Convergence of Point Processes with Weakly Dependent Points

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Abstract

For each $n \geq 1$, let $\{X_{j,n}\}_{1 \leq j \leq n}$ be a sequence of strictly stationary random variables. In this article, we give some asymptotic weak dependence conditions for the convergence in distribution of the point process $N_n = \sum_{j=1}^{n} \delta_{X_{j,n}}$ to an infinitely divisible point process. From the point process convergence, we obtain the convergence in distribution of the partial sum sequence $S_n = \sum_{j=1}^{n} X_{j,n}$ to an infinitely divisible random variable, whose Lèvy measure is related to the canonical measure of the limiting point process. As applications, we discuss the case of triangular arrays which possess known (row-wise) dependence structures, like the strong mixing property, the association, or the dependence structure of a stochastic volatility model.

Keywords: weak limit theorem; point process; infinitely divisible law; strong mixing; association

1 Introduction

In this article we examine the convergence in distribution of the sequence $\{N_n = \sum_{j=1}^{n} \delta_{X_{j,n}}, n \geq 1\}$ of point processes, whose points $(X_{j,n})_{1 \leq j \leq n, n \geq 1}$ form a triangular array of strictly stationary weakly dependent random variables, with values in a locally compact Polish space E. As it is well-known in the literature, by a mere invocation of the continuous mapping theorem, the convergence of the point processes $(N_n)_{n \geq 1}$ becomes a rich source for numerous other limit theorems, which describe the asymptotic behavior of various functions of N_n, provided these functions are continuous with respect to the vague topology (in the space of Radon measures in which N_n lives). This turns out to be a very useful approach, provided that one has a handle on the limiting point process N, which usually comes from its "cluster representation", namely a representation of the form $N \stackrel{d}{=} \sum_{i,j \geq 1} \delta_{T_{ij}}$ for some carefully chosen random points T_{ij}. In principle, one can obtain via this route the convergence of the partial sum sequence $\{S_n = \sum_{j=1}^{n} X_{j,n}, n \geq 1\}$ to the sum $X := \sum_{i,j \geq 1} T_{ij}$ of the points. (However, as it is usually the case in mathematics, this works only "in principle", meaning that the details are not to be ignored.)

On the other hand, a classical result in probability theory says that the class of all limiting distributions for the partial sum sequence $(S_n)_{n \geq 1}$ associated with a triangular array of independent random variables coincides with the class of all infinitely divisible distributions (see e.g. Theorem 4.2, [16]). This result has been extended recently in [10] and [11] to some similar results for arrays of weakly dependent random variables with finite variances. One of the goals of the present article is to investigate if such a limit theorem can be obtained via the more powerful approach of point process convergence, which does not require any moment assumptions.

Our work is a continuation of the line of research initiated by Davis and Hsing in their magistral article [6], in which they consider an array of random variables with values in $\mathbb{R} \backslash \{0\}$, of the form $X_{j,n} = X_j / a_n$, where $(X_j)_{j \geq 1}$ is a strictly stationary sequence with heavy tails, and a_n is the $(1 - 1/n)$-quantile of X_1. In this case, there is no surprise that the limiting distribution of the sequence $(S_n)_{n \geq 1}$ coincides the stable law.

The main asymptotic dependence structure in our array (called (AD-1)) is inherited from condition $\mathcal{A}(\{a_n\})$ of [6] (see also [7]), but unlike these authors, we do not require that $X_{1,n}$ lie in the domain of attraction of

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the stable law. This relaxation will allow us later to obtain a general (possibly non-stable) limit distribution for the partial sum sequence $(S_n)_{n\geq 1}$. The asymptotic dependence structure (AD-1) is based on a simple technique, which requires that we "split" each row of the array into k_n blocks of length r_n, and then we ask the newly produced block-vectors $Y_{i,n} = (X_{(i-1)r_n+1,n}, \ldots, X_{ir_n,n}), 1 \leq i \leq k_n$ to behave asymptotically as their independent copies $\tilde{Y}_{i,n} = (\tilde{X}_{(i-1)r_n+1,n}, \ldots, \tilde{X}_{ir_n,n}), 1 \leq i \leq k_n$. Note that this procedure does not impose any restrictions on the dependence structure within the blocks, it only specifies (asymptotically) the dependence structure *among* the blocks. (Since each row of the array has length n, as a by-product, this procedure necessarily yields a remainder number $n - r_n k_n$ of terms, which will be taken care of by an asymptotic negligibility condition called (AN).) The origins of this technique can be traced back to Jakubowski's thorough investigation of the minimal asymptotic dependence conditions in the stable limit theorems (see [18], [19]). However, in Jakubowski's condition (B) the number of blocks is assumed to be 2, whereas in our condition (AD-1), as well as in condition $\mathcal{A}(\{a_n\})$, the number k_n of blocks explodes to infinity.

