of flows can be taken to have a single element).

In this section we address the question of existence of a $\mathcal{Q}$-Markov process given its generator. More precisely, we start with a collection $\{\mathcal{G}^f; f \in \mathcal{S}\}$, each $\mathcal{G}^f$ being the generator of a given semigroup $\mathcal{T}^f$. To simplify the problem *we will assume that each $\mathcal{T}^f$ is the semigroup asociated to a transition system $\mathcal{Q}^f$*. The consistency conditions that have to be imposed on the collection $\{\mathcal{Q}^f; f \in \mathcal{S}\}$ so that it leads to a set-indexed transition system $\mathcal{Q}$ (and hence to a $\mathcal{Q}$-Markov process) are given in Section 4 (Assumptions 2 and 3). Using an integral form of the Kolomogorov-Feller equations, we prove that these conditions can be expressed equivalently in terms of the generators $\mathcal{G}^f$ and the semigroups $\mathcal{T}^f$. (*We conjecture that the same formulas will be valid in the general case when there is no transition system $\mathcal{Q}^f$ associated to the semigroup $\mathcal{T}^f$.*)

Throughout, we shall illustrate our results with three examples of set-Markov processes: processes with independent increments, the empirical process, and the Dirichlet process.

We note in passing that set-Markov processes satisfy a type of strong Markov property which is studied in a separate paper [2].

2 The Set-Indexed Framework

This section introduces the general definitions, properties and assumptions that are used in the theory of set-indexed processes, as presented in [11]. We will also give several examples, in addition to the multiparameter case.

Let $\mathcal{A}$ be a **semilattice** of closed subsets of a compact Hausdorff topological space T (i.e. $\mathcal{A}$ is closed under arbitrary intersections), which contains the empty set and the space T itself, but does not contain disjoint (non-empty) sets. In addition, we assume that the collection $\mathcal{A}$ is **separable from above**, in the sense that any set $A \in \mathcal{A}$ can be approximated from above as

$$A = \cap_n g_n(A); \quad g_{n+1}(A) \subseteq g_n(A), A \subseteq g_n(A)^0 \ \forall n$$

where the approximation set $g_n(A)$ can be written as a finite union of sets that lie in a finite sub-semilattice $\mathcal{A}_n$ of $\mathcal{A}$; moreover, $\mathcal{A}_n \subseteq \mathcal{A}_{n+1} \ \forall n$ and g_n preserves arbitrary intersections and finite unions i.e. $g_n(\cap_{\alpha\in\Lambda} A_\alpha) = \cap_{\alpha\in\Lambda} g_n(A_\alpha)$, $\forall A_\alpha \in \mathcal{A}$; and $\cup_{i=1}^k A_i = \cup_{j=1}^m A'_j \Rightarrow \cup_{i=1}^k g_n(A_i) = \cup_{j=1}^m g_n(A'_j)$, $\forall A_i, A'_j \in \mathcal{A}$. By convention, $g_n(\emptyset) = \emptyset$.

There are many examples of classes of sets which have these properties.

Examples 1.

1. $T = [0,1]^d, \mathcal{A} = \{[0,z]; z \in T\} \cup \{\emptyset\}$.
2. $T = [0,1]^d, \mathcal{A} = \{A; A$ a compact lower layer in $T\} \cup \{\emptyset\}$. (A is a lower layer if $z \in A \Rightarrow [0,z] \subseteq A$)

3. $T = [a,b]^d, a < 0 < b, \mathcal{A} = \{[0,z]; z \in T\} \cup \{\emptyset\}$.

4. $T = [a,b]^d, a < 0 < b, \mathcal{A} = \{A; A \text{ a compact lower layer in } T\} \cup \{\emptyset\}$.

5. $T = \overline{B(0,t_0)}$ (compact ball in $\mathbf{R}^3$), $\mathcal{A} = \{A_{R,t}; R := `[a,b] \times [c,d]', 0 \leq a < b < 2\pi, -\pi \leq c < d \leq \pi, t \in [0,t_0]\}$, where the set

$$A_{R,t} := \{(r\cos\theta\cos\tau, r\sin\theta\cos\tau, r\sin\tau); \theta \in [a,b], \tau \in [c,d], r \in [0,t]\}$$

can be interpreted as the history of the region

$$R := `[a,b] \times [c,d]' = \{(\cos\theta\cos\tau, \sin\theta\cos\tau, \sin\tau); \theta \in [a,b], \tau \in [c,d]\}$$

of the Earth from the beginning until time t. (Here θ represents the longitude of the generic point in the region R, while τ is the latitude.) Hence, $\mathcal{A}$ can be identified with the history of the world until time t_0.

Let $\emptyset' := \cap_{A \in \mathcal{A}\setminus\{\emptyset\}} A$ be the minimal set in $\mathcal{A}$ ($\emptyset' \neq \emptyset$). The role played by $\emptyset'$ will be similar to the role played by 0 in the classical theory.

We will consider the following classes of sets generated by $\mathcal{A}$:

- $\mathcal{A}(u)$ is the class of all finite unions of sets in $\mathcal{A}$
- $\mathcal{C}$ is the class of all sets of the form $C = A\setminus B$ with $A \in \mathcal{A}, B \in \mathcal{A}(u)$
- $\mathcal{C}(u)$ is the class of all finite unions of sets in $\mathcal{C}$

Note that $\mathcal{A}(u)$ is closed under finite intersections or finite unions, $\mathcal{C}$ is a semialgebra and $\mathcal{C}(u)$ is the algebra generated by $\mathcal{C}$. The value $X_C = X_{A\cup B} - X_B$ of an (additive) set-indexed process $(X_A)_{A\in\mathcal{A}}$ over the set $C = A\setminus B = (A \cup B)\setminus B$ will play the role of the increment $X_t - X_s, s < t$ of a one-dimensional process $(X_t)_{t\in[0,a]}$.

Any set $B \in \mathcal{A}(u)$ admits at least one **extremal representation** of the form $B = \cup_{i=1}^n A_i, A_i \in \mathcal{A}$ with $A_i \not\subseteq \cup_{j\neq i} A_j \ \forall i$. For a set $C \in \mathcal{C}$ we will say that the representation $C = A\setminus B$ is extremal if the representation of B is extremal.

Since the finite sub-semilattices of the indexing collection $\mathcal{A}$ play a very important role in the theory of set-indexed processes, having an appropriate ordering on the sets of a finite sub-semilattice proves to be a very useful tool in handling these objects. The ordering that we have in mind will be defined in such a manner that a set is never numbered before any of its subsets. We will call such an ordering **consistent** (with the strong past). More precisely, if $\mathcal{A}'$ is a finite sub-semilattice of $\mathcal{A}$ we set $A_0 := \emptyset'$ and $A_1 := \cap_{A\in\mathcal{A}'} A$ (note that the sets A_0 and A_1 are not necessarily distinct). Proceeding inductively, assuming that the distinct sets $A_1, \ldots, A_i \in \mathcal{A}'$ have already been counted, choose $A_{i+1} \in \mathcal{A}'\setminus\{A_1 \ldots, A_i\}$ such that if there exists a set $A \in \mathcal{A}'$ with

$A \subseteq A_{i+1}, A \neq A_{i+1}$ then $A = A_j$ for some $j \leq i$. It is clear that such an ordering always exists although in general, it is not unique.

If $\{A_0 = \emptyset', A_1, \ldots, A_n\}$ is a consistent ordering of a finite sub-semilattice $\mathcal{A}'$, the set $C_i = A_i \setminus \cup_{j=0}^{i-1} A_j \in \mathcal{C}$ is called **the left neighbourhood** of the set A_i (we make the convention that the left neighbourhood of $A_0 = \emptyset'$ is itself). The definition of the left neighbourhood does not depend on the ordering since one can show that $C_i = A_i \setminus (\cup_{A \in \mathcal{A}', A_i \not\subseteq A} A)$.

Comments 1.

1. If $B_1, \ldots, B_m \in \mathcal{A}(u)$ are such that $B_1 \subseteq \cdots \subseteq B_m$ then there exists a finite sub-semilattice $\mathcal{A}'$ of $\mathcal{A}$ and a consistent ordering $\{A_0 = \emptyset', A_1, \ldots, A_n\}$ of $\mathcal{A}'$ such that $B_l = \cup_{j=0}^{i_l} A_j$, for some $0 < i_1 \leq \ldots \leq i_m = n$.

2. If $\mathcal{A}', \mathcal{A}''$ are two finite sub-semilattices of $\mathcal{A}$ such that $\mathcal{A}' \subseteq \mathcal{A}''$, there exists a consistent ordering $\{A_0 = \emptyset', A_1, \ldots, A_n\}$ of $\mathcal{A}''$ such that if $\mathcal{A}' = \{A_{i_0} = \emptyset', A_{i_1}, \ldots, A_{i_m}\}$ with $0 = i_0 < i_1 \leq \ldots \leq i_m$, then $\cup_{s=1}^{l} A_{i_s} = \cup_{j=1}^{i_l} A_j$ for any $l = 1, \ldots, m$.

Let us consider now a complete probability space $(\Omega, \mathcal{F}, P)$ and let $X := (X_A)_{A \in \mathcal{A}}$ be an $\mathbf{R}$-valued process defined on this space, indexed by the class $\mathcal{A}$.

We say that the process X **has an (almost surely) unique additive extension to** $\mathcal{A}(u)$ if whenever the set $B \in \mathcal{A}(u)$ can be written as $B = \cup_{i=1}^{n} A_i = \cup_{j=1}^{m} A'_j$ with $A_1, \ldots, A_n, A'_1, \ldots, A'_m \in \mathcal{A}$ we have

$$\sum_{i=1}^{n} X_{A_i} - \sum_{1 \leq i_1 < i_2 \leq n} X_{A_{i_1} \cap A_{i_2}} + \cdots + (-1)^{n+1} X_{A_1 \cap \ldots \cap A_n} =$$
$$\sum_{j=1}^{m} X_{A'_j} - \sum_{1 \leq j_1 < j_2 \leq m} X_{A'_{j_1} \cap A'_{j_2}} + \cdots + (-1)^{m+1} X_{A'_1 \cap \ldots \cap A'_m} \text{ a.s.}$$

In this case, outside a set of measure 0, we can define X_B as being either one of the members of the above almost sure equality. In order to verify that a process has a unique additive extension to $\mathcal{A}(u)$ it is enough to prove the previous almost sure equality only in the case when $n = 1$. The general case will follow since if $\cup_{i=1}^{n} A_i = \cup_{j=1}^{m} A'_j$, then each of the sets $A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k} \in \mathcal{A}$ can be written as the union $\cup_{j=1}^{m} (A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k} \cap A'_j)$.

Similarly, the process X is said to **have an (almost surely) unique additive extension to** $\mathcal{C}$ if whenever the set $C \in \mathcal{C}$ can be written as $C = A \setminus B = A' \setminus B'$ with $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{A}(u)$ we have

$$X_A - X_{A \cap B} = X_{A'} - X_{A' \cap B'} \text{ a.s.}$$

The additive extension to $\mathcal{C}(u)$ is defined in the obvious manner.

A set-indexed process $X := (X_A)_{A\in\mathcal{A}}$ with a unique additive extension to $\mathcal{C}(u)$, for which $X_{C_1}, \ldots, X_{C_n}$ are independent whenever the sets $C_1, \ldots, C_n \in \mathcal{C}$ are disjoint, is called a **process with independent increments**.

An example of a process which has a unique additive extension to $\mathcal{C}(u)$ is the **empirical process** of size n, corresponding to a probability measure F on T, defined by $X_A := \sum_{j=1}^n I_{\{Z_j\in A\}}, A \in \mathcal{A}$, where $(Z_j)_{j\geq 1}$ are i.i.d. T-valued random variables with common distribution F.

Another example is the **Dirichlet process** with parameter measure α, where α is a finite positive measure on $\sigma(\mathcal{A})$ (see Ferguson [9]). This process is in fact almost surely *countably* additive on $\sigma(\mathcal{A})$ and takes values in $[0, 1]$; moreover $X_T = 1$ a.s. Its finite dimensional distribution of this process over any disjoint sets $A_1, \ldots, A_k$ with $A_i \in \sigma(\mathcal{A})$ is the (non-singular) Dirichlet distribution with parameters $(\alpha(A_1), \ldots, \alpha(A_k); \alpha((\cup_{i=1}^k A_i)^c))$ (see Ferguson [9]).

In what follows we will examine the information structure that can be associated with a set-indexed process.

A collection $(\mathcal{F}_A)_{A\in\mathcal{A}}$ of sub-σ-fields of $\mathcal{F}$ is called a **filtration** if $\mathcal{F}_A \subseteq \mathcal{F}_{A'}$ whenever $A, A' \in \mathcal{A}, A \subseteq A'$. We will consider only complete filtrations. An $\mathcal{A}$-indexed filtration $(\mathcal{F}_A)_{A\in\mathcal{A}}$ can be extended to a filtration indexed by $\mathcal{A}(u)$ by defining for each $B \in \mathcal{A}(u)$,

$$\mathcal{F}_B := \bigvee_{A\in\mathcal{A}, A\subseteq B} \mathcal{F}_A \tag{1}$$

A set-indexed process X is **adapted** with respect to the filtration $(\mathcal{F}_A)_{A\in\mathcal{A}}$ if X_A is $\mathcal{F}_A$ -measurable for any $A \in \mathcal{A}$. If X is adapted and has a unique additive extension to $\mathcal{A}(u)$ then X_B is $\mathcal{F}_B$-measurable for any $B \in \mathcal{A}(u)$. Given a set-indexed process X, the **minimal filtration** with respect to which X is adapted is given by $\mathcal{F}_B := \sigma(\{X_A; A \in \mathcal{A}, A \subseteq B\})$.

Any map $f : [0, a] \to \mathcal{A}(u)$ which is increasing with respect to the partial order induced by the set-inclusion is called a **flow**.

