Intermittency for stochastic heat and wave equations with fractional noise

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(Homogenous) Heat equation

\[
\begin{aligned}
\frac{\partial u}{\partial t}(t, x) &= \frac{1}{2} \Delta u(t, x) \quad t > 0, \, x \in \mathbb{R}^d \\
u(0, x) &= u_0(x) \quad x \in \mathbb{R}^d
\end{aligned}
\]

where \( \Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} \). This equation has a unique solution:

\[
u(t, x) = (G(t, \cdot) \ast u_0)(x) = \int_{\mathbb{R}^d} G(t, x - y) u_0(y) dy
\]

where \( G \) is the Green function of the heat operator \( L = \frac{\partial}{\partial t} - \frac{1}{2} \Delta \):

\[
G(t, x) = \frac{1}{(2\pi t)^{d/2}} \exp \left( -\frac{\|x\|^2}{2t} \right)
\]
Some classical PDEs

(Homogeneous) Wave equation

\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2} (t, x) = \Delta u(t, x) & t > 0, x \in \mathbb{R}^d \\
u(0, x) = u_0(x) & x \in \mathbb{R}^d \\
\frac{\partial u}{\partial t} (0, x) = v_0(x) & x \in \mathbb{R}^d
\end{cases}
\]

This equation has a unique solution given by

\[
u(t, x) = (G(t, \cdot) \ast v_0)(x) + \frac{\partial}{\partial t} (G(t, \cdot) \ast u_0)(x)
\]

where \( G \) is the Green function of the wave operator \( L = \frac{\partial^2}{\partial t^2} - \Delta \):

\[
G(t, x) = \begin{cases}
\frac{1}{2} 1_{\{ |x| < t \}} & \text{if } d = 1 \\
\frac{1}{2\pi} \frac{1}{\sqrt{t^2 - \|x\|^2}} 1_{\{ \|x\| < t \}} & \text{if } d = 2
\end{cases}
\]
Non-homogenous equations

$L$ is the heat or wave operator: $L = \frac{\partial}{\partial t} - \frac{1}{2} \Delta$ or $L = \frac{\partial^2}{\partial t^2} - \Delta$

$F$ is a "forcing" term

\[
\begin{cases}
Lu(t, x) = F(t, x) & t > 0, x \in \mathbb{R}^d \\
\text{same initial conditions as above}
\end{cases}
\]  

(1)

The solution of (1) is given by

\[
 u(t, x) = w(t, x) + \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y)F(s, y)dyds
\]

- $w$ is the solution of the corresponding homogeneous equation
- $G$ is the Green function of $L$
Stochastic PDEs (SPDEs)

We replace $F$ in (1) by a randomly forced term $\dot{W}$ (called “the noise”). We obtain the following (linear) SPDE:

\[
\begin{cases}
Lu(t, x) = \dot{W}(t, x) & t > 0, x \in \mathbb{R}^d \\
\text{same initial conditions as above}
\end{cases}
\] (2)

Intuitively, the solution of (2) should be given by:

\[
u(t, x) = w(t, x) + \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y) \dot{W}(s, y) dy ds
\]

\[
= w(t, x) + \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y) W(ds, dy)
\]

where the last integral is interpreted as a stochastic integral.
Generalized derivative of BM

Let $W = \{ W(t); \ t \in \mathbb{R} \}$ be a standard Brownian Motion (BM). Define

$$\dot{W}(\varphi) := - \int_{\mathbb{R}} \varphi'(t) W(t) dt$$

for all $\varphi \in C_0^\infty(\mathbb{R})$.

If the map $t \mapsto W(t)$ was differentiable, then we would have:

$$\dot{W}(\varphi) = \int_{\mathbb{R}} \varphi(t) W'(t) dt = \int_{\mathbb{R}} \varphi(t) W(dt) =: W(\varphi)$$

But the map $t \mapsto W(t)$ is nowhere differentiable!