A fundamental result of [6] states that under $\mathcal{A}(\{a_n\})$, *if* the sequence of point processes $(N_n)_{n\geq 1}$ converges, then its limit N admits a very nice cluster representation of the form $N = \sum_{i,j\geq 1} \delta_{P_i Q_{ij}}$, with independent components $(P_i)_{i\geq 1}$ and $(Q_{ij})_{i\geq 1, j\geq 1}$ (see Theorem 2.3, [6]). The key word in this statement is "if". In the present article, we complement Theorem 2.3, [6] by supplying a new asymptotic dependence condition (called (AD-2)) which along with conditions (AD-1) and (AN), ensure that the convergence of $(N_n)_{n\geq 1}$ *does* happen, for an arbitrary triangular array of random variables (not necessarily of the form $X_{j,n} = X_j / a_n$, with $a_n \sim n^{1/\alpha} L(n)$ for some $\alpha \in (0, 2)$ and a slowly varying function L). Under this new condition, we are able to find a new formulation for Kallenberg's necessary and sufficient condition for the convergence of $(N_n)_{n\geq 1}$, in terms of the incremental differences between the Laplace functionals of the processes $N_{m,n} = \sum_{j=1}^{m} \delta_{X_{j,n}}, m \leq n$.

The new condition (AD-2) is an "anti-clustering" condition, which does not allow the jump times of the partial sum process $S_n(t) = \sum_{j=1}^{[nt]} X_{j,n}, t \in [0,1]$, whose size exceed in modulus an arbitrary fixed value $\eta > 0$ (and are located at a minimum distance of m/n of each other), to get condensed in a "small" interval of time of length $r_n/n \sim k_n^{-1}$. This happens with a probability which is asymptotically 1, when n gets large and m either stabilizes around a finite value m_0, or gets large as well. From the mathematical point of view, condition (AD-2) treats the dependence structure *within* the blocks of length r_n, which was left open by condition (AD-1). Condition (AD-2) is automatically satisfied when the rows of the array are m-depedent. The asymptotic negligibility condition (AN) forces the rate of the convergence in probability to 0 of $X_{1,n}$ to be at most n^{-1}.

Of course, there are many instances in the literature in which the sequence $(N_n)_{n\geq 1}$ converges. The most notable example is probably the case of the moving average sequences $X_{i,n} = a_n^{-1} \sum_{j=0}^{\infty} C_{i,j} Z_{i-j}, i \geq 1$: the classical Theorem 2.4.(i), [9] treats the case of constant coefficients $C_{i,j} = c_j$, whereas the recent Theorem 3.1, [23] allows for random coefficients $C_{i,j}$. Another important example is given by Theorem 3.1, [28], in which $X_{j,n} = n^{-1/\alpha} X_j$ and $(X_j)_{j\geq 1}$ is a symmetric α-stable process.

With the convergence of the sequence $(N_n)_{n\geq 1}$ in hand, we can prove a general (non-stable) limit theorem for the partial sum sequence $(S_n)_{n\geq 1}$, and the hypothesis of this new theorem are indeed verified by the moving average sequences (even with random coefficients). The infinitely divisible law that we obtain as the limit of $(S_n)_{n\geq 1}$ must have a Lévy measure ρ which satisfies the condition $\int_0^1 x\rho(dx) < \infty$. This is a limitation which has to do with the method that we use, based on the Ferguson-Klass [15] representation of an infinitely divisible law. It remains an open problem to see how one could recover a general infinitely divisible law as the limit of $(S_n)_{n\geq 1}$, using point process techniques.

There is a large amount of literature dedicated to limit theorems for the partial sum sequence associated to a triangular array, based on point process techniques. For a comprehensive account on this subject in the independent case, we refer the reader to the expository article [25]. In the case of arrays which possess a row-wise dependence structure, the first systematic application of point process techniques for obtaining limit theorems for the sequence $(S_n)_{n\geq 1}$, has been developed in [20]. The article [20] identifies the necessary conditions for the general applicability of point process techniques, including cases which are not covered in the present article (e.g. the case of the α-stable limit distribution, with $\alpha \in [1, 2)$), and contains the first general limit theorem for sums of m-dependent random variables with heavy tails. Without aiming at exhausting the entire list of contributions to this area, we should also mention the article [22], which includes the necessary and sufficient conditions for the convergence in distribution of sums of m-dependent random variables, to a generalized Poisson distribution.

The present article is organized as follows. In Section 2, we introduce the asymptotic dependence conditions and we prove the main theorem which gives the convergence of the sequence $(N_n)_{n\geq 1}$ of point processes. Section 3 is dedicated to the convergence of the partial sum sequence $(S_n)_{n\geq 1}$. In Section 4, we give a direct consequence of the main theorem, which can be viewed as a complement of Theorem 2.3, [6], specifying some conditions which guarantee that the limit N of the sequence $(N_n)_{n\geq 1}$ exists (and admits a "product-type" cluster representation). Section 5 is dedicated to the analysis of condition (AD-1) in the case of an array whose row-wise dependence structure is given by the strong mixing property, the association, or is that of a stochastic volatility sequence. Appendix A gives a necessary and sufficient condition for a product-type cluster representation of a Poisson process. Appendix B gives a technical construction needed in the strongly mixing case.

2 Weak convergence of point processes

We begin by introducing the point process background. Our main references are [21], [26] and [27].