Definition 1. *A flow $f : [0, a] \to \mathcal{A}(u)$ is*

a) continuous *if for any $t \in [0, a]$ and for any decreasing sequence $(t_n)_n$ with $\lim_{n\to\infty} t_n = t$ we have $f(t) = \cap_n f(t_n)$, and for any increasing sequence $(t_n)_n$ with $\lim_{n\to\infty} t_n = t$ we have $f(t) = \overline{\cup_n f(t_n)}$.*

b) simple *if it is continuous and there exists a partition $0 = t_0 < t_1 < \ldots < t_n = a$ and flows $f_{i+1} : [t_i, t_{i+1}] \to \mathcal{A}$, $i = 0, \ldots n-1$ such that $f(0) = \emptyset'$ and $f(t) = \cup_{j=1}^i f_j(t_j) \cup f_{i+1}(t), t \in [t_i, t_{i+1}], i = 0, \ldots, n-1$. (In other words, a simple flow is piecewise $\mathcal{A}$-valued.)*

If $\mathcal{A}'$ is a finite sub-semilattice and ord$= \{\emptyset' = A_0, A_1, \ldots, A_n\}$ is a consistent ordering of $\mathcal{A}'$, we say that a simple flow f **connects the sets of the semilattice** $\mathcal{A}'$, in the sense of the ordering ord, if $f(t) = \cup_{j=1}^i A_j \cup f_{i+1}(t), t \in [t_i, t_{i+1}]$, where $t_0 = 0 < t_1 < \ldots < t_n = a$ is a partition of the domain of definition of f and $f_{i+1} : [t_i, t_{i+1}] \to \mathcal{A}$ are continuous flows with $f_{i+1}(t_i) = A_i, f_{i+1}(t_{i+1}) = A_{i+1}$.

Lemma 1. *(Lemma 5.1.7, [11]) For every finite sub-semilattice $\mathcal{A}'$ and for each consistent ordering ord of $\mathcal{A}'$, there exists a simple flow f which connects the sets of the semilattice $\mathcal{A}'$, in the sense of the ordering ord.*

Comment 2. A consequence of the previous lemma is the following: given $B_1, \ldots, B_m \in \mathcal{A}(u)$ such that $B_1 \subseteq \ldots \subseteq B_m$ there exists a simple flow f and $t_1 \leq \ldots \leq t_m$ such that $B_i = f(t_i); i = 1, \ldots, m$.

3 Set-Markov Processes

In this section we will introduce the definition of the set-Markov property. Two immediate consequences of the definition will be: (a) any process with independent increments is set-Markov; and (b) the set-Markov property is equivalent to the classical Markov property on every flow. Finally we will prove that any set-Markov process is also sharp Markov.

If $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are three sub-σ-fields of the same probabilistic space, we will use the notation $\mathcal{F} \perp \mathcal{H} \mid \mathcal{G}$ if $\mathcal{F}$ and $\mathcal{H}$ are conditionally independent given $\mathcal{G}$.

Definition 2. *Let $X := (X_A)_{A \in \mathcal{A}}$ be a set-indexed process with a unique additive extension to $\mathcal{C}(u)$ and $(\mathcal{F}_A)_{A \in \mathcal{A}}$ its minimal filtration. We say that the process X is* **set-Markov** *if $\forall A \in \mathcal{A},\ \forall B \in \mathcal{A}(u),\ \mathcal{F}_B \perp \sigma(X_{A \setminus B}) \mid \sigma(X_B)$.*

It is easy to see that in the classical case the set-Markov property is equivalent to the usual Markov property.

Examples 2.

1. Any process $X := (X_A)_{A \in \mathcal{A}}$ with independent increments is set-Markov since $\forall A \in \mathcal{A},\ \forall B \in \mathcal{A}(u),\ X_{A \setminus B}$ is independent of $\mathcal{F}_B$.

2. The empirical process of size n (corresponding to F) is set-Markov since $\forall A \in \mathcal{A},\ \forall B \in \mathcal{A}(u)$, for any partition $B = \cup_{i=1}^p C_i, C_i \in \mathcal{C}$ and for any $l, k_1, \ldots, k_p \in \{0, 1, \ldots, n\}; k := \sum_{i=1}^p k_i$ with $k + l \leq n$

$$P[X_{A \setminus B} = \frac{l}{n} | X_{C_i} = \frac{k_i}{n}; i = 1, \ldots, p] = P[X_{A \setminus B} = \frac{l}{n} | X_B = \frac{k}{n}]$$

   both sides being equal to the value at l of the binomial distribution with $n - k$ trials and $\frac{F(A \setminus B)}{1 - F(B)}$ probability of success.

3. The Dirichlet process $X := (X_A)_{A \in \sigma(\mathcal{A})}$ with parameter measure α is set-Markov since $\forall A \in \mathcal{A},\ \forall B \in \mathcal{A}(u)$, for any partition $B = \cup_{i=1}^p C_i, C_i \in \mathcal{C}$ and for any $y, x_1, \ldots, x_p \in [0, 1]; x := \sum_{i=1}^p x_i$ with $x + y \leq 1$

$$P[X_{A \setminus B} \leq y | X_{C_i} = x_i; i = 1, \ldots, p] = P[X_{A \setminus B} \leq y | X_B = x]$$

both sides being equal to the value at $\frac{y}{1-x}$ of the Beta distribution with parameters $(\alpha(A\backslash B);\alpha((A\cup B)^c))$ (we use property **7.7.3**, p. 180, [21] of the Dirichlet distribution).

We shall make repeated use of the following elementary result.

Lemma 2. *Let $\mathcal{G}'\subseteq\mathcal{G}$ be two σ-fields in the same probability space and X,Y two random vectors on this space such that Y is $\mathcal{G}$-measurable. Suppose that $E[f(X)|\mathcal{G}]=E[f(X)|\mathcal{G}']$ for every bounded measurable function f. Then $E[h(X,Y)|\mathcal{G}]=E[h(X,Y)|\mathcal{G}',Y]$ for every bounded measurable function h.*

Proposition 1. *Let $X:=(X_A)_{A\in\mathcal{A}}$ be a set-indexed process with a unique additive extension to $\mathcal{C}(u)$ and $(\mathcal{F}_A)_{A\in\mathcal{A}}$ its minimal filtration. The process X is set-Markov if and only if $\forall B,B'\in\mathcal{A}(u),B\subseteq B'$, $\mathcal{F}_B\perp\sigma(X_{B'})\mid\sigma(X_B)$.*

Proof: Using Lemma 2 and the additivity of the process, it follows that X is set-Markov if and only if $\forall A\in\mathcal{A}\ \forall B\in\mathcal{A}(u)$, $\mathcal{F}_B\perp\sigma(X_{A\cup B})\mid\sigma(X_B)$. (Write $X_{A\cup B}=X_{A\backslash B}+X_B$ and use the fact that X_B is $\mathcal{F}_B$-measurable.) Let $B,B'\in\mathcal{A}(u)$ be such that $B\subseteq B'$. Say $B'=\cup_{i=1}^k A_i, A_i\in\mathcal{A}$ is an arbitrary representation. Then $B'=B\cup B'=B\cup\cup_{i=1}^k A_i$ and the result follows by induction on k. □

Comment 3. Using Proposition 1 and Lemma 2, we can say that a process $X:=(X_A)_{A\in\mathcal{A}}$ is set-Markov if and only if $\forall B,B'\in\mathcal{A}(u),B\subseteq B'$, $\mathcal{F}_B\perp\sigma(X_{B'\backslash B})\mid\sigma(X_B)$.

The following result says that a process is set-Markov if and only if it is Markov (in the usual sense) along any flow. Moreover, it suffices to restrict our attention only to simple flows.

Proposition 2. *Let $X:=(X_A)_{A\in\mathcal{A}}$ be a set-indexed process with a unique additive extension to $\mathcal{C}(u)$ and $(\mathcal{F}_A)_{A\in\mathcal{A}}$ its minimal filtration. Then X is set-Markov if and only if for every simple flow $f:[0,a]\to\mathcal{A}(u)$ the process $X^f:=(X_{f(t)})_{t\in[0,a]}$ is Markov with respect to the filtration $(\mathcal{F}_{f(t)})_{t\in[0,a]}$. (For necessity, we can consider any flow, not only the simple ones.)*

Proof: The process X^f is Markov with respect to the filtration $(\mathcal{F}_{f(t)})_t$ if and only if $\forall s,t\in[0,a],s<t,\mathcal{F}_{f(s)}\perp\sigma(X_{f(t)})\mid\sigma(X_{f(s)})$. (We note that the filtration $(\mathcal{F}_{f(t)})_{t\in[0,a]}$ is not the minimal filtration associated to the process X^f.) This is equivalent to the set-Markov property since we know that whenever the sets $B,B'\in\mathcal{A}(u)$ are such that $B\subseteq B'$ there exists a simple flow f and some $s<t$ such that $f(s)=B$ and $f(t)=B'$ (Comment 2). □

The preceding proposition, while simple, is crucial since (as noted in the Introduction) it provides us with the means to define the generator of a set-Markov process. This will be done in Section 5.

We now show that every set-Markov process satisfies the sharp-Markov property defined in [10]. In analogy with the minimal filtration $(\mathcal{F}_B)_{B\in\mathcal{A}(u)}$ we define the following σ-fields, for an arbitrary set $B \in \mathcal{A}(u)$:

$$\mathcal{F}_{\partial B} := \sigma(\{X_A; A \in \mathcal{A}, A \subseteq B, A \not\subseteq B^0\})$$

$$\mathcal{F}_{B^c} := \sigma(\{X_A; A \in \mathcal{A}, A \not\subseteq B\})$$

Definition 3. *Let $X := (X_A)_{A\in\mathcal{A}}$ be a set-indexed process with a unique additive extension to $\mathcal{C}(u)$ and $(\mathcal{F}_A)_{A\in\mathcal{A}}$ its minimal filtration. We say that the process X is* **sharp-Markov** *if $\forall B \in \mathcal{A}(u)$, $\mathcal{F}_B \perp \mathcal{F}_{B^c} \mid \mathcal{F}_{\partial B}$.*

The next lemma will be essential in proving that a set-Markov process is sharp Markov.

Lemma 3. *If $X := (X_A)_{A\in\mathcal{A}}$ is a set-Markov process, then $\forall B \in \mathcal{A}(u)$ $\forall A_i \in \mathcal{A}; i = 1, \dots, n$, $\mathcal{F}_B \perp \sigma(X_{A_1\setminus B}, \dots, X_{A_n\setminus B}) \mid \sigma(X_B)$.*

Proof: Let $B \in \mathcal{A}(u)$,$A_1, \dots A_n \in \mathcal{A}$ and $h : \mathbf{R}^n \to \mathbf{R}$ an arbitrary bounded measurable function. Without loss of generality we can assume that $A_i \not\subseteq B$ $\forall i$; say $B = \cup_{j=n+1}^m A_j$, $A_j \in \mathcal{A}$. Let $\mathcal{A}'$ be the smallest finite sub-semilattice which contains $A_1, \dots A_m$, $\{A'_0 = \emptyset', A'_1, \dots, A'_p\}$ a consistent ordering of $\mathcal{A}'$, and C'_i the left neighborhood of A'_i in $\mathcal{A}'$. Say $A_j = A'_{i_j}$ for $j = 1 \dots n$; then $A_j\setminus B = A'_{i_j}\setminus B = \cup_{i\in I_j} C'_i$ with $I_j \subseteq \{1, \dots, i_j\}$ and $X_{A_j\setminus B} = \sum_{i\in I_j} X_{C'_i}$.

Therefore we can say that $h(X_{A_1\setminus B}, \dots, X_{A_n\setminus B}) = h_1(X_{C'_{l_1}}, \dots, X_{C'_{l_s}})$, for a certain bounded measurable function h_1 and some $l_1 \le \dots \le l_s, C'_{l_i} \not\subseteq B$. In order to simplify the notation, let us denote $D_i := C'_{l_i}, i = 1, \dots, s$. Let $B_i = B \cup \cup_{k=1}^i D_k$ for $i = 1, \dots, s$. Then $D_i = B_i\setminus B_{i-1}$ and $X_{D_i} = X_{B_i} - X_{B_{i-1}}$; hence $X_{D_1}, \dots, X_{D_{s-1}}$ are $\mathcal{F}_{B_{l_{s-1}}}$-measurable. Because each D_i is the left neighbourhood of A'_{l_i}, we also have $B_i = B \cup \cup_{k=1}^i A'_{l_k}$ and therefore $D_i = (A'_{l_i} \cup B_{i-1})\setminus B_{i-1} = A'_{l_i}\setminus B_{i-1}$. Using the equivalent definition of the set-Markov property given by Comment 3, we have $E[f(X_{D_s})|\mathcal{F}_{B_{s-1}}] = E[f(X_{D_s})|X_{B_{s-1}}]$ for any bounded measurable function f. Now we are in the position to apply Lemma 2 to get

$$E[h_1(X_{D_1}, \dots, X_{D_s})|\mathcal{F}_{B_{s-1}}] = E[h_1(X_{D_1}, \dots, X_{D_s})|X_{B_{s-1}}, X_{D_1}, \dots, X_{D_{s-1}}].$$

Hence $E[h(X_{A_1\setminus B}, \dots, X_{A_n\setminus B})|\mathcal{F}_B] = E[E[h_1(X_{D_1}, \dots, X_{D_s})|\mathcal{F}_{B_{s-1}}]|\mathcal{F}_B] = E[E[h_1(X_{D_1}, \dots, X_{D_s})|X_{B_{s-1}}, X_{D_1}, \dots, X_{D_{s-1}}]|\mathcal{F}_B]$. Writing $X_{B_{s-1}} = X_B + \sum_{k=1}^{s-1} X_{D_k}$ we get $E[h(X_{A_1\setminus B}, \dots, X_{A_k\setminus B})|\mathcal{F}_B] = E[h_2(X_{D_1}, \dots, X_{D_{s-1}}, X_B)|\mathcal{F}_B]$. Continuing in the same manner, reducing at each step another set D_i, we finally get $E[h(X_{A_1\setminus B}, \dots, X_{A_n\setminus B})|\mathcal{F}_B] = E[h(X_{A_1\setminus B}, \dots, X_{A_n\setminus B})|X_B]$. □

Proposition 3. *Any set-Markov process is sharp Markov.*

Proof: Let $B \in \mathcal{A}(u), A_i \in \mathcal{A}, A_i \not\subseteq B; i = 1, \dots, n$ and $h : \mathbf{R}^n \to \mathbf{R}$ an arbitrary bounded measurable function. Writing $X_{A_i} = X_{A_i\cap B} + X_{A_i\setminus B}$ and using Lemma 2 we can say that $E[h(X_{A_1}, \dots, X_{A_n})|\mathcal{F}_B] = E[h(X_{A_1}, \dots, X_{A_n})|X_B, X_{A_i\cap B}, i = 1, \dots n]$, which is $\mathcal{F}_{\partial B}$-measurable, by Lemma 2.4, [10]. □

In what follows we will give an important characterization of the set-Markov processes that will be instrumental for the construction of these processes.

We will need the following elementary result.