It can be shown that $\{ \dot{W}(\varphi); \ \varphi \in C_0^\infty(\mathbb{R}) \}$ is a zero-mean Gaussian process with covariance

$$E[\dot{W}(\varphi) \dot{W}(\psi)] = \int_{\mathbb{R}} \varphi(t) \psi(t) dt$$

(Itô, 1954; Gelfand and Vilenkin, 1964)
**Space-time white noise**

The **space-time white noise** is a zero-mean Gaussian process

\[ W = \{ W(\varphi); \varphi \in C^\infty_0(\mathbb{R}^{d+1}) \} \]

with covariance

\[
E[W(\varphi)W(\psi)] = \int_{\mathbb{R}^{d+1}} \varphi(t,x)\psi(t,x)\,dt\,dx
\]

Note: We consider only \( W(\varphi) \) with \( \varphi \in C^\infty_0(\mathbb{R}_+ \times \mathbb{R}^d) \).

It can be shown that: (Walsh, 1986)

\[
W(\varphi) \overset{d}{=} \dot{W}(\varphi) := (-1)^{d+1} \int_{\mathbb{R}^{d+1}} \frac{\partial^{d+1}\varphi}{\partial t\partial x_1 \ldots \partial x_d}(t,x)W(t,x)\,dt\,dx
\]

where \( \{ W(t,x); t \in \mathbb{R}, x \in \mathbb{R}^d \} \) is a Brownian sheet.

**Stochastic Integral**

The map \( C^\infty_0(\mathbb{R}^{d+1}) \ni \varphi \mapsto W(\varphi) \in L^2(\Omega) \) is an isometry, which can be extended to \( L^2(\mathbb{R}^{d+1}) \).
Stochastic heat equation (linear case)

\[
\begin{cases}
\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) + \dot{W}(t, x), & t > 0, x \in \mathbb{R}^d \\
u(0, x) = 0 & x \in \mathbb{R}^d
\end{cases}
\]

Equation (3) has a random-field solution \( u(t, x) \) if and only if

\[
u(t, x) = \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y) W(ds, dy)
\]
is well-defined, i.e.

\[
\int_0^t \int_{\mathbb{R}^d} G^2(t - s, x - y) dyds = c_d \int_0^t (t - s)^{-d/2} ds < \infty
\]

This forces \( d = 1 \).

**Moments of the solution:** \( \{u(t, x); t > 0, x \in \mathbb{R}\} \) is Gaussian

\[
E|u(t, x)|^p = c_p (E|u(t, x)|^2)^{p/2} = c_p t^{p/4}, \quad p > 0
\]
The Lyapunov exponent of a random field $u = \{u(t, x)\}$ is defined by:

$$\gamma(p) := \lim_{t \to \infty} \frac{1}{t} \log E|u(t, x)|^p \quad \text{(if it exists)}$$

(Suppose that $u$ is stationary in $x$.) By Hölder’s inequality,

$$(E|u(t, x)|^p)^{1/p} \leq (E|u(t, x)|^q)^{1/q} \quad \text{if} \quad p < q$$

The map $p \mapsto \frac{1}{p} \gamma(p)$ is non-decreasing. $u$ is called intermittent if

$$p \mapsto \frac{1}{p} \gamma(p) \quad \text{is strictly increasing.}$$

Carmona and Molchanov (1994)

If $\gamma(p) < \infty$ for some $p > 2$ and $\gamma(2) > 0$, then $u$ is intermittent.
Parabolic Anderson Model (P. W. Anderson, 1958)

\[
\begin{aligned}
\frac{\partial u}{\partial t}(t, x) &= \Delta u(t, x) + u(t, x) \dot{W}(t, x), \quad t > 0, x \in \mathbb{R} \\
u(0, x) &= 1 \\
\end{aligned}
\] (PAM)

The solution of (PAM) (if it exists) satisfies the integral equation:

\[
u(t, x) = 1 + \int_0^t \int_{\mathbb{R}} G(t - s, x - y) u(s, y) W(ds, dy)
\]

(4)

Write \(u(s, y)\) using (4). Iterating this procedure, we obtain:

\[
u(t, x) = 1 + \int_0^t \int_{\mathbb{R}} G(t - s, x - y) W(ds, dy) + \\
\int_0^t \int_{\mathbb{R}} \int_0^s \int_{\mathbb{R}} G(t - s, x - y) G(s - r, y - z) W(dr, dz) W(ds, dy) + \ldots
\]
Second moment of the solution of (PAM) (Khoshnevisan, 2014)

Note that $E(u(t, x)) = 1$ (multiple Wiener integrals have zero mean).

$$E|u(t, x) - 1|^2 = E \left| \int_0^t \int_{\mathbb{R}} G(t - s, x - y)u(s, y)W(ds, dy) \right|^2$$

$$= \int_0^t \int_{\mathbb{R}} G^2(t - s, x - y)E|u(s, y)|^2 dyds$$

$M(t) = E|u(t, x)|^2$ does not depend on $x$!