Let E be a locally compact Polish space, $\mathcal{E}$ its Borel σ-algebra and $\mathcal{B}$ the class of bounded Borel sets in $\mathcal{E}$. A measure μ on E is called Radon if $\mu(B) < \infty$ for all $B \in \mathcal{B}$. If $E = \mathbb{R}\backslash\{0\}$ or $E = (0, \infty)$, the class $\mathcal{B}$ contains the Borel sets in E which are bounded away from 0 and $\pm\infty$, respectively from 0 and ∞.

Let $M_p(E)$ be the class of all Radon measures on E such that $\mu(B) \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ for all $B \in \mathcal{B}$. The space $M_p(E)$ is endowed with the topology of vague convergence. The corresponding Borel σ-field is denoted by $\mathcal{M}_p(E)$. (Recall that a sequence $(\mu_n)_{n\geq 1} \subset M_p(E)$ converges vaguely to μ if $\mu_n(B) \to \mu(B)$ for any $B \in \mathcal{B}$ with $\mu(\partial B) = 0$.) For each $B_1, \ldots, B_k \in \mathcal{E}$, we define $\pi_{B_1,\ldots,B_k} : M_p(E) \to \mathbb{Z}_+^k$ by $\pi_{B_1,\ldots,B_k}(\mu) = (\mu(B_1), \ldots, \mu(B_k))$. We denote by δ_x the Dirac measure at $x \in E$, and by o the null measure in $M_p(E)$. For any $\mu \in M_p(E)$ and for any measurable non-negative function f on E, we let $\mu(f) = \int_E f(x)\mu(dx)$.

A point process N is an $M_p(E)$-valued random variable, defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Its Laplace functional is defined by $L_N(f) = E(e^{-N(f)})$, for any measurable non-negative function f on E. If N_1 and N_2 are two point processes on the same probability space, we use the notation $N_1 \stackrel{d}{=} N_2$ if $P \circ N_1^{-1} = P \circ N_2^{-1}$; this is equivalent to the fact that $L_{N_1}(f) = L_{N_2}(f)$, for any measurable non-negative function f on E.

If $N, (N_n)_{n\geq 1}$ are point processes, we say that $(N_n)_{n\geq 1}$ converges in distribution to N (and we write $N_n \stackrel{d}{\to} N$), if $\{P \circ N_n^{-1}\}_{n\geq 1}$ converges weakly to $P \circ N^{-1}$. By the continuous mapping theorem, if $N_n \stackrel{d}{\to} N$, then $\{h(N_n)\}_{n\geq 1}$ converges in distribution to $h(N)$, for every continuous function $h : M_p(E) \to \mathbb{R}$. By Theorem 4.2, [21], $(N_n)_{n\geq 1}$ converges in distribution to N if and only if $L_{N_n}(f) \to L_N(f), \forall f \in C_K^+(E)$, where $C_K^+(E)$ denotes the class of all continuous non-negative functions f on E, with compact support.

A point process N is said to be infinitely divisible if for every $n \geq 1$, there exists some i.i.d. point processes $N_1, \ldots, N_n$ such that $N \stackrel{d}{=} N_1 + \ldots + N_n$. By Theorem 6.1, [21], if N is an infinitely divisible process, then there exists a unique measure λ on $M_p(E)\backslash\{o\}$ (called the canonical measure of N) such that

$$\int_{M_p(E)\backslash\{o\}} (1 - e^{-\mu(B)})\lambda(d\mu) < \infty, \quad \forall B \in \mathcal{B}, \tag{1}$$

$$-\log L_N(f) = \int_{M_p(E)\backslash\{o\}} (1 - e^{-\mu(f)})\lambda(d\mu). \tag{2}$$

We begin to introduce our framework. For each $n \geq 1$, let $(X_{j,n})_{1\leq j\leq n}$ be a strictly stationary sequence of E-valued random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

We introduce our first asymptotic dependence condition. A similar condition was considered in [6].

Definition 2.1 *We say that the triangular array $(X_{j,n})_{1\leq j\leq n, n\geq 1}$ satisfies* **condition (AD-1)** *if there exists a sequence $(r_n)_n \subset \mathbb{Z}_+$ with $r_n \to \infty$ and $k_n = [n/r_n] \to \infty$ as $n \to \infty$, such that:*

$$\lim_{n\to\infty} \left| E\left(e^{-\sum_{j=1}^n f(X_{j,n})}\right) - \left\{E\left(e^{-\sum_{j=1}^{r_n} f(X_{j,n})}\right)\right\}^{k_n} \right| = 0, \quad \forall f \in C_K^+(E). \tag{3}$$

In Section 5, we will examine condition (AD-1) in the case of arrays which possess a well-known dependence structure on each row.

To see the intuitive meaning of condition (AD-1), let us consider the point process $N_{m,n} = \sum_{j=1}^{m} \delta_{X_{j,n}}$, whose Laplace functional is denoted by $L_{m,n}$, for each $m \leq n$. By convention, we let $L_{0,n} = 1$. We denote $N_n = N_{n,n}$. Note that $L_{m,n}(f) = E(e^{-N_{m,n}(f)}) = E(e^{-\sum_{j=1}^{m} f(X_{j,n})})$.