Lemma 4. *Let $\mathcal{F}_i, i = 1, \ldots, 4$ be four σ-fields in the same probability space. If $\mathcal{F}_1 \perp \mathcal{F}_3 \mid \mathcal{F}_2$ and $\mathcal{F}_1 \vee \mathcal{F}_2 \perp \mathcal{F}_4 \mid \mathcal{F}_3$, then $\mathcal{F}_1 \perp \mathcal{F}_3 \vee \mathcal{F}_4 \mid \mathcal{F}_2$.*

Proposition 4. *A set-indexed process $X := (X_A)_{A\in\mathcal{A}}$ with a unique additive extension to $\mathcal{C}(u)$, is set-Markov if and only if for every finite sub-semilattice $\mathcal{A}'$ and for every consistent ordering $\{A_0 = \emptyset', A_1, \ldots, A_n\}$ of $\mathcal{A}'$*

$$\sigma(X_{A_0}, X_{A_0\cup A_1}, \ldots, X_{\cup_{j=0}^{i-1}A_j}) \perp \sigma(X_{\cup_{j=0}^{i+1}A_j}) | \sigma(X_{\cup_{j=0}^{i}A_j}) \ \forall i = 1, \ldots, n-1 \quad (2)$$

Proof: We will use the characterization of the set-Markov property given by Proposition 1. Necessity follows immediately.

For sufficiency note first that equation (2) implies a similar equation where the union of the first $i+1$ sets is replaced by the union of the first $i+p$ sets. In fact it is easily shown by induction on p and using Lemma 4, that

$$\sigma(X_{A_0}, X_{A_0\cup A_1}, \ldots, X_{\cup_{j=0}^{i-1}A_j}) \perp \sigma(X_{\cup_{j=0}^{i+k}A_j}; k = 1, \ldots, p) | \sigma(X_{\cup_{j=0}^{i}A_j}) \ \forall i \quad (3)$$

Consider now arbitrary sets $B, B' \in \mathcal{A}(u)$ with $B \subseteq B'$. By a monotone class argument it is enough to show that $\forall A'_l \in \mathcal{A}, A'_l \subseteq B; l = 1, \ldots, m$, $\sigma(X_{A'_1}, \ldots, X_{A'_m}) \perp \sigma(X_{B'}) | \sigma(X_B)$. Without loss of generality we may assume that $\{A'_0 = \emptyset', A'_1, \ldots, A'_m\}$ is a finite sub-semilattice and that the ordering $\{A'_0 = \emptyset', A'_1, \ldots, A'_m\}$ is consistent. Because the process X has a unique additive extension to $\mathcal{A}(u)$, there exists a bijective map ψ such that $(X_{A'_1}, X_{A'_2}, \ldots, X_{A'_m}) = \psi(X_{A'_1}, X_{A'_1\cup A'_2}, \ldots, X_{\cup_{l=1}^{m}A'_l})$ a.s.. Consequently, we have $\sigma(X_{A'_1}, X_{A'_2}, \ldots, X_{A'_m}) = \sigma(X_{A'_1}, X_{A'_1\cup A'_2}, \ldots, X_{\cup_{l=1}^{m}A'_l})$.

By Comment 1.1, there exists a finite sub-semilattice $\mathcal{A}'$ of $\mathcal{A}$ and a consistent ordering $\{A_0 = \emptyset', A_1, \ldots, A_n\}$ of $\mathcal{A}'$, such that $A'_1 = \cup_{j=0}^{i_1}A_j, A'_1 \cup A'_2 = \cup_{j=0}^{i_2}A_j, \ldots, \cup_{l=1}^{m}A'_l = \cup_{j=0}^{i_m}A_j, B = \cup_{j=0}^{i_{m+1}}A_j, B' = \cup_{j=0}^{i_{m+2}}A_j$ for some $i_1 \leq i_2 \leq \ldots \leq i_{m+2}$. Using (3), it follows that $\sigma(X_{A'_1}, X_{A'_1\cup A'_2}, \ldots, X_{\cup_{l=1}^{m}A'_l}) \perp \sigma(X_{B'}) | \sigma(X_B)$. $\square$

4 Construction of the Process

In this section we will introduce a special class of set-Markov processes for which the mechanism of transition from one state to another is completely known, and we will construct such a process. In light of Proposition 2, we will also determine the necessary and sufficient conditions that have to be imposed on a family of one-dimensional transition systems, indexed by a collection of simple flows, such that on each simple flow f from the chosen collection, the corresponding Markov process has the law of X^f, where X is set-Markov (i.e., under what circumstances a class of one dimensional transition systems determines a set-Markov process).

Let $\mathcal{B}(\mathbf{R})$ denote the Borel subsets of $\mathbf{R}$. We begin with the definition of the transition system.

Definition 4.

(a) *For each $B, B' \in \mathcal{A}(u), B \subseteq B'$ let $Q_{BB'}(x;\Gamma), x \in \mathbf{R}, \Gamma \in \mathcal{B}(\mathbf{R})$, be a transition probability on $\mathbf{R}$ i.e., $Q_{BB'}(x;\cdot)$ is a probability measure $\forall x$, and $Q_{BB'}(\cdot;\Gamma)$ is measurable $\forall \Gamma$. The family $\mathcal{Q} := (Q_{BB'})$ of all these transition probabilities is called a* **transition system** *if $\forall B \in \mathcal{A}(u), Q_{BB}(x;\cdot) = \delta_x$ and $\forall B, B', B'' \in \mathcal{A}(u), B \subseteq B' \subseteq B''$*

$$Q_{BB''}(x;\Gamma) = \int_{\mathbf{R}} Q_{B'B''}(y;\Gamma) Q_{BB'}(x;dy) \ \forall x \in \mathbf{R}, \forall \Gamma \in \mathcal{B}(\mathbf{R})$$

(b) *Let $\mathcal{Q} := (Q_{BB'})_{B,B' \in \mathcal{A}(u); B \subseteq B'}$ be a transition system. A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ with a unique additive extension to $\mathcal{C}(u)$ is called* $\mathcal{Q}$-**Markov** *if $\forall B, B' \in \mathcal{A}(u), B \subseteq B'$*

$$P[X_{B'} \in \Gamma | \mathcal{F}_B] = Q_{BB'}(X_B;\Gamma) \ \forall \Gamma \in \mathcal{B}(\mathbf{R})$$

where $(\mathcal{F}_A)_{A \in \mathcal{A}}$ is the minimal filtration of the process X.

In other words, a $\mathcal{Q}$-Markov process is a set-Markov process for which $Q_{BB'}$ is a version of the conditional distribution of $X_{B'}$ given X_B, for every $B, B' \in \mathcal{A}(u), B \subseteq B'$.

Examples 3.

1. Any process $X := (X_A)_{A \in \mathcal{A}}$ with independent increments is $\mathcal{Q}$-Markov with $Q_{BB'}(x;\Gamma) := F_{B' \setminus B}(\Gamma - x)$, where F_C is the distribution of $X_C, C \in \mathcal{C}(u)$. The Poisson process and the Brownian motion are the particular cases for which F_C is a Poisson distribution with mean Λ_C, respectively, a normal distribution with mean 0 and variance Λ_C.

2. The empirical process of size n, corresponding to a probability measure F, is $\mathcal{Q}$-Markov with $Q_{BB'}(\frac{k}{n};\{\frac{m}{n}\}); k, m \in \{0,1,\ldots,n\}, k \le m$ given by the value at $m-k$ of the binomial distribution with $n-k$ trials and $\frac{F(B' \setminus B)}{1-F(B)}$ probability of success.

3. The Dirichlet process with parameter measure α is $\mathcal{Q}$-Markov with $Q_{BB'}(x;[0,z]); x, z \in [0,1]$ given by the value at $\frac{y-x}{1-x}$ of the Beta distribution with parameters $(\alpha(B' \setminus B); \alpha(B'^c))$.

The following result gives equivalent characterizations of a $\mathcal{Q}$-Markov process.

Proposition 5. *Let $\mathcal{Q} := (Q_{BB'})_{B,B' \in \mathcal{A}(u); B \subseteq B'}$ be a transition system, $X := (X_A)_{A \in \mathcal{A}}$ a set-indexed process with a unique additive extension to $\mathcal{C}(u)$ and initial distribution μ, and $(\mathcal{F}_A)_{A \in \mathcal{A}}$ the minimal filtration of the process X. The following statements are equivalent:*

(a) *The process X is $\mathcal{Q}$-Markov.*

(b) *For every simple flow $f : [0,a] \to \mathcal{A}(u)$ the process $X^f := (X_{f(t)})_{t \in [0,a]}$ is $\mathcal{Q}^f$-Markov (with respect to $(\mathcal{F}_{f(t)})_{t \in [0,a]}$), where $Q^f_{st} := Q_{f(s),f(t)}$.*

(c) *For every finite sub-semilattice $\mathcal{A}'$, for every consistent ordering $\{A_0 = \emptyset', A_1, \ldots, A_n\}$ of $\mathcal{A}'$ and for every $i = 1, \ldots, n-1$*

$$\sigma(X_{A_0}, X_{A_1}, X_{A_1 \cup A_2}, \ldots, X_{\cup_{j=1}^{i-1} A_j}) \perp \sigma(X_{\cup_{j=1}^{i+1} A_j}) \mid \sigma(X_{\cup_{j=1}^{i} A_j})$$

and $Q_{\cup_{j=1}^{i} Aj, \cup_{j=1}^{i+1} A_j}$ is a version of the conditional distribution of $X_{\cup_{j=1}^{i+1} A_j}$ given $X_{\cup_{j=1}^{i} A_j}$.

(d) *For every finite sub-semilattice $\mathcal{A}'$ and for every consistent ordering $\{A_0 = \emptyset', A_1, \ldots, A_n\}$ of $\mathcal{A}'$*

$$P(X_{A_0} \in \Gamma_0, X_{A_1} \in \Gamma_1, X_{A_1 \cup A_2} \in \Gamma_2, \ldots, X_{\cup_{j=1}^{n} A_j} \in \Gamma_n) =$$

$$\int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) \prod_{i=1}^{n} I_{\Gamma_i}(x_i) Q_{\cup_{j=1}^{n-1} A_j, \cup_{j=1}^{n} A_j}(x_{n-1}; dx_n) \ldots$$

$$Q_{A_1, A_1 \cup A_2}(x_1; dx_2) Q_{\emptyset' A_1}(x_0; dx_1) \mu(dx_0)$$

for every $\Gamma_0, \Gamma_1, \ldots, \Gamma_n \in \mathcal{B}(\mathbf{R})$.

(e) *For every finite sub-semilattice $\mathcal{A}'$, for every consistent ordering $\{A_0 = \emptyset', A_1, \ldots, A_n\}$ of $\mathcal{A}'$, if we denote with C_i the left neighbourhood of the set A_i, then*

$$P(X_{C_0} \in \Gamma_0, X_{C_1} \in \Gamma_1, X_{C_2} \in \Gamma_2, \ldots, X_{C_n} \in \Gamma_n) =$$

$$\int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_1) \prod_{i=2}^{n} I_{\Gamma_i}(x_i - x_{i-1}) Q_{\cup_{j=1}^{n-1} A_j, \cup_{j=1}^{n} A_j}(x_{n-1}; dx_n) \ldots$$

$$Q_{A_1, A_1 \cup A_2}(x_1; dx_2) Q_{\emptyset' A_1}(x_0; dx_1) \mu(dx_0)$$

for every $\Gamma_0, \Gamma_1, \ldots, \Gamma_n \in \mathcal{B}(\mathbf{R})$.

Proof: The equivalences (a)-(b), (a)-(c) follow by arguments similar to those used to prove Proposition 2, respectively, Proposition 4. The equivalence (c)-(d) follows exactly as in the classical case. Finally, the equivalence (d)-(e) follows by a change of variables, since $X_{C_i} = X_{\cup_{j=1}^{i} A_j} - X_{\cup_{j=1}^{i-1} A_j}$ a.s. □

The general construction of a $\mathcal{Q}$-Markov process will be made using increments: i.e., the sets in $\mathcal{C}$. The following assumption is necessary. It requires that the distribution of the process over the left-neighbourhoods $C_0, C_1, \ldots, C_n$ of a finite sub-semilattice, does not depend on the consistent ordering of the semilattice.

Assumption 1. *If $\{A_0 = \emptyset', A_1, \ldots, A_n\}$ and $\{A_0 = \emptyset', A'_1, \ldots, A'_n\}$ are two consistent orderings of the same finite sub-semilattice $\mathcal{A}'$ and π is the permutation of $\{1, \ldots, n\}$ with $\pi(1) = 1$ such that $A_i = A'_{\pi(i)}$ $\forall i$, then*

$$\int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_1) \prod_{i=2}^{n} I_{\Gamma_i}(x_i - x_{i-1}) Q_{\cup_{j=1}^{n-1} A_j, \cup_{j=1}^{n} A_j}(x_{n-1}; dx_n) \ldots$$
$$Q_{A_1, A_1 \cup A_2}(x_1; dx_2) Q_{\emptyset' A_1}(x_0; dx_1) \mu(dx_0) =$$
$$\int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(y_0) I_{\Gamma_1}(y_1) \prod_{i=2}^{n} I_{\Gamma_i}(y_{\pi(i)} - y_{\pi(i)-1}) Q_{\cup_{j=1}^{n-1} A'_j, \cup_{j=1}^{n} A'_j}(y_{n-1}; dy_n) \ldots$$
$$Q_{A'_1, A'_1 \cup A'_2}(y_1; dy_2) Q_{\emptyset' A'_1}(y_0; dy_1) \mu(dy_0)$$

for every $\Gamma_0, \Gamma_1, \ldots, \Gamma_n \in \mathcal{B}(\mathbf{R})$.

The finite dimensional distributions of a $\mathcal{Q}$-Markov process over the sets in $\mathcal{C}$ have to be defined so that they ensure the (almost sure) additivity of the process. The next result gives the definition of the finite dimensional distribution (of an additive $\mathcal{Q}$-Markov process) over an arbitrary k-tuple of sets in $\mathcal{C}$ and shows that, if the transition system $\mathcal{Q}$ satisfies Assumption 1, then the definition will not depend on the extremal representations of these sets.