Hence $M$ satisfies the renewal equation: (Feller, 1971)

$$M(t) = 1 + \int_0^t M(s)g(t - s)ds$$

where $g(t - s) = \int_{\mathbb{R}} G^2(t - s, x - y)dy = c(t - s)^{-1/2}$. From renewal theory,

$$\gamma(2) = \lim_{t \to \infty} \frac{1}{t} \log E|u(t, x)|^2 = \frac{1}{4}$$
Higher moments of the solution of (PAM)

**Bertini and Cancrini (1995):** For any integer \( k \geq 2 \)

\[
E|u(t, x)|^k = 2 \exp \left\{ \frac{k(k^2 - 1)}{4!} t \right\} \Phi \left( \sqrt{\frac{k(k^2 - 1)}{12}} t \right)
\]

where \( \Phi(z) = P(Z \leq z) \) and \( Z \sim N(0, 1) \). Hence

\[
\gamma(k) = \lim_{t \to \infty} \frac{1}{t} \log E|u(t, x)|^k = \frac{1}{4!} k(k^2 - 1).
\]

**Khoshnevisan (2014):** For any \( p \geq 2 \),

\[
\overline{\gamma}(p) := \limsup_{t \to \infty} \frac{1}{t} \log E|u(t, x)|^p = O(p^3)
\]

**Hyperbolic Anderson Model (HAM)**

Similar results exist for the wave equation (Chen and Dalang, 2014).
Fractional Brownian Motion (fBM)

The fBM is a zero-mean Gaussian process \( \{B(t); t \geq 0\} \) with

\[
E(B(t)B(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}) =: R_H(t, s)
\]

where \( H \in (0, 1) \) is the Hurst index.

- When \( H = 1/2 \), the fBM is a standard BM.
- When \( H \neq 1/2 \), the fBM is not a semimartingale. We can use Malliavin calculus instead of Itô calculus.
- When \( H > 1/2 \),

\[
R_H(t, s) = \alpha_H \int_0^t \int_0^s |u - v|^{2H-2} du dv \quad \text{with} \quad \alpha_H = H(2H - 1)
\]
Fractional-colored noise (B. and Tudor, 2008)

\( \{ W(\varphi); \varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d) \} \) is a zero-mean Gaussian process with

\[
E[W(\varphi)W(\psi)] = \alpha_H \int_{(\mathbb{R}_+ \times \mathbb{R}^d)^2} \varphi(t, x)\psi(s, y)|t-s|^{2H-2}\|x-y\|^{-\alpha} dt dx ds dy
\]

with \( \frac{1}{2} < H < 1 \) and \( 0 < \alpha < d \). In addition, we impose the condition:

\[
\int_{\mathbb{R}^d} \frac{1}{1 + \|\xi\|^2} \|\xi\|^{-d+\alpha} d\xi < \infty, \quad \text{i.e.} \quad \alpha < 2
\]

Potential Analysis

If \( 0 < \alpha < d \), then for any \( \varphi, \psi \in S(\mathbb{R}^d) \)

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x)\psi(y)\|x - y\|^{-\alpha} dx dy = c_{\alpha,d} \int_{\mathbb{R}^d} |\mathcal{F}\varphi(\xi)|\mathcal{F}\psi(\xi)\|\xi\|^{-d+\alpha} d\xi
\]

where \( \mathcal{F}\varphi \) is the Fourier transform of \( \varphi \)
(PAM) or (HAM) with fractional-colored noise

\[ Lu(t, x) = u(t, x) \dot{W}(t, x) \]  \hspace{1cm} (5)

\( L \) is the heat or wave operator (assume \( w = 1 \))