For each $n \geq 1$, let $(\tilde{N}_{i,n})_{1 \leq i \leq k_n}$ be a sequence of i.i.d. point processes with the same distribution as $N_{r_n,n}$, and let $\tilde{N}_n = \sum_{i=1}^{k_n} \tilde{N}_{i,n}$. Then $L_{\tilde{N}_n}(f) = \{L_{r_n,n}(f)\}^{k_n}$ and (3) becomes: $|L_{N_n}(f) - L_{\tilde{N}_n}(f)| \to 0, \forall f \in C_K^+(E)$. This shows that under (AD-1), the asymptotic behavior of the sequence $(N_n)_n$ is the same as that of $(\tilde{N}_n)_n$, i.e. $(N_n)_n$ converges in distribution if and only if $(\tilde{N}_n)_n$ does, and in this case, the limits are the same.

We now introduce an asymptotic negligibility condition, in probability.

Definition 2.2 *We say that* $(X_{j,n})_{j \leq n, n \geq 1}$ *satisfies* **condition (AN)** *if* $\limsup_{n \to \infty} nP(X_{1,n} \in B) < \infty, \forall B \in \mathcal{B}$.

Under (AN), the triangular array $(\tilde{N}_{i,n})_{1 \leq i \leq k_n,\ n \geq 1}$ becomes a "null-array", i.e. $P(\tilde{N}_{1,n}(B) > 0) \to 0$ for all $B \in \mathcal{B}$. To see this, note that $P(\tilde{N}_{1,n}(B) > 0) = P(\bigcup_{j=1}^{r_n} \{X_{j,n} \in B\}) \leq (n/k_n) P(X_{1,n} \in B) \to 0$. By invoking Theorem 6.1, [21], we infer that the sequence $(\tilde{N}_n)_{n \geq 1}$ (or equivalently, the sequence $(N_n)_{n \geq 1}$) converges in distribution to *some* point process N if and only if there exists a measure λ satisfying (1) such that

$$k_n(1 - L_{r_n,n}(f)) \to \int_{M_p(E) \setminus \{o\}} (1 - e^{-\mu(f)}) \lambda(d\mu), \quad \forall f \in C_K^+(E). \tag{4}$$

In this case, N is an infinitely divisible point process with canonical measure λ, i.e. (2) holds. By writing

$$L_{\tilde{N}_n}(f) = \{L_{r_n,n}(f)\}^{k_n} = \left\{1 - \frac{k_n(1 - L_{r_n,n}(f))}{k_n}\right\}^{k_n}$$

and using the fact that $(1 + x_n/n)^n \to e^x$ iff $x_n \to x$, we see that condition (4) requires that $L_{\tilde{N}_n}(f) \to L_N(f)$.

In conclusion, when dealing with triangular arrays which satisfy (AD-1) and (AN), the *only* possible limit (if it exists) for the sequence $(N_n)_{n \geq 1}$ of point processes is an infinitely divisible point process.

As in [10], if $(r_n)_n$ is an arbitrary sequence of positive integers with $r_n \to \infty$, we let $\mathcal{S} = \mathcal{S}_{(r_n)_n}$ be the set of all positive integers m such that: $\limsup_{n \to \infty} n \sum_{j=m+1}^{r_n} E[f(X_{1,n}) f(X_{j,n})] = 0,\ \forall f \in C_K^+(E)$. Let m_0 be the smallest integer in $\mathcal{S}$. By convention, we let $m_0 = \infty$ if $\mathcal{S} = \emptyset$. For an arbitrary function ϕ, we denote

$$\lim_{m \to m_0} \phi(m) = \begin{cases} \phi(m_0), & \text{if } m_0 < \infty \\ \lim_{m \to \infty} \phi(m), & \text{if } m_0 = \infty \end{cases}$$

We are now ready to introduce our second asymptotic dependence condition.

Definition 2.3 *We say that the triangular array* $(X_{j,n})_{1 \leq j \leq n, n \geq 1}$ *satisfies* **condition (AD-2)** *if there exists a sequence* $(r_n)_n \subset \mathbb{Z}_+$ *with* $r_n \to \infty$, *such that*

$$\lim_{m \to m_0} \limsup_{n \to \infty} n \sum_{j=m+1}^{r_n} E[f(X_{1,n}) f(X_{j,n})] = 0, \quad \forall f \in C_K^+(E).$$

Specifying the row-wise dependence structure of the array does not guarantee that condition (AD-2) is satisfied, but it may help to understand its meaning.

Example 2.4 (m-dependent random variables) Suppose that for every $n \geq 1$, $(X_{j,n})_{1 \leq j \leq n}$ are *m-dependent*, i.e. $(X_{1,n}, \ldots, X_{j,n})$ and $(X_{j+r,n}, X_{j+r+1,n}, \ldots, X_{n,n})$ are independent, for all $j, r \leq n$ with $j + r \leq n$ and $r \geq m$. Suppose that $(X_{j,n})_{1 \leq j \leq n, n \geq 1}$ satisfies condition (AN). Then for any sequence $(r_n)_n \subset \mathbb{Z}_+$ with $r_n \to \infty$, $\mathcal{S}_{(r_n)_n} = \{l \in \mathbb{Z}_+; l \geq m\}$, since for any $l \geq m$ and for any $f \in C_K^+(E)$,

$$n \sum_{j=l+1}^{r_n} E[f(X_{1,n}) f(X_{j,n})] = n r_n \{E[f(X_{1,n})]\}^2 \leq k_n r_n^2 \|f\|_\infty^2 P(X_{1,n} \in K)^2 \leq \frac{C}{k_n} \|f\|_\infty^2 \to 0.$$

(Here K is the compact support of f.) Therefore $m_0 = m$ and condition (AD-2) is satisfied. In particular, if $(X_{j,n})_{1 \leq j \leq n}$ are 1-dependent (or i.i.d.), then $\mathcal{S} = \mathbb{Z}_+$ and $m_0 = 1$.