Lemma 5. *Let $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ be a transition system satisfying Assumption 1. Let $(C_1, \ldots, C_k)$ be a k-tuple of distinct sets in $\mathcal{C}$ and suppose that each set C_i admits two extremal representations $C_i = A_i \setminus \cup_{j=1}^{n_i} A_{ij} = A'_i \setminus \cup_{j=1}^{m_i} A'_{ij}$. Let $\mathcal{A}', \mathcal{A}''$ be the minimal finite sub-semilattices of $\mathcal{A}$ which contain the sets A_i, A_{ij}, respectively A'_i, A'_{ij}, $\{B_0 = \emptyset', B_1, \ldots, B_n\}, \{B'_0 = \emptyset', B'_1, \ldots, B'_m\}$ two consistent orderings of $\mathcal{A}', \mathcal{A}''$ and D_j, D'_l the left neighbourhoods of the sets B_j, B'_l for $j = 1, \ldots, n; l = 1, \ldots, m$. If each set $C_i; i = 1, \ldots, k$ can be written as $C_i = \cup_{j \in J_i} D_j = \cup_{l \in L_i} D'_l$ for some $J_i \subseteq \{1, \ldots, n\}, L_i \subseteq \{1, \ldots, m\}$, then*

$$\int_{\mathbf{R}^{n+1}} \prod_{i=1}^{k} I_{\Gamma_i}\Big(\sum_{j \in J_i} (x_j - x_{j-1})\Big) Q_{\cup_{j=1}^{n-1} B_j, \cup_{j=1}^{n} B_j}(x_{n-1}; dx_n) \ldots$$
$$Q_{B_1, B_1 \cup B_2}(x_1; dx_2) Q_{\emptyset' B_1}(x; dx_1) \mu(dx) =$$
$$\int_{\mathbf{R}^{m+1}} \prod_{i=1}^{k} I_{\Gamma_i}\Big(\sum_{l \in L_i} (y_l - y_{l-1})\Big) Q_{\cup_{l=1}^{m-1} B'_l, \cup_{l=1}^{m} B'_l}(y_{m-1}; dy_m) \ldots$$

$$Q_{B'_1, B'_1 \cup B'_2}(y_1; dy_2) Q_{\emptyset' B'_1}(y; dy_1) \mu(dy)$$

for every $\Gamma_1, \ldots, \Gamma_k \in \mathcal{B}(\mathbf{R})$, *with the convention* $x_0 = y_0 = 0$.

Proof: Let $\tilde{\mathcal{A}}$ be the minimal finite sub-semilattice of $\mathcal{A}$ determined by the sets in $\mathcal{A}'$ and $\mathcal{A}''$; clearly $\mathcal{A}' \subseteq \tilde{\mathcal{A}}, \mathcal{A}'' \subseteq \tilde{\mathcal{A}}$. Let $\mathrm{ord}^1 = \{E_0 = \emptyset', E_1, \ldots, E_N\}$ and $\mathrm{ord}^2 = \{E'_0 = \emptyset', E'_1, \ldots, E'_N\}$ be two consistent orderings of $\tilde{\mathcal{A}}$ such that, if $B_j = E_{i_j}; j = 1, \ldots, n$ and $B'_l = E'_{k_l}; l = 1, \ldots, m$ for some indices $i_1 < i_2 < \ldots < i_n$, respectively $k_1 < k_2 < \ldots < k_m$, then

$$B_1 = \cup_{p=1}^{i_1} E_p, \; B_1 \cup B_2 = \cup_{p=1}^{i_2} E_p, \ldots, \cup_{j=1}^{n} B_j = \cup_{p=1}^{i_n} E_p$$

$$B'_1 = \cup_{q=1}^{k_1} E'_q, \; B'_1 \cup B'_2 = \cup_{q=1}^{k_2} E'_q, \ldots, \cup_{l=1}^{m} B'_l = \cup_{q=1}^{k_m} E'_q$$

Let π be the permutation of $\{1, \ldots, N\}$ such that $E_p = E'_{\pi(p)}; p = 1, \ldots, N$. Denote by H_p, H'_q the left neighbourhoods of E_p, E'_q with respect to the orderings $\mathrm{ord}^1, \mathrm{ord}^2$; clearly $H_p = H'_{\pi(p)}$, $\forall p = 1, \ldots, N$. Note that $D_j = (\cup_{v=1}^{j} B_v) \backslash (\cup_{v=1}^{j-1} B_v) = (\cup_{p=1}^{i_j} E_p) \backslash (\cup_{p=1}^{i_{j-1}} E_p) = \cup_{p=i_{j-1}+1}^{i_j} H_p; j = 1, \ldots, n$ and similarly $D'_l = \cup_{q=k_{l-1}+1}^{k_l} H'_q; l = 1, \ldots, m$. Hence

$$\begin{aligned} C_i &= \cup_{j \in J_i} D_j = \cup_{j \in J_i} \cup_{p=i_{j-1}+1}^{i_j} H_p = \cup_{j \in J_i} \cup_{p=i_{j-1}+1}^{i_j} H'_{\pi(p)} \\ &= \cup_{l \in L_i} D'_l = \cup_{l \in L_i} \cup_{q=k_{l-1}+1}^{k_l} H'_q \end{aligned}$$

and we can conclude that

$$\{\pi(p); p \in \cup_{j \in J_i} \{i_{j-1}+1, i_{j-1}+2, \ldots, i_j\}\} = \cup_{l \in L_i} \{k_{l-1}+1, k_{l-1}+2, \ldots, k_l\}$$

This implies that

$$\int_{\mathbf{R}^{N+1}} \prod_{i=1}^{k} I_{\Gamma_i} \Big(\sum_{j \in J_i} \sum_{p=i_{j-1}+1}^{i_j} (x_p - x_{p-1})\Big) Q_{\cup_{p=1}^{N-1} E_p, \cup_{p=1}^{N} E_p}(x_{N-1}; dx_N) \ldots$$

$$Q_{E_1, E_1 \cup E_2}(x_1; dx_2) Q_{\emptyset' E_1}(x; dx_1) \mu(dx)$$

$$= \int_{\mathbf{R}^{N+1}} \prod_{i=1}^{k} I_{\Gamma_i} \Big(\sum_{l \in L_i} \sum_{q=k_{l-1}+1}^{k_l} (y_q - y_{q-1})\Big) Q_{\cup_{q=1}^{N-1} E'_q, \cup_{q=1}^{N} E'_q}(y_{N-1}; dy_N) \ldots$$

$$Q_{E'_1, E'_1 \cup E'_2}(y_1; dy_2) Q_{\emptyset' E'_1}(y; dy_1) \mu(dy)$$

with the convention $x_0 = y_0 = 0$. This gives us the desired relationship, because $\sum_{p=i_{j-1}+1}^{i_j} (x_p - x_{p-1}) = x_{i_j} - x_{i_{j-1}}$, the left-hand side integral collapses to an integral with respect to $Q_{\cup_{j=1}^{n} B_j, \cup_{j=1}^{n-1} B_j}(x_{i_{n-1}}; dx_{i_n}) \ldots Q_{B_1, B_1 \cup B_2}(x_{i_1}; dx_{i_2})$ $Q_{\emptyset' B_1}(x; dx_{i_1}) \mu(dx)$, and a similar phenomenon happens in the right-hand side. □

We are now ready to prove the main result of this section, which says that in fact the previous assumption is also sufficient to construct a $\mathcal{Q}$-Markov process.

Theorem 1. *If μ is a probability measure on $\mathbf{R}$ and $\mathcal{Q} := (Q_{BB'})_{B\subseteq B'}$ is a transition system which satisfies the consistency Assumption 1, then there exists a $\mathcal{Q}$-Markov process $X := (X_A)_{A\in\mathcal{A}}$ with initial distribution μ.*

Proof: Let $(\mathbf{R}^{\mathcal{C}}, \mathcal{R}^{\mathcal{C}}) := \prod_{C\in\mathcal{C}}(R_C, \mathcal{R}_C)$ where $(R_C, \mathcal{R}_C), C \in \mathcal{C}$ represent $\mathcal{C}$ copies of the real space $\mathbf{R}$ with its Borel subsets. For each k-tuple $(C_1, \ldots, C_k)$ of distinct sets in $\mathcal{C}$ we will define a probability measure $\mu_{C_1\ldots C_k}$ on $(\mathbf{R}^k, \mathcal{R}^k)$ such that the system of all these probability measures satisfy the following two consistency conditions:

(C1) If $(C'_1 \ldots C'_k)$ is another ordering of the k-tuple $(C_1 \ldots C_k)$, say $C_i = C'_{\pi(i)}$, π is a permutation of $\{1, \ldots, k\}$, then

$$\mu_{C_1\ldots C_k}(\Gamma_1 \times \ldots \times \Gamma_k) = \mu_{C'_1\ldots C'_k}(\Gamma_{\pi^{-1}(1)} \times \ldots \times \Gamma_{\pi^{-1}(k)})$$

for every $\Gamma_1, \ldots, \Gamma_k \in \mathcal{B}(\mathbf{R})$.

(C2) If $C_1, \ldots, C_k, C_{k+1}$ are $k+1$ distinct sets in $\mathcal{C}$, then

$$\mu_{C_1\ldots C_k}(\Gamma_1 \times \ldots \times \Gamma_k) = \mu_{C_1\ldots C_k C_{k+1}}(\Gamma_1 \times \ldots \times \Gamma_k \times \mathbf{R})$$

for every $\Gamma_1, \ldots, \Gamma_k \in \mathcal{B}(\mathbf{R})$.

By Kolmogorov's extension theorem there exists a probability measure P on $(\mathbf{R}^{\mathcal{C}}, \mathcal{R}^{\mathcal{C}})$ such that the coordinate-variable process $X := (X_C)_{C\in\mathcal{C}}$ defined by $X_C(x) := x_C$ has the measures $\mu_{C_1\ldots C_k}$ as its finite dimensional distributions. We will prove that the process X has an (almost surely) unique additive extension to $\mathcal{C}(u)$. The $\mathcal{Q}$-Markov property of this process will follow from the way we choose its finite dimensional distributions $\mu_{C_0C_1\ldots C_n}$ over the left neighbourhoods of a finite sub-semilattice, according to Proposition 5, (e).

Step 1 (Construction of the finite dimensional distributions)
We define $\mu_\emptyset := \delta_0$. Let $(C_1, \ldots, C_k)$ be a k-tuple of distinct nonempty sets in $\mathcal{C}$ and $C_i = A_i \setminus \cup_{j=1}^{n_i} A_{ij}; i = 1, \ldots, k$ some extremal representations. Let $\mathcal{A}'$ be the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets A_i, A_{ij}, $\{B_0 = \emptyset', B_1, \ldots, B_n\}$ a consistent ordering of $\mathcal{A}'$ and D_j the left neighbourhood of the set B_j for $j = 1, \ldots, n$. Define

$$\mu_{D_0D_1\ldots D_n}(\Gamma_0 \times \Gamma_1 \times \ldots \times \Gamma_n) := \int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_1) \prod_{i=2}^{n} I_{\Gamma_i}(x_i - x_{i-1})$$

$$Q_{\cup_{j=1}^{n-1} B_j, \cup_{j=1}^{n} B_j}(x_{n-1}; dx_n) \ldots Q_{B_1, B_1\cup B_2}(x_1; dx_2) Q_{\emptyset' B_1}(x_0; dx_1)\mu(dx_0)$$

for every $\Gamma_0, \Gamma_1, \ldots, \Gamma_n \in \mathcal{B}(\mathbf{R})$.

Say $C_i = \cup_{j \in J_i} D_j$ for some $J_i \subseteq \{1, \dots, n\}; i = 1, \dots, k$, let $\alpha : \mathbf{R}^{n+1} \to \mathbf{R}^k, \alpha(x_0, x_1, \dots, x_n) := (\sum_{j \in J_1} x_j, \dots, \sum_{j \in J_k} x_j)$ and define

$$\mu_{C_1 \dots C_k} := \mu_{D_0 D_1 \dots D_n} \circ \alpha^{-1}$$

The fact that $\mu_{C_1 \dots C_k}$ does not depend on the ordering of the semilattice $\mathcal{A}'$ is a consequence of Assumption 1. The fact that $\mu_{C_1 \dots C_k}$ does not depend on the extremal representations of the sets C_i is also a consequence of Assumption 1, using Lemma 5. Finally, we observe that the finite dimensional distributions are additive.

Step 2 (Consistency condition (C1))
Let $(C'_1 \dots C'_k)$ be another ordering of the k-tuple $(C_1 \dots C_k)$, with $C_i = C'_{\pi(i)}$, π being a permutation of $\{1, \dots, k\}$. Let $C_i = A_i \setminus \cup_{j=1}^{n_i} A_{ij}; i = 1, \dots, k$ be extremal representations, $\mathcal{A}'$ the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets A_i, A_{ij}, $\{B_0 = \emptyset', B_1, \dots, B_n\}$ a consistent ordering of $\mathcal{A}'$ and D_j the left neighbourhood of the set B_j for each $j = 1, \dots, n$. Say $C_i = \cup_{j \in J_i} D_j, C'_i = \cup_{j \in J'_i} D_j$ and let $\alpha(x_0, x_1, \dots, x_n) := (\sum_{j \in J_1} x_j, \dots, \sum_{j \in J_k} x_j)$, $\beta(x_0, x_1, \dots, x_n) := (\sum_{j \in J'_1} x_j, \dots, \sum_{j \in J'_k} x_j)$. We have $J_i = J'_{\pi(i)}$. Hence

$$\begin{aligned} \mu_{C_1 \dots C_k}(\Gamma_1 \times \dots \times \Gamma_k) &= \mu_{D_0 D_1 \dots D_n}(\alpha^{-1}(\Gamma_1 \times \dots \times \Gamma_k)) \\ &= \mu_{D_0 D_1 \dots D_n}(\beta^{-1}(\Gamma_{\pi^{-1}(1)} \times \dots \times \Gamma_{\pi^{-1}(k)})) \\ &= \mu_{C'_1 \dots C'_k}(\Gamma_{\pi^{-1}(1)} \times \dots \times \Gamma_{\pi^{-1}(k)}) \end{aligned}$$

for every $\Gamma_1, \dots, \Gamma_k \in \mathcal{B}(\mathbf{R})$.