The solution of equation (5) satisfies the integral equation:

\[ u(t, x) = 1 + \int_0^t \int_{\mathbb{R}} G(t - s, x - y) u(s, y) W(\delta s, \delta y) \]

Stochastic integral is given by divergence operator (Malliavin calculus). \( u(t, x) \) is written a series of multiple Wiener integrals:

\[ u(t, x) = \sum_{n \geq 0} J_n(t, x) \quad \text{with} \quad J_n(t, x) = I_n(f_n(\cdot, t, x)) \]

\[ f_n(t_1, x_1, \ldots, t_n, x_n, t, x) = G(t - t_n, x - x_n) \ldots G(t_2 - t_1, x_2 - x_1) \]

if \( 0 < t_1 < \ldots < t_n < t \).
Existence of the solution and its second moment (B. 2012)

Solution of (5) exists if and only if the series converges in \( L^2(\Omega) \), i.e.

\[
E \left| u(t, x) \right|^2 = \sum_{n \geq 0} E \left| J_n(t, x) \right|^2 < \infty
\]

It can be shown that

\[
E \left| J_n(t, x) \right|^2 \leq C^n \frac{t^{(2H-\alpha)/2} n}{(n!)^{1-\alpha/2}} \quad \text{for the heat equation}
\]

\[
E \left| J_n(t, x) \right|^2 \leq C^n \frac{t^{(2H+2-\alpha)} n}{(n!)^{3-\alpha}} \quad \text{for the wave equation}
\]

\( \Rightarrow \) If \( \alpha < 2 \), both heat and wave equations have unique solutions.
Higher moments of the solution (B. and Conus, 2014)

**Important Fact** (Malliavin calculus): On the *same* Wiener chaos, the $L^p(\Omega)$-norms are equivalent:

$$\| J_n(t, x) \|_p \leq (p - 1)^{n/2} \| J_n(t, x) \|_2$$

where $\| X \|_p = (E|X|^p)^{1/p}$. By Minkowski’s inequality

$$\| u(t, x) \|_p \leq \sum_{n \geq 0} \| J_n(t, x) \|_p \leq \sum_{n \geq 0} (p - 1)^{n/2} \| J_n(t, x) \|_2$$

We use the previous estimates for $\| J_n(t, x) \|_2$ and the elementary inequality: for any $a > 0$

$$\sum_{n \geq 0} \frac{x^n}{(n!)^a} \leq C \exp(Cx^{1/a}) \quad \text{for all } x > 0$$

($C$ depends on $a$)
(PAM) with fractional-colored noise

\[ E|u(t, x)|^p \leq C_p \exp(C_p^{(4-\alpha)/(2-\alpha)} t^{\rho}) \]

where \( C \) depends of \( H \) and \( \alpha \), and

\[ \rho = \frac{4H - \alpha}{2 - \alpha} \]

(HAM) with fractional colored noise

\[ E|u(t, x)|^p \leq C_p \exp(C_p^{(4-\alpha)/(3-\alpha)} t^{\rho}) \]

where \( C \) depends of \( H \) and \( \alpha \), and

\[ \rho = \frac{2H + 2 - \alpha}{3 - \alpha} \]

Note: Intuitively, the case \( H = 1/2 \) and \( \alpha = d = 1 \) should correspond to \( W = \)space-time white noise. In this case, \( \rho = 1 \).
Weak intermittency

$u$ is called \textbf{weakly $\rho$-intermittent} if

$$\overline{\gamma}(p) := \limsup_{t \to \infty} \frac{1}{t^\rho} \log E|u(t, x)|^p < \infty \quad \text{for all } p \geq 2$$

$$\underline{\gamma}(2) := \liminf_{t \to \infty} \frac{1}{t^\rho} \log E|u(t, x)|^2 > 0$$

Lower bound for the second moment (B. and Conus, 2014)

For both heat and wave equations,

$$E|u(t, x)|^2 \geq C \exp(C t^\rho)$$

Hence, $u$ is weakly $\rho$-intermittent.

**Idea:** Develop a Feynman-Kac-type representation for $E|u(t, x)|^2$. A similar idea was used in Dalang and Mueller (2009) when the noise is \textit{white in time} and colored in space.
References


Thank you!