Remark 2.5 The following slightly stronger form of condition (AD-2) has a clearer intuitive meaning. We say that the triangular array $(X_{j,n})_{1\leq j\leq n,n\geq 1}$ satisfies **condition (AD-2')** if

$$\lim_{m\to m_0}\limsup_{n\to\infty} n\sum_{j=m+1}^{r_n} P(X_{1,n}\in B, X_{j,n}\in B)=0, \quad \forall B\in\mathcal{B}.$$

Note that, due to the stationarity of the array, we have:

$$P\left(\bigcup_{i=1}^{r_n-m}\bigcup_{k=m+i}^{r_n}\{X_{i,n}\in B, X_{k,n}\in B\}\right)\leq r_n\sum_{j=m+1}^{r_n}P(X_{1,n}\in B, X_{j,n}\in B). \tag{5}$$

Therefore, if condition (AD-2') holds, then

$$\lim_{m\to m_0}\limsup_{n\to\infty} k_n P(\exists\ i<k\leq r_n \text{ with } k-i\geq m \text{ such that } X_{i,n}\in B, X_{k,n}\in B)=0.$$

In particular, if condition (AD-2') holds with $m_0=1$, then

$$k_n P(N_{r_n,n}(B)>1)=k_n P(\exists\ i<k\leq r_n \text{ such that } X_{i,n}\in B, X_{k,n}\in B)\to 0.$$

For each $B\in\mathcal{B}$ and for each $t\in[0,1]$, define $M_n^B([0,t])=N_{[nt],n}(B)$. Condition (AD-2') with $m_0=1$ forces

$$\lim_{n\to\infty}\frac{n}{r_n}P\left(M_n^B\left(\left[0,\frac{r_n}{n}\right]\right)>1\right)=0.$$

Intuitively, we can view this as an "asymptotic orderly" property of the sequence $(M_n^B)_n$. (According to p. 30, [4], a point process N is called *orderly* if $\lim_{t\to 0}t^{-1}P(N([0,t])>1)=0$.)

The following theorem gives a necessary and sufficient condition for the convergence in distribution of the sequence $(N_n)_n$. As mentioned earlier, the limit process must be an infinitely divisible point process.

As it was pointed out by an anonymous referee, our approach to identify the limit in the theorem below, is closely related to the method used in the proof of Theorem 3.1 of [19].

Theorem 2.6 *For each $n\geq 1$, let $(X_{j,n})_{1\leq j\leq n}$ be a strictly stationary sequence of E-valued random variables. Suppose that the triangular array $(X_{j,n})_{1\leq j\leq n,n\geq 1}$ satisfies condition (AN), as well as conditions (AD-1) and (AD-2) (with the same sequence $(r_n)_n$).*

Then the sequence $(N_n)_{n\geq 1}$ converges in distribution to some point process N if and only if there exists a measure λ on $M_p(E)\setminus\{0\}$ which satisfies (1), such that

$$\lim_{m\to m_0}\limsup_{n\to\infty}\left|n(L_{m-1,n}(f)-L_{m,n}(f))-\int_{M_p(E)\setminus\{0\}}(1-e^{-\mu(f)})\lambda(d\mu)\right|=0, \quad \forall f\in C_K^+(E). \tag{6}$$

In this case, N is an infinitely divisible point process with canonical measure λ.

In view of (4), we see that the second term appearing in the limit of (6) is the limit of $k_n(1-L_{r_n,n}(f))$. Since $n\sim r_n k_n$, the intuition behind condition (6) is that we are forcing $r_n(L_{m-1,n}(f)-L_{m,n}(f))$, to behave asymptotically as $1-L_{r_n,n}(f)=\sum_{m=1}^{r_n}(L_{m-1,n}(f)-L_{m,n}(f))$. In other words, the incremental differences $L_{m-1,n}(f)-L_{m,n}(f)$ with $1\leq m\leq r_n$, are forced to have the same asymptotic behavior as their average.