Step 3 (Consistency Condition (C2))
Let $C_1, \dots, C_k, C_{k+1}$ be $k+1$ distinct sets in $\mathcal{C}$ and $C_i = A_i \setminus \cup_{j=1}^{n_i} A_{ij}; i = 1, \dots, k+1$ some extremal representations. Let $\mathcal{A}', \mathcal{A}''$ be the minimal finite sub-semilattices of $\mathcal{A}$ which contain the sets $A_i, A_{ij}; i = 1, \dots, k; j = 1, \dots, n_i$, respectively $A_i, A_{ij}; i = 1, \dots, k+1; j = 1, \dots, n_i$. Clearly $\mathcal{A}' \subseteq \mathcal{A}''$. Using Comment 1.2 there exists a consistent ordering $\{B_0 = \emptyset', B_1, \dots, B_n\}$ of $\mathcal{A}''$ such that, if $\mathcal{A}' = \{B_{i_0} = \emptyset', B_{i_1}, \dots, B_{i_m}\}$ with $0 = i_0 < i_1 < \dots < i_m$, then $\cup_{s=1}^{l} B_{i_s} = \cup_{j=1}^{i_l} B_j$ for all $l = 1, \dots, m$. For each $j = 1, \dots, n; l = 1, \dots, m$, let D_j, E_l be the left neighbourhoods of B_j in $\mathcal{A}''$, respectively of B_{i_l} in $\mathcal{A}'$ and note that $E_l = \cup_{j=i_{l-1}+1}^{i_l} D_j$ for each $l = 1, \dots, m$. Let $\gamma(x_0, x_1, \dots, x_n) := (x_0, \sum_{j=1}^{i_1} x_j, \sum_{j=i_1+1}^{i_2} x_j, \dots, \sum_{j=i_{m-1}+1}^{i_m} x_j)$ and for the moment suppose that

$$\mu_{E_0 E_1 \dots E_m} = \mu_{D_0 D_1 \dots D_n} \circ \gamma^{-1} \tag{4}$$

Say $C_i = \cup_{j \in J_i} D_j; i = 1, \dots, k+1$ for some $J_i \subseteq \{1, \dots, n\}$ and define $\alpha(x_0, x_1, \dots, x_n) := (\sum_{j \in J_1} x_j, \dots, \sum_{j \in J_{k+1}} x_j)$. On the other hand, if we say that for each $i = 1, \dots, k$ we have $C_i = \cup_{l \in L_i} E_l$ for some $L_i \subseteq \{1, \dots, m\}$, then $J_i = \cup_{l \in L_i} \{i_{l-1} + 1, i_{l-1} + 2, \dots, i_l\}$; define $\beta(y_0, y_1, \dots, y_m) := (\sum_{l \in L_1} y_l, \dots, \sum_{l \in L_k} y_l)$. Then

$$\mu_{C_1 \dots C_k}(\Gamma_1 \times \dots \times \Gamma_k) = \mu_{E_0 E_1 \dots E_m}(\beta^{-1}(\Gamma_1 \times \dots \times \Gamma_k))$$

$$\begin{aligned} &= \mu_{D_0D_1\ldots D_n}(\gamma^{-1}(\beta^{-1}(\Gamma_1\times\ldots\times\Gamma_k))) \\ &= \mu_{D_0D_1\ldots D_n}(\alpha^{-1}(\Gamma_1\times\ldots\times\Gamma_k\times\mathbf{R})) \\ &= \mu_{C_1\ldots C_kC_{k+1}}(\Gamma_1\times\ldots\times\Gamma_k\times\mathbf{R}) \end{aligned}$$

for every $\Gamma_1,\ldots,\Gamma_k\in\mathcal{B}(\mathbf{R})$.

In order to prove (4) let $\Gamma_0,\Gamma_1,\ldots,\Gamma_m\in\mathcal{B}(\mathbf{R})$ be arbitrary. Then

$$\mu_{D_0D_1\ldots D_n}(\gamma^{-1}(\Gamma_0\times\Gamma_1\times\ldots\times\Gamma_m)) =$$

$$\int_{\mathbf{R}^{n+1}} I_{\Gamma_0\times\Gamma_1\times\ldots\times\Gamma_m}(\gamma(x_0,x_1,x_2-x_1,\ldots,x_n-x_{n-1}))$$

$$Q_{\cup_{j=1}^{n-1}B_j,\cup_{j=1}^{n}B_j}(x_{n-1};dx_n)\ldots Q_{B_1,B_1\cup B_2}(x_1;dx_2)Q_{\emptyset' B_1}(x_0;dx_1)\mu(dx_0)$$

Note that $\gamma(x_0,x_1,x_2-x_1,\ldots,x_n-x_{n-1})=(x_0,x_{i_1},x_{i_2}-x_{i_1},\ldots,x_{i_m}-x_{i_{m-1}})$ and hence the integrand does not depend on the variables $x_j, j\notin\{i_0,i_1,\ldots,i_m\}$. By the definition of the transition system, the above integral collapses to

$$\int_{\mathbf{R}^{m+1}} I_{\Gamma_0}(x_0)I_{\Gamma_1}(x_{i_1})\prod_{l=2}^{m} I_{\Gamma_l}(x_{i_l}-x_{i_{l-1}})Q_{\cup_{j=1}^{i_{m-1}}B_j,\cup_{j=1}^{i_m}B_j}(x_{i_{m-1}};dx_{i_m})\ldots$$

$$Q_{\cup_{j=1}^{i_1}B_j,\cup_{j=1}^{i_2}B_j}(x_{i_1};dx_{i_2})Q_{\emptyset',\cup_{j=1}^{i_1}B_j}(x_0;dx_{i_1})\mu(dx_0)$$

which is exactly the definition of $\mu_{E_0E_1\ldots E_m}(\Gamma_0\times\Gamma_1\times\ldots\times\Gamma_m)$ because $\cup_{j=1}^{i_l}B_j=\cup_{s=1}^{l}B_{i_s}$ for every $l=1,\ldots,m$. This concludes the proof of (4).

Step 4 (Almost Sure Additivity of the Canonical Process)
We will show that the canonical process X on the space $(\mathbf{R}^{\mathcal{C}},\mathcal{R}^{\mathcal{C}})$ has an (almost surely) unique additive extension to $\mathcal{C}(u)$ (with respect to the probability measure P given by Kolmogorov's extension theorem). Let $C,C_1,\ldots,C_k\in\mathcal{C}$ be such that $C=\cup_{i=1}^{k}C_i$, and suppose that $C_i=A_i\setminus\cup_{j=1}^{n_i}A_{ij}; i=1,\ldots,k$ are extremal representations. Let $\mathcal{A}'$ be the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets A_i,A_{ij}, $\{B_0=\emptyset',B_1,\ldots,B_n\}$ a consistent ordering of $\mathcal{A}'$ and D_j the left neighbourhood of B_j. Assume that each $C_i=\cup_{j\in J_i}D_j$ for some $J_i\subseteq\{1,\ldots,n\}$. Because the finite dimensional distributions of X were chosen in an additive way, we have $X_{C_i}=\sum_{j\in J_i}X_{D_j}$ a.s., $X_{C_{i_1}\cap C_{i_2}}=\sum_{j\in I_{i_1}\cap I_{i_2}}X_{D_j}$ a.s., ... , $X_{\cap_{i=1}^{k}C_i}=\sum_{j\in\cap_{i=1}^{k}I_i}X_{D_j}$ a.s., $X_C=\sum_{j\in\cup_{i=1}^{k}J_i}X_{D_j}$ a.s. Hence $\sum_{i=1}^{k}X_{C_i}-\sum_{i_1<i_2}X_{C_{i_1}\cap C_{i_2}}+\ldots+(-1)^kX_{\cap_{i=1}^{k}C_i}=\sum_{j\in\cup_{i=1}^{k}J_i}X_{D_j}=X_C$ a.s.
□

Finally we will translate the preceding result in terms of flows. Let μ be an arbitrary probability measure on $\mathbf{R}$.

For each finite sub-semilattice $\mathcal{A}'$ and for each consistent ordering ord$=\{A_0=\emptyset',A_1,\ldots,A_n\}$ of $\mathcal{A}'$ pick one simple flow $f:=f_{\mathcal{A}',\mathrm{ord}}$ which connects the sets of $\mathcal{A}'$ in the sense of the ordering ord. Let $\mathcal{S}$ be the collection of all the simple flows $f_{\mathcal{A}',\mathrm{ord}}$ and $\{\mathcal{Q}^f:=(Q_{st}^f)_{s<t}; f\in\mathcal{S}\}$ a collection of one-dimensional transition systems indexed by $\mathcal{S}$.

The next assumption will provide a necessary and sufficient condition which will allow us to reconstruct a set-indexed transition system $\mathcal{Q}$ from a class of one-dimensional transition systems $\{Q^f\}$. It requires that whenever we have two simple flows $f, g \in \mathcal{S}$ such that $f(s) = g(u), f(t) = g(v)$ for some $s < t, u < v$, we must have $Q^f_{st} = Q^g_{uv}$.

Assumption 2. *If ord1*$= \{A_0 = \emptyset', A_1, \ldots, A_n\}$ *and ord2*$= \{A_0 = \emptyset', A'_1, \ldots, A'_m\}$ *are two consistent orderings of some finite semilattices* $\mathcal{A}', \mathcal{A}''$ *such that* $\cup_{j=1}^n A_j = \cup_{j=1}^m A'_j$, $\cup_{j=1}^k A_j = \cup_{j=1}^l A'_j$ *for some* $k < n, l < m$, *and we denote* $f := f_{\mathcal{A}',ord1}$, $g := f_{\mathcal{A}'',ord2}$ *with* $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$, *then*

$$Q^f_{0t_1} = Q^g_{0u_1} \text{ and } Q^f_{t_k t_n} = Q^g_{u_l u_m}$$

The following assumption is easily seen to be equivalent to Assumption 1.

Assumption 3. *If ord*$1 = \{A_0 = \emptyset', A_1, \ldots, A_n\}$ *and ord*$2 = \{A_0 = \emptyset', A'_1, \ldots, A'_n\}$ *are two consistent orderings of the same finite semilattice* $\mathcal{A}'$ *with* $A_i = A'_{\pi(i)}$, *where* π *is a permutation of* $\{1, \ldots, n\}$ *with* $\pi(1) = 1$, *and we denote* $f := f_{\mathcal{A}',ord1}$, $g := f_{\mathcal{A}',ord2}$ *with* $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$, *then*

$$\int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_1) \prod_{i=2}^n I_{\Gamma_i}(x_i - x_{i-1}) Q^f_{t_{n-1}t_n}(x_{n-1}; dx_n) \ldots \tag{5}$$

$$Q^f_{t_1 t_2}(x_1; dx_2) Q^f_{0t_1}(x_0; dx_1) \mu(dx_0) =$$

$$\int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(y_0) I_{\Gamma_1}(y_1) \prod_{i=2}^n I_{\Gamma_i}(y_{\pi(i)} - y_{\pi(i)-1}) Q^g_{u_{n-1}u_n}(y_{n-1}; dy_n) \ldots$$

$$Q^g_{u_1 u_2}(y_1; dy_2) Q^g_{0u_1}(y_0; dy_1) \mu(dy_0)$$

for every $\Gamma_0, \Gamma_1, \ldots, \Gamma_n \in \mathcal{B}(\mathbf{R})$.

The following result is immediate.

Corollary 1. *If* μ *is a probability measure on* $\mathbf{R}$ *and* $\{\mathcal{Q}^f := (Q^f_{st})_{s<t}; f \in \mathcal{S}\}$ *is a collection of one-dimensional transition systems which satisfies the matching Assumption 2 and the consistency Assumption 3, then there exist a set-indexed transition system* $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ *and a* $\mathcal{Q}$*-Markov process* $X := (X_A)_{A \in \mathcal{A}}$ *with initial distribution* μ, *such that* $\forall f \in \mathcal{S}$, $X^f := (X_{f(t)})_t$ *is* $\mathcal{Q}^f$*-Markov.*

Proof: Let $B, B' \in \mathcal{A}(u)$ be such that $B \subseteq B'$. Let $B = \cup_{j=1}^m A'_j, B' = \cup_{l=1}^p A''_l$ be some extremal representations, $\mathcal{A}'$ the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets A'_j, A''_l and ord=$\{A_0 = \emptyset', A_1, \ldots, A_n\}$ a consistent ordering of $\mathcal{A}'$ such that $B = \cup_{j=1}^k A_j$ and $B' = \cup_{j=1}^n A_j$. Denote $f := f_{\mathcal{A}',\text{ord}}$ with $f(t_i) = \cup_{j=1}^i A_j$. We define $Q_{BB'} := Q^f_{t_k t_n}$.

The definition of $Q_{BB'}$ does not depend on the extremal representations of B, B', because of Assumption 2.

Finally it is not hard to see that the family $\mathcal{Q} := (Q_{BB'})_{B\subseteq B'}$ is a transition system which satisfies Assumption 1. □

5 The Generator

In this section we will make use of flows to introduce the generator of a $\mathcal{Q}$-Markov process. Corollary 1 allows us to characterize a $\mathcal{Q}$-Markov process by a class $\{\mathcal{Q}^f; f \in \mathcal{S}\}$ of one-dimensional transition systems. These transition sytems in turn are characterized by their corresponding generators. This observation permits us to define a generator for a $\mathcal{Q}$-Markov process as a class of generators indexed by a suitable class of flows. The generator completely characterizes the distribution of a $\mathcal{Q}$-Markov process. We will also determine necessary and sufficient conditions that will ensure that a family of one-dimensional generators, indexed by a collection of simple flows, is the generator of a $\mathcal{Q}$-Markov process.

Let $B(\mathbf{R})$ be the Banach space of all bounded measurable functions $h : \mathbf{R} \to \mathbf{R}$ with the supremum norm.

Let $\mathcal{Q} := (Q_{BB'})_{B\subseteq B'}$ be a transition system and $X := (X_A)_{A\in\mathcal{A}}$ a $\mathcal{Q}$-Markov process with initial distribution μ. For each finite sub-semilattice $\mathcal{A}'$ and for each consistent ordering ord= $\{A_0 = \emptyset', A_1, \ldots, A_n\}$ of $\mathcal{A}'$ pick one simple flow $f := f_{\mathcal{A}',\mathrm{ord}}$ which connects the sets of $\mathcal{A}'$ in the sense of the ordering ord. Let $\mathcal{S}$ be the collection of all the simple flows $f_{\mathcal{A}',\mathrm{ord}}$.

For each $f \in \mathcal{S}$, let $\mathcal{T}^f := (T^f_{st})_{s<t}$ be the semigroup associated to the transition system $\mathcal{Q}^f$ and $\mathcal{G}^f_s, \mathcal{G}^{*f}_s$ the backward, respectively the forward generator of the process X^f at time s, with domains $\mathcal{D}(\mathcal{G}^f_s), \mathcal{D}(\mathcal{G}^{*f}_s)$. We will make the usual assumption that all the domains $\mathcal{D}(\mathcal{G}^f_s)$ and $\mathcal{D}(\mathcal{G}^{*f}_s)$ have a common subspace $\mathcal{D}$, which is dense in $B(\mathbf{R})$, such that for every $s < t$, $T^f_{st}(\mathcal{D}) \subseteq \mathcal{D}$ and for every $h \in \mathcal{D}$, the function $T^f_{st}h$ is strongly continuously differentiable with respect to s and t with

$$-(\frac{\partial^-}{\partial r}T^f_{rs}h)|_{r=s} = (\frac{\partial^+}{\partial t}T^f_{st}h)|_{t=s} \quad \forall s$$

The operator $\mathcal{G}^f_s = \mathcal{G}^{*f}_s$ defined on $\mathcal{D}$, will be called *the generator* of the process X^f at time s. A consequence of Kolmogorov-Feller equations is that

$$T^f_{st}h - h = \int_s^t \mathcal{G}^f_v T^f_{vt}h \; dv, \; \forall h \in \mathcal{D} \tag{6}$$

Definition 5. *The collection $\{\mathcal{G}^f := (\mathcal{G}^f_s)_s; f \in \mathcal{S}\}$, where $\mathcal{G}^f_s$ is the generator of the one-dimensional process X^f at time s, is called the* **generator** *of the set-indexed process X.*

Corollary 2. *The generator of a $\mathcal{Q}$-Markov process determines its distribution.*

Proof: Since $\mathcal{G}^f$ determines $\mathcal{T}^f$ and $\mathcal{Q}^f$, the result is an immediate consequence of Corollary 1. $\square$

Examples 4.

1. The generator of a process with independent increments, which is stochastically continuous and weakly differentiable on every simple flow $f \in \mathcal{S}$, is given by

$$(\mathcal{G}_s^f h)(x) = (\gamma_s^f)' h'(x) + \frac{1}{2}(\Lambda_s^f)' h''(x) + \int_{\{|y|>1\}} (h(x+y) - h(x))(\Pi_s^f)'(dx) + \int_{\{|y|\leq 1\}} (h(x+y) - h(x) - yh'(x))(\Pi_s^f)'(dx)$$

where $\gamma_t^f, \Lambda_t^f, \Pi_t^f(dy)$ are respectively, the translation function, the variance measure, and the Lévy measure of the process on the flow f and $(\gamma_s^f)', (\Lambda_s^f)', (\Pi_s^f)'(dy)$ denote the derivatives at s of these functions. The domain of $\mathcal{G}_s^f$ contains the dense subspace C_b^2 of twice continuously differentiable functions $h : \mathbf{R} \to \mathbf{R}$ such that h, h', h'' are bounded.

2. The generator of the empirical process of size n, corresponding to a probability measure F which has the property that $\forall f \in \mathcal{S}\ F \circ f$ is differentiable, is given by

$$(\mathcal{G}_s^f h)\left(\frac{k}{n}\right) = (n-k)\left(h\left(\frac{k+1}{n}\right) - h\left(\frac{k}{n}\right)\right)\frac{(F\circ f)'(s)}{1 - F\circ f(s)},\ k < n,$$

where $(F \circ f)'(s)$ denotes the derivative at s of $F \circ f$. The domain of $\mathcal{G}_s^f$ is the space of all finite arrays $h := (h(k))_{k=0,\ldots,n}$.

3. The generator of the Dirichlet process with parameter measure α which has the property that $\forall f \in \mathcal{S}\ \alpha \circ f$ is differentiable, is given by

$$(\mathcal{G}_s^f h)(x) = (\alpha \circ f)'(s) \int_0^{1-x} \frac{h(x+y) - h(x)}{y} \left(\frac{1-x-y}{1-x}\right)^{\alpha(f(s)^c)-1} dy$$

where $(\alpha \circ f)'(s)$ denotes the derivative at s of $\alpha \circ f$. The domain of $\mathcal{G}_s^f$ contains the dense subspace C_b^1 of continuously differentiable functions $h : \mathbf{R} \to \mathbf{R}$ such that h, h' are bounded.

The goal is to find the conditions that have to be satisfied by the collection $\{\mathcal{G}^f := (\mathcal{G}_s^f)_s; f \in \mathcal{S}\}$ of generators such that the associated collection $\{\mathcal{Q}^f; f \in \mathcal{S}\}$ of one-parameter transition systems satisfies the matching Assumption 2 and the consistency Assumption 3. By invoking Corollary 1, we will be able to conclude next that there exists a set-indexed transition system $\mathcal{Q}$ and a $\mathcal{Q}$-Markov process $X := (X_A)_{A\in\mathcal{A}}$ with initial distribution μ, whose generator is exactly the collection $\{\mathcal{G}^f := (\mathcal{G}^f_s)_s; f \in \mathcal{S}\}$.

Using the integral equation (6), we will first give the equivalent form in terms of generators of the matching Assumption 2.

Assumption 4. *If ord*$1 = \{A_0 = \emptyset', A_1, \ldots, A_n\}$ *and ord*$2 = \{A_0 = \emptyset', A'_1, \ldots, A'_m\}$ *are two consistent orderings of some finite semilattices* $\mathcal{A}', \mathcal{A}''$, *such that* $\cup_{j=1}^n A_j = \cup_{j=1}^m A'_j, \cup_{j=1}^k A_j = \cup_{j=1}^l A'_j$ *for some* $k < n, l < m$, *and we denote* $f := f_{\mathcal{A}',ord1}$, $g := f_{\mathcal{A}'',ord2}$ *with* $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$, *then for every* $h \in \mathcal{D}$

$$\int_0^{t_1} \mathcal{G}^f_v T^f_{vt_1} h dv = \int_0^{u_1} \mathcal{G}^g_v T^g_{vu_1} h dv \text{ and } \int_{t_k}^{t_n} \mathcal{G}^f_v T^f_{vt_n} h dv = \int_{u_l}^{u_m} \mathcal{G}^g_v T^g_{vu_m} h dv$$

We will need the following notational convention.

If $Q_1(x_1; dx_2), Q_2(x_2; dx_3), \ldots, Q_{n-1}(x_{n-1}; dx_n)$ are transition probabilities on $\mathbf{R}$, $T_1, T_2, \ldots, T_{n-1}$ are their associated bounded linear operators, and $h : \mathbf{R}^n \to \mathbf{R}$ is a bounded measurable function, then

$$T_1T_2 \ldots T_{n-1}\ [h(x_1, x_2, \ldots, x_n)](x_1) \stackrel{\text{def}}{:=}$$

$$\int_{\mathbf{R}^{n-1}} h(x_1, x_2, \ldots, x_n) Q_{n-1}(x_{n-1}; dx_n) \ldots Q_2(x_2; dx_3) Q_1(x_1; dx_2)$$

If in addition, $\mathcal{G}$ is a linear operator on $B(\mathbf{R})$ with domain $\mathcal{D}(\mathcal{G})$, and the function h is chosen such that for every $x_1, \ldots, x_k \in \mathbf{R}$, the function

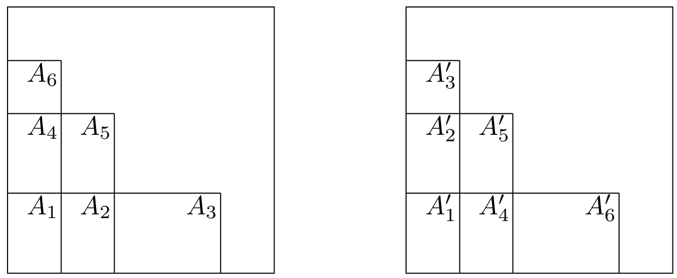
$$\int_{\mathbf{R}^{n-k}} h(x_1, x_2, \ldots, x_n) Q_{n-1}(x_{n-1}; dx_n) \ldots Q_{k+1}(x_{k+1}; dx_{k+2}) Q_k(\cdot; dx_{k+1})$$

is in $\mathcal{D}(\mathcal{G})$, then

$$T_1T_2 \ldots T_{k-1}\mathcal{G}T_k \ldots T_{n-1}\ [h(x_1, x_2, \ldots, x_n)](x_1) \stackrel{\text{def}}{:=}$$

$$\int_{\mathbf{R}^{k-1}} (\mathcal{G} \int_{\mathbf{R}^{n-k}} h(x_1, x_2, \ldots, x_n) Q_{n-1}(x_{n-1}; dx_n) \ldots Q_{k+1}(x_{k+1}; dx_{k+2})$$
$$Q_k(\cdot; dx_{k+1}))(x_k) Q_{k-1}(x_{k-1}; dx_k) \ldots Q_2(x_2; dx_3) Q_1(x_1; dx_2).$$

In order to understand the consistency Assumption 3 and to see how we can express it in terms of the generators, we consider a simple example. Let $\mathcal{A}'$ be a finite semilattice consisting of 6 sets, ord1 $= \{A_0 = \emptyset', A_1, \ldots, A_6\}$ and ord2 $= \{A_0 = \emptyset', A'_1, \ldots, A'_6\}$ two consistent orderings of $\mathcal{A}'$ with $A_i = A'_{\pi(i)}$, where π is the permutation (1)(24)(36)(5). Denote with C_i, C'_i the left neighbourhoods of A_i respectively A'_i.



Set $f := f_{\mathcal{A}',ord1}$, $g := f_{\mathcal{A}',ord2}$ with $f(t_i) = \cup_{j=1}^{i} A_j$, $g(u_i) = \cup_{j=1}^{i} A'_j$ for $i = 1, \ldots, 6$. Suppose that a $\mathcal{Q}$-Markov process $(X_A)_{A \in \mathcal{A}}$ exists; denote $X := X^f$ and $Y := X^g$. We have

$$X_{t_2} - X_{t_1} = Y_{u_4} - Y_{u_3}; \quad X_{t_3} - X_{t_2} = Y_{u_6} - Y_{u_5}; \quad X_{t_4} - X_{t_3} = Y_{u_2} - Y_{u_1}$$

$$X_{t_5} - X_{t_4} = Y_{u_5} - Y_{u_4}; \quad X_{t_6} - X_{t_5} = Y_{u_3} - Y_{u_2}$$

Using these equalities we will first find the relationship between $T^f_{t_{i-1}t_i}$ and $T^f_{u_{\pi(i)-1},u_{\pi(i)}}$ and then between the generators $(\mathcal{G}^f_w)_{w \in [t_{i-1},t_i]}$ and $(\mathcal{G}^g_v)_{v \in [u_{\pi(i)-1},u_{\pi(i)}]}$ for $i = 2, \ldots, 6$.

For $i = 2$ we have $P[X_{t_2} - X_{t_1} \in \Gamma | X_{t_1} = x_1] = P[Y_{u_4} - Y_{u_3} \in \Gamma | Y_{u_1} = x_1]$ which leads us to the equation: $\forall h \in B(\mathbf{R})$

$$\int_{\mathbf{R}} h(x_2) Q^f_{t_1 t_2}(x_1; dx_2) = \int_{\mathbf{R}^2} h(x_1 + y_4 - y_3) Q^g_{u_3 u_4}(y_3; dy_4) Q^g_{u_1 u_3}(x_1; dy_3)$$

Using the above notational convention, this can be written as

$$(T^f_{t_1 t_2} h)(x_1) = T^g_{u_1 u_3} T^g_{u_3 u_4}[h(x_1 + y_4 - y_3)](x_1). \tag{7}$$

Let $h \in \mathcal{D}$ and subtract $h(x_1)$ from both sides of this equation. Using (6) and Fubini's theorem, we get

$$\int_{t_1}^{t_2} (\mathcal{G}^f_w T^f_{w t_2} h)(x_1) dw = \int_{u_3}^{u_4} T^g_{u_1 u_3} \mathcal{G}^g_v T^g_{v u_4}[h(x_1 + y_4 - y_3)](x_1) dv \tag{8}$$

Note that (7) and (8) are equivalent (since $\mathcal{D}$ is dense in $B(\mathbf{R})$).

For $i = 3$, $P[X_{t_2} - X_{t_1} \in \Gamma_1, X_{t_3} - X_{t_2} \in \Gamma_2 | X_{t_1} = x_1] = P[Y_{u_4} - Y_{u_3} \in \Gamma_1, Y_{u_6} - Y_{u_5} \in \Gamma_2 | Y_{u_1} = x_1]$ which leads us to the equation: $\forall h_1, h_2 \in B(\mathbf{R})$

$$T^g_{u_1 u_3} T^g_{u_3 u_4}[h_1(y_4 - y_3)(T^f_{t_2 t_3} h_2)(x_1 + y_4 - y_3)](x_1) \tag{9}$$

$$= T^g_{u_1 u_3} T^g_{u_3 u_4} T^g_{u_4 u_5} T^g_{u_5 u_6}[h_1(y_4 - y_3) h_2(x_1 + y_4 - y_3 + y_6 - y_5)](x_1)$$

Let $h_1, h_2 \in \mathcal{D}$ and subtract $T^g_{u_1u_3}T^g_{u_3u_4}[h_1(y_4-y_3)h_2(x_1+y_4-y_3)](x_1)$ from both sides of this equation. On the left-hand side we get

$$\int_{t_2}^{t_3} T^g_{u_1u_3}T^g_{u_3u_4}[h_1(y_4-y_3)(\mathcal{G}^f_w T^f_{wt_3}h_2)(x_1+y_4-y_3)](x_1)$$

On the right-hand side we have

$$T^g_{u_1u_3}T^g_{u_3u_4}T^g_{u_4u_5}[(T^g_{u_5u_6}h'(x_1,y_3,y_4,y_5,\cdot))(y_5)-h(x_1,y_3,y_4)](x_1)$$

with $h'(x_1,y_3,y_4,y_5,y_6) := h_1(y_4-y_3)h_2(x_1+y_4-y_3+y_6-y_5)$ and $h(x_1,y_3,y_4) = h_1(y_4-y_3)h_2(x_1+y_4-y_3)$, which becomes

$$\int_{u_5}^{u_6} T^g_{u_1u_3}T^g_{u_3u_4}T^g_{u_4u_5}\mathcal{G}^g_v T^g_{vu_6}[h_1(y_4-y_3)h_2(x_1+y_4-y_3+y_6-y_5)](x_1)$$

because $h'(x_1,y_3,y_4,y_5,y_5) = h(x_1,y_3,y_4)$.
Therefore, the equivalent form of (9) in terms of the generators is

$$\int_{t_2}^{t_3} T^g_{u_1u_3}T^g_{u_3u_4}[h_1(y_4-y_3)(\mathcal{G}^f_w T^f_{wt_3}h_2)(x_1+y_4-y_3)](x_1) \tag{10}$$

$$= \int_{u_5}^{u_6} T^g_{u_1u_3}T^g_{u_3u_4}T^g_{u_4u_5}\mathcal{G}^g_v T^g_{vu_6}[h_1(y_4-y_3)h_2(x_1+y_4-y_3+y_6-y_5)](x_1)$$

Continuing in the same manner for $i = 4$ we will get the necessary relationship between $T^f_{t_3t_4}$ and $T^g_{u_1u_2}$. (When we write down this relationship we have to specify the ordering of $\pi(2)-1, \pi(2), \pi(3)-1, \pi(3), \pi(4)-1, \pi(4)$.) This relationship will have an equivalent form in terms of the generators $(\mathcal{G}^f_w)_{w\in[t_3,t_4]}$ and $(\mathcal{G}_v)_{v\in[u_1,u_2]}$.