Proof: The proof of the theorem will follow from (4), once we show the following relation:

$$\lim_{m\to m_0}\limsup_{n\to\infty}|k_n(1-L_{r_n,n}(f))-n(L_{m-1,n}(f)-L_{m,n}(f))|=0, \quad \forall f\in C_K^+(E),$$

which can be expressed equivalently as follows, letting $h(x)=1-e^{-x}$:

$$\lim_{m\to m_0}\limsup_{n\to\infty} k_n|E[h(N_{r_n,n}(f))]-r_nE[h(N_{m,n}(f))-h(N_{m-1,n}(f))]|=0, \quad \forall f\in C_K^+(E). \tag{7}$$

In the remaining part of the proof we show that (7) holds. Note that only conditions (AD-2) and (AN), and the stationarity of the array, will be needed for this. We have

$$\begin{aligned} E[h(N_{r_n,n}(f))] &= E[h(N_{m-1,n}(f))] + \sum_{k=0}^{r_n-m} E[h(N_{k+m,n}(f)) - h(N_{k+m-1,n}(f))] \\ r_n E[h(N_{m,n}(f)) - h(N_{m-1,n}(f))] &= (m-1)E[h(N_{m,n}(f)) - h(N_{m-1,n}(f))] + \\ &\quad \sum_{k=0}^{r_n-m} E[h(N_{k+m,n}(f) - N_{k,n}(f)) - h(N_{k+m-1,n}(f) - N_{k,n}(f))], \end{aligned}$$

where the second equality is due to the strict stationarity of the sequence $(X_{j,n})_{1\le j\le n}$. Taking the difference between the previous two equalities, we get:

$$E[h(N_{r_n,n}(f))] - r_n E[h(N_{m,n}(f)) - h(N_{m-1,n}(f))] = mE[h(N_{m-1,n}(f))] - (m-1)E[h(N_{m,n}(f))]$$

$$+ \sum_{k=0}^{r_n-m} E\{[h(N_{k+m,n}(f)) - h(N_{k+m,n}(f) - N_{k,n}(f))] - [h(N_{k+m-1,n}(f)) - h(N_{k+m-1,n}(f) - N_{k,n}(f))]\}. \quad (8)$$

We now apply Taylor's expansion formula: $h(a) - h(a-b) = b\int_0^1 h'(a-xb)dx$. We get

$$\begin{aligned} h(N_{k+m,n}(f)) - h(N_{k+m,n}(f) - N_{k,n}(f)) &= N_{k,n}(f)\int_0^1 h'(N_{k+m,n}(f) - xN_{k,n}(f))dx \\ h(N_{k+m-1,n}(f)) - h(N_{k+m-1,n}(f) - N_{k,n}(f)) &= N_{k,n}(f)\int_0^1 h'(N_{k+m-1,n}(f) - xN_{k,n}(f))dx \end{aligned}$$

Taking the difference of the previous two equalities and applying Taylor's formula again, we obtain:

$$\begin{aligned} &E|[h(N_{k+m,n}(f)) - h(N_{k+m,n}(f) - N_{k,n}(f))] - [h(N_{k+m-1,n}(f)) - h(N_{k+m-1,n}(f) - N_{k,n}(f))]| \\ &\quad = E\left|N_{k,n}(f)\int_0^1 [h'(N_{k+m,n}(f) - xN_{k,n}(f)) - h'(N_{k+m-1,n}(f) - xN_{k,n}(f))]dx\right| \\ &\quad = E\left|N_{k,n}(f)(N_{k+m,n}(f) - N_{k+m-1,n}(f))\int_0^1 h''(\theta_{k,m,n}(x))dx\right| = E\left|N_{k,n}(f)f(X_{k+m,n})\int_0^1 h''(\theta_{k,m,n}(x))dx\right| \\ &\quad \le E[N_{k,n}(f)f(X_{k+m,n})], \qquad (9) \end{aligned}$$

where $\theta_{k,m,n}(x) \ge 0$ is a (random) value between $N_{k+m-1,n}(f) - xN_{k,n}(f)$ and $N_{k+m,n}(f) - xN_{k,n}(f)$, and we used the fact that $|h''(\theta)| = e^{-\theta} \le 1$ if $\theta \ge 0$. Coming back to (8), and using (9), we get:

$$\begin{aligned} k_n|E[h(N_{r_n,n}(f))] - r_n E[h(N_{m,n}(f)) - h(N_{m-1,n}(f))]| &\le mk_n E[h(N_{m-1,n}(f))] + (m-1)k_n E[h(N_{m,n}(f))] \\ &\quad + k_n \sum_{k=0}^{r_n-m} E[N_{k,n}(f)f(X_{k+m,n})]. \qquad (10) \end{aligned}$$

We claim that condition (AN) implies:

$$\lim_{n\to\infty} k_n E[h(N_{m,n}(f))] = 0, \quad \forall m \ge 1. \qquad (11)$$

To see this, we use the fact that $h(x) \le x$ if $x \ge 0$. If K is the (compact) support of f, then

$$k_n E[h(N_{m,n}(f))] \le k_n E\left[\sum_{j=1}^m f(X_{j,n})\right] = mk_n E[f(X_{1,n})] \le mk_n \|f\|_\infty P(X_{1,n} \in K) \le C\frac{m}{r_n}\|f\|_\infty \to 0.$$

On the other hand, by the stationarity, $k_n \sum_{k=0}^{r_n-m} E[N_{k,n}(f)f(X_{k+m,n})] = k_n \sum_{k=0}^{r_n-m} \sum_{i=1}^{k} E[f(X_{i,n})f(X_{k+m,n}) = k_n \sum_{j=m+1}^{r_n} (r_n - j + 1) E[f(X_{1,n})f(X_{j,n})] \leq n \sum_{j=m+1}^{r_n} E[f(X_{1,n})f(X_{j,n})]$. Hence, (AD-2) implies that:

$$\lim_{m \to m_0} \limsup_{n \to \infty} k_n \sum_{k=0}^{r_n - m} E[N_{k,n}(f) f(X_{k+m,n})] = 0. \tag{12}$$

Relation (7) follows from (10), (11) and (12). □

The next result shows that if conditions (AD-2) and (6) hold with $m_0 = 1$, then N is a Poisson process.