At the end of this procedure we discover a collection of 5 necessary relationships that have to be satisfied by the generators $\mathcal{G}^f$ and $\mathcal{G}^g$ of the process. A very important fact is that these relationships are also sufficient, i.e. if they hold then the finite dimensional distribution of the process over the semilattice $\mathcal{A}'$ is 'invariant' under the two orderings *ord1* and *ord2*.

We return now to the general context. Let $\mu^f_{t_1}$ be the probability measure defined by $\mu^f_{t_1}(\Gamma) := \int_{\mathbf{R}} Q^f_{0t_1}(x;\Gamma)\mu(dx)$. The following result is crucial and its non-trivial proof can be found in the appendix.

Lemma 6. *Let* $ord1 = \{A_0 = \emptyset', A_1, \ldots, A_n\}$ *and* $ord2 = \{A_0 = \emptyset', A'_1, \ldots, A'_n\}$ *be two consistent orderings of the same finite semilattice* $\mathcal{A}'$ *with* $A_i = A'_{\pi(i)}$*, where* π *is a permutation of* $\{1,\ldots,n\}$ *with* $\pi(1) = 1$*, and set* $f := f_{\mathcal{A}',ord1}$*,* $g := f_{\mathcal{A}',ord2}$ *with* $f(t_i) = \cup_{j=1}^i A_j$*,* $g(u_i) = \cup_{j=1}^i A'_j$.

Suppose that $Q^f_{0t_1} = Q^g_{0u_1}$*. The following statements are equivalent:*

(a) *the integral condition (5) of the consistency Assumption 3 holds;*

(b) *for each $i = 2, \ldots, n$, if we denote with $l_1 \le l_2 \le \ldots \le l_{2(i-1)}$ the increasing ordering of the values $\pi(2)-1, \pi(2), \pi(3)-1, \pi(3), \ldots, \pi(i)-1, \pi(i)$, and with p the index for which $\pi(i)-1 = l_{p-1}, \pi(i) = l_p$, then for every $h_2, \ldots, h_i \in \mathcal{D}$ and for $\mu^f_{t_1}$-almost all x_1*

(b1) *if $p = 2(i-1)$ we have*

$$\int_{t_{i-1}}^{t_i} T^g_{u_1 u_{l_1}} T^g_{u_{l_1} u_{l_2}} \ldots T^g_{u_{l_{p-3}} u_{l_{p-2}}}$$

$$[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1})(\mathcal{G}^f_w T^f_{wt_i} h_i)(x_1 + \sum_{j=2}^{i-1}(y_{\pi(j)} - y_{\pi(j)-1}))](x_1) dw$$

$$= \int_{u_{l_{p-1}}}^{u_{l_p}} T^g_{u_1 u_{l_1}} T^g_{u_{l_1} u_{l_2}} \ldots T^g_{u_{l_{p-3}} u_{l_{p-2}}} T^g_{u_{l_{p-2}} u_{l_{p-1}}} \mathcal{G}^g_v T^g_{v u_{l_p}}$$

$$[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(x_1 + \sum_{j=2}^{i}(y_{\pi(j)} - y_{\pi(j)-1}))](x_1) dv;$$

(b2) *if $p < 2(i-1)$ we have*

$$\int_{t_{i-1}}^{t_i} T^g_{u_1 u_{l_1}} \ldots T^g_{u_{l_{p-3}} u_{l_{p-2}}} T^g_{u_{l_{p-2}} u_{l_{p+1}}} T^g_{u_{l_{p+1}} u_{l_{p+2}}} \ldots T^g_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}$$

$$[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1})(\mathcal{G}^f_w T^f_{wt_i} h_i)(x_1 + \sum_{j=2}^{i-1}(y_{\pi(j)} - y_{\pi(j)-1}))](x_1) dw$$

$$= \int_{u_{l_{p-1}}}^{u_{l_p}} T^g_{u_1 u_{l_1}} T^g_{u_{l_1} u_{l_2}} \ldots T^g_{u_{l_{p-3}} u_{l_{p-2}}} T^g_{u_{l_{p-2}} u_{l_{p-1}}} \mathcal{G}^g_v T^g_{v u_{l_p}}$$

$$T^g_{u_{l_p} u_{l_{p+1}}} T^g_{u_{l_{p+1}} u_{l_{p+2}}} \ldots T^g_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}} \{\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) \cdot$$

$$[h_i(x_1 + \sum_{j=2}^{i}(y_{\pi(j)} - y_{\pi(j)-1})) - h_i(x_1 + \sum_{j=2}^{i-1}(y_{\pi(j)} - y_{\pi(j)-1}))]\}(x_1) dv.$$

The following assumption is the equivalent form in terms of generators of the consistency Assumption 3.

Assumption 5. *If $ord1 = \{A_0 = \emptyset', A_1, \ldots, A_n\}$ and $ord2 = \{A_0 = \emptyset', A'_1, \ldots, A'_n\}$ are two consistent orderings of the same finite semilattice $\mathcal{A}'$ with $A_i = A'_{\pi(i)}$, where π is a permutation of $\{1, \ldots, n\}$ with $\pi(1) = 1$, and we denote $f := f_{\mathcal{A}', ord1}$, $g := f_{\mathcal{A}', ord2}$ with $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$, then the generators $\mathcal{G}^f$ and $\mathcal{G}^g$ satisfy condition (b) stated in Lemma 6.*

Clearly Assumptions 4 and 5 are necessary conditions satisfied by the generator of any set-indexed $\mathcal{Q}$-Markov process. The following result is an immediate consequence of Corollary 1 which says that in fact, these two assumptions are also sufficient for the construction of the process.

Theorem 2. *Let μ be a probability measure on $\mathbf{R}$ and $\{\mathcal{G}^f := (\mathcal{G}^f_s)_s; f \in \mathcal{S}\}$ a collection of families of linear operators on $B(\mathbf{R})$ such that each operator $\mathcal{G}^f_s$ is defined on a dense subspace $\mathcal{D}$ of $B(\mathbf{R})$ and is the generator at time s of a semigroup $\mathcal{T}^f := (T^f_{st})_{s<t}$ associated with a transition system $\mathcal{Q}^f := (Q^f_{st})_{s<t}$. If the family $\{\mathcal{G}^f; f \in \mathcal{S}\}$ satisfies the matching Assumption 4 and the consistency Assumption 5, then there exist a set-indexed transition system $\mathcal{Q} := (Q_{BB'})_{B,B' \in \mathcal{A}(u); B \subseteq B'}$ and a $\mathcal{Q}$-Markov process $X := (X_A)_{A \in \mathcal{A}}$ with initial distribution μ, whose generator is exactly the collection $\{\mathcal{G}^f_s; f \in \mathcal{S}\}$.*

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A Proof of Lemma 6

We will prove the desired equivalence by means of an intermediate condition **(b')**. We will show that **(a)** $\Leftrightarrow$ **(b')** and **(b')** $\Leftrightarrow$ **(b)**. Here is this condition.

(b') *For each $i = 2, \ldots, n$, if we denote with $l_1 \leq l_2 \leq \ldots \leq l_{2(i-1)}$ the increasing ordering of the values $\pi(2)-1, \pi(2), \pi(3)-1, \pi(3), \ldots, \pi(i)-1, \pi(i)$, and with p the index for which $\pi(i)-1 = l_{p-1}, \pi(i) = l_p$, then for every $h_2, \ldots, h_i \in B(\mathbf{R})$ and for $\mu^f_{t_1}$-almost all x_1*

$$T^g_{u_1 u_{l_1}} T^g_{u_{l_1} u_{l_2}} \ldots T^g_{u_{l_{p-3}} u_{l_{p-2}}} T^g_{u_{l_{p-2}} u_{l_{p+1}}} T^g_{u_{l_{p+1}} u_{l_{p+2}}} \ldots T^g_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}} \tag{11}$$
$$[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1})(T^f_{t_{i-1} t_i} h_i)(x_1 + \sum_{j=2}^{i-1}(y_{\pi(j)} - y_{\pi(j)-1}))](x_1)$$
$$= T^g_{u_1 u_{l_1}} T^g_{u_{l_1} u_{l_2}} \ldots T^g_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}$$
$$[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(x_1 + \sum_{j=2}^{i}(y_{\pi(j)} - y_{\pi(j)-1}))](x_1);$$

Proof of (a) $\Rightarrow$ (b'): Let X be a $\mathcal{Q}^f$-Markov process and Y a $\mathcal{Q}^g$-Markov process with the same initial distribution μ. Since $Q^f_{0t_1} = Q^g_{0u_1}$, both X_{t_1} and Y_{u_1} have the same distribution $\mu^f_{t_1}$. The integral condition (5) of the consistency Assumption 6 is equivalent to saying that for $\mu^f_{t_1}$-almost all x_1, the conditional distribution of $(X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}})$ given $X_{t_1} = x_1$ coincide with the conditional distribution of $(Y_{u_{\pi(2)}} - Y_{u_{\pi(2)-1}}, \ldots, Y_{u_{\pi(n)}} - Y_{u_{\pi(n)-1}})$ given $Y_{u_1} = x_1$.

For $i = 2$ we will use the following relationship: for every $\Gamma_2 \in \mathcal{B}(\mathbf{R})$ and for $\mu^f_{t_1}$-almost all x_1

$$P[X_{t_2} - X_{t_1} \in \Gamma_2 | X_{t_1} = x_1] = P[Y_{u_{\pi(2)}} - Y_{u_{\pi(2)-1}} \in \Gamma_2 | Y_{u_1} = x_1]$$

Using the Markov property, the left-hand side is $\int_{\mathbf{R}} I_{\Gamma_2 + x_1}(x_2) Q^f_{t_1 t_2}(x_1; dx_2)$, whereas on the right-hand side we have

$$\int_{\mathbf{R}^2} I_{\Gamma_2}(y_{u_{\pi(2)}} - y_{u_{\pi(2)-1}}) Q^g_{u_{\pi(2)-1} u_{\pi(2)}}(y_{\pi(2)-1}; dy_{\pi(2)}) Q^g_{u_1 u_{\pi(2)-1}}(x_1; dy_{\pi(2)-1})$$

By a monotone class argument we can conclude that for every $h_2 \in B(\mathbf{R})$ and for $\mu^f_{t_1}$-almost all x_1

$$\int_{\mathbf{R}} h_2(x_2) Q^f_{t_1 t_2}(x_1; dx_2) = \int_{\mathbf{R}^2} h_2(x_1 + y_{u_{\pi(2)}} - y_{u_{\pi(2)-1}}) \tag{12}$$

$$Q^g_{u_{\pi(2)-1} u_{\pi(2)}}(y_{\pi(2)-1}; dy_{\pi(2)}) Q^g_{u_1 u_{\pi(2)-1}}(x_1; dy_{\pi(2)-1})$$

which is the desired relationship for $i = 2$.

For $i = 3$ we will use the following relationship: for every $\Gamma_2, \Gamma_3 \in \mathcal{B}(\mathbf{R})$ and for $\mu^f_{t_1}$-almost all x_1

$$P[X_{t_2} - X_{t_1} \in \Gamma_2, X_{t_3} - X_{t_2} \in \Gamma_3 | X_{t_1} = x_1] =$$
$$P[Y_{u_{\pi(2)}} - Y_{u_{\pi(2)-1}} \in \Gamma_2, Y_{u_{\pi(3)}} - Y_{u_{\pi(3)-1}} \in \Gamma_3 | Y_{u_1} = x_1]$$

The left-hand side can be written as

$$\int_{\mathbf{R}^2} I_{\Gamma_2 + x_1}(x_2) I_{\Gamma_3 + x_2}(x_3) Q^f_{t_2 t_3}(x_2; dx_3) Q^f_{t_1 t_2}(x_1; dx_2)$$

which becomes

$$\int_{\mathbf{R}^3} I_{\Gamma_2}(y_{\pi(2)} - y_{\pi(2)-1}) I_{\Gamma_3 + x_1 + y_{\pi(2)} - y_{\pi(2)-1}}(x_3) Q^f_{t_2 t_3}(x_1 + y_{\pi(2)} - y_{\pi(2)-1}; dx_3)$$

$$Q^g_{u_{\pi(2)-1} u_{\pi(2)}}(y_{\pi(2)-1}; dy_{\pi(2)})^g Q^g_{u_1 u_{\pi(2)-1}}(x_1; dy_{\pi(2)-1})$$

using equation (12).

The right-hand side can be written as

$$\int_{\mathbf{R}^4} I_{\Gamma_2}(y_{\pi(2)} - y_{\pi(2)-1}) I_{\Gamma_3}(y_{\pi(3)} - y_{\pi(3)-1}) Q^g_{u_{l_3}, u_{l_4}}(y_{l_3}; dy_{l_4}) Q^g_{u_{l_2}, u_{l_3}}(y_{l_2}; dy_{l_3})$$

$$Q^g_{u_{l_1}, u_{l_2}}(y_{l_1}; dy_{l_2}) Q^g_{u_1, u_{l_1}}(x_1; dy_{l_1})$$

where $l_1 \leq l_2 \leq l_3 \leq l_4$ is the increasing ordering of the values $\pi(2)-1, \pi(2), \pi(3) - 1, \pi(3)$.