Proposition 2.7 *For each $n \geq 1$, let $(X_{j,n})_{1 \leq j \leq n}$ be a strictly stationary sequence of E-valued random variables. Suppose that the triangular array $(X_{j,n})_{1 \leq j \leq n, n \geq 1}$ satisfies conditions (AN) and (AD-1), as well as condition (AD-2) with $m_0 = 1$, i.e. $\limsup_{n \to \infty} n \sum_{j=2}^{r_n} E[f(X_{1,n})f(X_{j,n})] = 0$, $\forall f \in C_K^+(E)$.*

If there exists a measure λ on $M_p(E) \setminus \{0\}$ which satisfies (1), such that

$$\lim_{n \to \infty} n(1 - E(e^{-f(X_{1,n})}) = \int_{M_p(E) \setminus \{0\}} (1 - e^{-\mu(f)}) \lambda(d\mu), \quad \forall f \in C_K^+(E), \tag{13}$$

then $(N_n)_{n \geq 1}$ converges in distribution to a Poisson process with intensity $\nu(B) := \lambda(\{\mu \in M_p(E); \mu(B) = 1\})$.

Proof: By Theorem 2.6, $N_n \xrightarrow{d} N$, where N is an infinitely divisible process N with canonical measure λ.

For each $n \geq 1$, let $(X_{j,n}^*)_{1 \leq j \leq n}$ be an i.i.d. sequence with the same distribution as $X_{1,n}$. Let $N_n^* = \sum_{j=1}^{n} N_{j,n}^*$, where $N_{j,n}^* = \delta_{X_{j,n}^*}$. Then $(N_{j,n}^*)_{1 \leq j \leq n, n \geq 1}$ is a null-array, since $P(N_{1,n}^*(B) > 0) = P(X_{1,n} \in B) \to 0$ for all $B \in \mathcal{B}$. Note that $(N_{j,n}^*)_{1 \leq j \leq n}$ are i.i.d. point processes. By (13), we have:

$$\lim_{n \to \infty} \sum_{j=1}^{n} (1 - E(e^{-N_{j,n}^*(f)}) = \int_{M_p(E) \setminus \{0\}} (1 - e^{-\mu(f)}) \lambda(d\mu), \quad \forall f \in C_K^+(E).$$

Therefore, by Theorem 6.1, [21], it follows that $N_n^* \xrightarrow{d} N$, and $\{nP \circ [N_{1,n}^*(B_1), \ldots, N_{1,n}^*(B_k)]^{-1}\}_n$ converges weakly to $\lambda \circ \pi_{B_1, \ldots, B_k}^{-1}$, $\forall B_1, \ldots, B_k \in \mathcal{B}$. In particular, $nP(X_{1,n} \in B) = nP(N_{1,n}^*(B) = 1) \longrightarrow (\lambda \circ \pi_B^{-1})(\{1\}) = \nu(B)$, $\forall B \in \mathcal{B}$, and hence the sequence $\{nP \circ X_{1,n}^{-1}\}_{n \geq 1}$ converges vaguely to ν. Since λ satisfies (1), the measure ν is Radon. By Proposition 3.21, [26], it follows that $N_n^* \xrightarrow{d} N^*$, where N^* is a Poisson process of intensity ν. We conclude that $N \stackrel{d}{=} N^*$. □

3 Partial Sum Convergence

In this section we suppose that $E = (0, \infty)$. Let $N_n = \sum_{j=1}^{n} \delta_{X_{j,n}}$ and $S_n = \sum_{j=1}^{n} X_{j,n}$.

In Section 2, we have seen various asymptotic dependence conditions which guarantee the convergence in distribution of the sequence $(N_n)_{n \geq 1}$ to an infinitely divisible point process N. In the present section, we show that if the limit process N is "nice" (in a sense that will be specified below), this convergence, together with an asymptotic negligibility condition in the mean, implies the convergence in distribution of the partial sum sequence $(S_n)_n$ to an infinitely divisible random variable. In the literature, this has been a well-known recipe for obtaining the convergence in distribution of $(S_n)_n$ to the stable law (see e.g. [5], [12], [3], [6]). Our contribution consists in allowing the class of limiting distributions to include more general infinitely divisible laws.

Let N be an infinitely divisible point process on E, with canonical measure λ. By Lemma 6.5, [21], the distribution of N coincides with that of $\int_{M_p(E)} \mu \xi(d\mu)$, where ξ is a Poisson process on $M_p(E)$ with intensity λ. Let us denote by $N_i = \sum_{j \geq 1} \delta_{T_{ij}}, i \geq 1$ the points of ξ, i.e. $\xi = \sum_{i \geq 1} \delta_{N_i}$, Then the distribution of N coincides with that of $\sum_{i \geq 1} N_i = \sum_{i,j \geq 1} \delta_{T_{ij}}$. (This is called the "cluster representation" of N.)

The following assumption explains what we meant earlier by a "nice" point process N.