By a monotone class argument we can conclude that for every $h_2, h_3 \in B(\mathbf{R})$ and for $\mu^f_{t_1}$-almost all x_1

$$\int_{\mathbf{R}^3} h_2(y_{\pi(2)} - y_{\pi(2)-1}) h_3(x_3) Q^f_{t_2 t_3}(x_1 + y_{\pi(2)} - y_{\pi(2)-1}; dx_3) \qquad (13)$$

$$Q^g_{u_{\pi(2)-1} u_{\pi(2)}}(y_{\pi(2)-1}; dy_{\pi(2)}) Q^g_{u_1 u_{\pi(2)-1}}(x_1; dy_{\pi(2)-1})$$

$$= \int_{\mathbf{R}^4} h_2(y_{\pi(2)} - y_{\pi(2)-1}) h_3(x_1 + y_{\pi(2)} - y_{\pi(2)-1} + y_{\pi(3)} - y_{\pi(3)-1}) Q^g_{u_{l_3}, u_{l_4}}(y_{l_3}; dy_{l_4})$$

$$Q^g_{u_{l_2}, u_{l_3}}(y_{l_2}; dy_{l_3}) Q^g_{u_{l_1}, u_{l_2}}(y_{l_1}; dy_{l_2}) Q^g_{u_1, u_{l_1}}(x_1; dy_{l_1})$$

which is the desired relationship for $i = 3$.

The inductive argument will be omitted since it is identical, but notationally complex.

Proof of (b') $\Rightarrow$ (a): Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_n \in \mathcal{B}(\mathbf{R})$ be arbitrary. Using the fact that $Q^f_{0t_1} = Q^g_{0u_1}$ and equation (12), the left-hand side of the integral condition (5) becomes

$$\int_{\mathbf{R}^{n+2}} I_{\Gamma_0}(y_0) I_{\Gamma_1}(y_1) I_{\Gamma_2}(y_{\pi(2)} - y_{\pi(2)-1}) I_{\Gamma_3 + y_1 + y_{\pi(2)} - y_{\pi(2)-1}}(x_3) I_{\Gamma_4 + x_3}(x_4) \ldots$$

$$I_{\Gamma_n + x_{n-1}}(x_n) Q^f_{t_{n-1} t_n}(x_{n-1}; dx_n) \ldots Q^f_{t_3 t_4}(x_3; dx_4) Q^f_{t_2 t_3}(y_1 + y_{\pi(2)} - y_{\pi(2)-1}; dx_3)$$

$$Q^g_{u_{\pi(2)-1} u_{\pi(2)}}(y_{\pi(2)-1}; dy_{\pi(2)}) Q^g_{u_1 u_{\pi(2)-1}}(y_1; dy_{\pi(2)-1}) Q^g_{0u_1}(y_0; dy_1) \mu(dy_0)$$

which in turn can be written as

$$\int_{\mathbf{R}^{n+3}} I_{\Gamma_0}(y_0) I_{\Gamma_1}(y_1) I_{\Gamma_2}(y_{\pi(2)} - y_{\pi(2)-1}) I_{\Gamma_3}(y_{\pi(3)} - y_{\pi(3)-1})$$

$$I_{\Gamma_4 + y_1 + \sum_{j=2}^3 (y_{\pi(j)} - y_{\pi(j)-1})}(x_4) \ldots I_{\Gamma_n + x_{n-1}}(x_n) Q^f_{t_{n-1} t_n}(x_{n-1}; dx_n) \ldots$$

$$Q^f_{t_3 t_4}(y_1 + \sum_{j=2}^{3} (y_{\pi(j)} - y_{\pi(j)-1}); dx_4) Q^g_{u_{l_3}, u_{l_4}}(y_{l_3}; dy_{l_4}) Q^g_{u_{l_2}, u_{l_3}}(y_{l_2}; dy_{l_3})$$

$$Q^g_{u_{l_1}, u_{l_2}}(y_{l_1}; dy_{l_2}) Q^g_{u_1, u_{l_1}}(y_1; dy_{l_1}) Q^g_{0u_1}(y_0; dy_1) \mu(dy_0)$$

using equation (13), where $l_1 \leq l_2 \leq l_3 \leq l_4$ is the increasing ordering of the values $\pi(2) - 1, \pi(2), \pi(3) - 1, \pi(3)$.

Continuing in the same manner at the last step we will get exactly the desired right-hand side of equation (5), since the increasing ordering of the values $\pi(2) - 1, \pi(2), \ldots, \pi(n) - 1, \pi(n)$ is exactly $1 \leq 2 \leq \ldots \leq n$.

Proof of (b') $\Leftrightarrow$ (b): The basic ingredient will be equation (6), which gives the integral expression of a semigroup in terms of its generator.

Since $\mathcal{D}$ is dense we can assume that the functions $h_2, \ldots, h_i$ are in $\mathcal{D}$ in the expression given by **(b')**. Subtract

$$T^g_{u_1 u_{l_1}} T^g_{u_{l_1} u_{l_2}} \ldots T^g_{u_{l_{p-3}} u_{l_{p-2}}} T^g_{u_{l_{p-2}} u_{l_{p+1}}} T^g_{u_{l_{p+1}} u_{l_{p+2}}} \ldots T^g_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}$$
$$[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(x_1 + \sum_{j=2}^{i-1}(y_{\pi(j)} - y_{\pi(j)-1}))](x_1)$$

from both sides of this expression.

On the left-hand side we will have

$$T^g_{u_1 u_{l_1}} T^g_{u_{l_1} u_{l_2}} \ldots T^g_{u_{l_{p-3}} u_{l_{p-2}}} T^g_{u_{l_{p-2}} u_{l_{p+1}}} T^g_{u_{l_{p+1}} u_{l_{p+2}}} \ldots T^g_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}$$
$$[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1})(T^f_{t_{i-1} t_i} h_i - h_i)(x_1 + \sum_{j=2}^{i-1}(y_{\pi(j)} - y_{\pi(j)-1}))](x_1)$$

which can be written as

$$\int_{t_{i-1}}^{t_i} T^g_{u_1 u_{l_1}} T^g_{u_{l_1} u_{l_2}} \ldots T^g_{u_{l_{p-3}} u_{l_{p-2}}} T^g_{u_{l_{p-2}} u_{l_{p+1}}} T^g_{u_{l_{p+1}} u_{l_{p+2}}} \ldots T^g_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}$$
$$[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1})(\mathcal{G}^f_w T^f_{w t_i} h_i)(x_1 + \sum_{j=2}^{i-1}(y_{\pi(j)} - y_{\pi(j)-1}))](x_1) dw$$

using the integral expression (6) and Fubini's theorem.

On the right-hand side we have

$$T^g_{u_1 u_{l_1}} T^g_{u_{l_1} u_{l_2}} \ldots T^g_{u_{l_{p-3}} u_{l_{p-2}}} T^g_{u_{l_{p-2}} u_{l_{p-1}}}$$
$$\{T^g_{u_{l_{p-1}} u_{l_p}} T^g_{u_{l_p} u_{l_{p+1}}} T^g_{u_{l_{p+1}} u_{l_{p+2}}} \ldots T^g_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}$$
$$[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(x_1 + \sum_{j=2}^{i}(y_{\pi(j)} - y_{\pi(j)-1}))](y_{l_{p-1}}) -$$
$$- T^g_{u_{l_{p-1}} u_{l_{p+1}}} T^g_{u_{l_{p+1}} u_{l_{p+2}}} \ldots T^g_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}$$
$$[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(x_1 + \sum_{j=2}^{i-1}(y_{\pi(j)} - y_{\pi(j)-1}))](y_{l_{p-1}})\}(x_1)$$

If we denote

$$h'(x_1, y_{l_1}, \ldots, y_{l_{p-1}}, y_{l_p}) :=$$
$$\int_{\mathbf{R}^{2(i-1)-p}} \prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(x_1 + \sum_{j=2}^{i}(y_{\pi(j)} - y_{\pi(j)-1}))$$

$$Q^g_{u_{l_{2(i-1)-1}}u_{l_{2(i-1)}}}(y_{l_{2(i-1)-1}};dy_{l_{2(i-1)}})\dots Q^g_{u_{l_{p+1}}u_{l_{p+2}}}(y_{l_{p+1}};dy_{l_{p+2}})$$
$$Q^g_{u_{l_p}u_{l_{p+1}}}(y_{l_p};dy_{l_{p+1}})$$

and

$$h(x_1,y_{l_1},\dots,y_{l_{p-1}}):=$$
$$\int_{\mathbf{R}^{2(i-1)-p}}\prod_{j=2}^{i-1}h_j(y_{\pi(j)}-y_{\pi(j)-1})h_i(x_1+\sum_{j=2}^{i-1}(y_{\pi(j)}-y_{\pi(j)-1}))$$
$$Q^g_{u_{l_{2(i-1)-1}}u_{l_{2(i-1)}}}(y_{l_{2(i-1)-1}};dy_{l_{2(i-1)}})\dots Q^g_{u_{l_{p+1}}u_{l_{p+2}}}(y_{l_{p+1}};dy_{l_{p+2}})$$
$$Q^g_{u_{l_{p-1}}u_{l_{p+1}}}(y_{l_{p-1}};dy_{l_{p+1}})$$

then the right-hand side becomes

$$T^g_{u_1u_{l_1}}T^g_{u_{l_1}u_{l_2}}\dots T^g_{u_{l_{p-3}}u_{l_{p-2}}}T^g_{u_{l_{p-2}}u_{l_{p-1}}}$$
$$[(T^g_{u_{l_{p-1}}u_{l_p}}h'(x_1,y_{l_1},\dots,y_{l_{p-1}},\cdot))(y_{l_{p-1}})-h(x_1,y_{l_1},\dots,y_{l_{p-1}})](x_1)$$
$$=T^g_{u_1u_{l_1}}T^g_{u_{l_1}u_{l_2}}\dots T^g_{u_{l_{p-3}}u_{l_{p-2}}}T^g_{u_{l_{p-2}}u_{l_{p-1}}}$$
$$[h'(x_1,y_{l_1},\dots,y_{l_{p-1}},y_{l_{p-1}})-h(x_1,y_{l_1},\dots,y_{l_{p-1}})+$$
$$+\int_{u_{l_{p-1}}}^{u_{l_p}}(\mathcal{G}^g_vT^g_{vu_{l_p}}h'(x_1,y_{l_1},\dots,y_{l_{p-1}},\cdot))(y_{l_{p-1}})dv](x_1)$$

since $h'(x_1,y_{l_1},\dots,y_{l_{p-1}},\cdot)\in\mathcal{D}$ for every $x_1,y_{l_1},\dots,y_{l_{p-1}}$. We have two cases:

Case 1) If $p=2(i-1)$, then all the integrals with respect to $Q^g_{u_{l_p}u_{l_{p+1}}}$, $Q^g_{u_{l_{p+1}}u_{l_{p+2}}},\dots,Q^g_{u_{l_{2(i-1)-1}}u_{l_{2(i-1)}}}$ disappear in the preceding expressions. Hence $h'(x_1,y_{l_1},\dots,y_{l_{p-1}},y_{l_{p-1}})=h(x_1,y_{l_1},\dots,y_{l_{p-1}})$ and the result follows.

Case 2) If $p<2(i-1)$, then

$$h'(x_1,y_{l_1},\dots,y_{l_{p-1}},y_{l_{p-1}})=(T^g_{u_{l_p}u_{l_{p+1}}}H(x_1,y_{l_1},\dots,y_{l_{p-1}},\cdot))(y_{l_{p-1}})$$
$$h(x_1,y_{l_1},\dots,y_{l_{p-1}})=(T^g_{u_{l_{p-1}}u_{l_{p+1}}}H(x_1,y_{l_1},\dots,y_{l_{p-1}},\cdot))(y_{l_{p-1}})$$

where

$$H(x_1,y_{l_1},\dots,y_{l_{p-1}},y_{l_{p+1}}):=$$
$$\int_{\mathbf{R}^{2(i-1)-p-1}}\prod_{j=2}^{i-1}h_j(y_{\pi(j)}-y_{\pi(j)-1})h_i(x_1+\sum_{j=2}^{i-1}(y_{\pi(j)}-y_{\pi(j)-1}))$$
$$Q^g_{u_{l_{2(i-1)-1}}u_{l_{2(i-1)}}}(y_{l_{2(i-1)-1}};dy_{l_{2(i-1)}})\dots Q^g_{u_{l_{p+1}}u_{l_{p+2}}}(y_{l_{p+1}};dy_{l_{p+2}})$$

To simplify the notation we will omit the arguments of the function H. Hence

$$h'(x_1, y_{l_1}, \ldots, y_{l_{p-1}}, y_{l_{p-1}}) - h(x_1, y_{l_1}, \ldots, y_{l_{p-1}})$$
$$= (T^g_{u_{l_p} u_{l_{p+1}}} H)(y_{l_{p-1}}) - [T^g_{u_{l_{p-1}} u_{l_p}} (T^g_{u_{l_p} u_{l_{p+1}}} H)](y_{l_{p-1}})$$
$$= -\int_{u_{l_{p-1}}}^{u_{l_p}} (\mathcal{G}^g_v T^g_{v u_{l_p}} (T^g_{u_{l_p} u_{l_{p+1}}} H))(y_{l_{p-1}}) dv$$

since $H(x_1, y_{l_1}, \ldots, y_{l_{p-1}}, \cdot) \in \mathcal{D}$, for every $x_1, y_{l_1}, \ldots, y_{l_{p-1}}$.

Using Fubini's theorem, the right-hand side becomes

$$\int_{u_{l_{p-1}}}^{u_{l_p}} T^g_{u_1 u_{l_1}} T^g_{u_{l_1} u_{l_2}} \ldots T^g_{u_{l_{p-3}} u_{l_{p-2}}} T^g_{u_{l_{p-2}} u_{l_{p-1}}}$$
$$[(\mathcal{G}^g_v T^g_{v u_{l_p}} U(x_1, y_{l_1}, \ldots, y_{l_{p-1}}, \cdot))(y_{l_{p-1}})](x_1) dv$$

where

$$U(x_1, y_{l_1}, \ldots, y_{l_{p-1}}, y_{l_p}) :=$$
$$= h'(x_1, y_{l_1}, \ldots, y_{l_{p-1}}, y_{l_p}) - (T^g_{u_{l_p} u_{l_{p+1}}} H(x_1, y_{l_1}, y_{l_2}, \ldots, y_{l_{p-1}}, \cdot))(y_{l_p})$$
$$= \int_{\mathbf{R}^{2(i-1)-p}} \prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1})$$
$$[h_i(x_1 + \sum_{j=2}^{i} (y_{\pi(j)} - y_{\pi(j)-1})) - h_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}))]$$
$$Q^g_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}} (y_{l_{2(i-1)-1}}; dy_{l_{2(i-1)}}) \ldots Q^g_{u_{l_p} u_{l_{p+1}}} (y_{l_p}; dy_{l_{p+1}})$$

This concludes the proof. □

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