Assumption 3.1 *The canonical measure λ has the support contained in the set $M_p^*(E)$, consisting of all measures $\mu \in M_p(E)$ whose points are summable, i.e. all measures $\mu = \sum_{j \geq 1} \delta_{t_j}$ with $\sum_{j \geq 1} t_j < \infty$.*

We define the map $T : M_p^*(E) \to (0,\infty)$ by $T(\mu) = \sum_{j\geq 1} t_j$ if $\mu = \sum_{j\geq 1} \delta_{t_j}$.

Assumption 3.1 is equivalent to saying that $N_i \in M_p^*(E)$ a.s. In turn, this is equivalent to saying that the random variables $U_i := \sum_{j\geq 1} T_{ij}, i \geq 1$ are finite a.s. Moreover, we have the following result.

Lemma 3.2 *Let N be a point process on $(0,\infty)$ with canonical measure λ, and the cluster representation: $N \stackrel{d}{=} \int_{M_p(E)} \mu \xi(d\mu) = \sum_{i\geq 1} N_i = \sum_{j\geq 1} \delta_{T_{ij}}$. (Here $\xi = \sum_{i\geq 1} \delta_{N_i}$ is a Poisson process on $M_p(E)$ with intensity λ, and $N_i = \sum_{j\geq 1} \delta_{T_{ij}}, i \geq 1$ are the points of ξ.)*

Suppose that λ satisfies Assumption 3.1, and set $U_i := \sum_{j\geq 1} T_{ij}, i \geq 1$. Then $N^* := \sum_{i\geq 1} \delta_{U_i}$ is a Poisson process with intensity $\rho := \lambda \circ T^{-1}$, i.e. $\rho(A) = \lambda(\{\mu = \sum_{j\geq 1} \delta_{t_j} \in M_p^*(E); \sum_{j\geq 1} t_j \in A\}), \forall A \in \mathcal{B}((0,\infty))$.

Proof: The lemma will be proved, once we show that for any measurable $f : E \to (0,\infty)$, we have

$$E\left(e^{-\sum_{i\geq 1} f(U_i)}\right) = \exp\left\{-\int_0^\infty (1 - e^{-f(x)})\rho(dx)\right\}.$$

Since $\xi = \sum_{i\geq 1} \delta_{N_i}$ is a Poisson process with intensity λ, for any $\psi : M_p(E) \to (0,\infty)$ measurable, we have $L_\xi(\psi) = E\left(e^{-\sum_{i\geq 1} \psi(N_i)}\right) = \exp\left\{-\int_{M_p^*(E)} (1 - e^{-\psi(\mu)})\lambda(d\mu)\right\}$. Let $\psi_f : M_p^*(E) \to (0,\infty)$ be given by $\psi_f(\mu) = f(T(\mu))$. Then $\psi_f(N_i) = f(T(N_i)) = f(\sum_{j\geq 1} T_{ij}) = f(U_i)$ and

$$E\left(e^{-\sum_{i\geq 1} f(U_i)}\right) = \exp\left\{-\int_{M_p^*(E)} (1 - e^{-\psi_f(\mu)})\lambda(d\mu)\right\} = \exp\left\{-\int_0^\infty (1 - e^{-y})(\lambda \circ \psi_f^{-1})(dy)\right\}.$$

Note that $\rho = \lambda \circ T^{-1}$. By the definitions of ψ_f and ρ, we have $\lambda \circ \psi_f^{-1} = \lambda \circ T^{-1} \circ f^{-1} = \rho \circ f^{-1}$. Hence $E\left(e^{-\sum_{i\geq 1} f(U_i)}\right) = \exp\left\{-\int_0^\infty (1 - e^{-y})(\rho \circ f^{-1})(dy)\right\} = \exp\left\{-\int_0^\infty (1 - e^{-f(x)})\rho(dx)\right\}$. $\Box$

The next lemma is of general interest and shows that the random variable X defined as the sum of the points of a Poisson process on $(0,\infty)$ has an infinitely divisible distribution. To ensure that X is finite a.s., some restrictions apply to the intensity ρ of the Poisson process. Recall that a measure ρ on $(0,\infty)$ is called a *Lévy measure* if $\int_{(0,1]} x^2 \rho(dx) < \infty$ and $\rho((1,\infty)) < \infty$.

Lemma 3.3 *Let $N^* = \sum_{i\geq 1} \delta_{U_i}$ be a Poisson process on $(0,\infty)$, whose intensity ρ is a Lévy measure and*

$$\int_{(0,1]} x\rho(dx) < \infty. \tag{14}$$

Then the random variable $X = \sum_{i\geq 1} U_i$ is finite a.s. and has an infinitely divisible distribution and

$$E(e^{iuX}) = \exp\left\{\int_0^\infty (e^{iux} - 1)\rho(dx)\right\}, \quad \forall u \in \mathbb{R}. \tag{15}$$

Proof: Without loss of generality, we can assume that $U_i = H_\rho^{-1}(\Gamma_i)$, where $\Gamma_i = \sum_{j=1}^i E_j$, $(E_j)_{j\geq 1}$ are i.i.d. Exponential(1) random variables, $H_\rho(x) = \rho(x,\infty)$, and $H_\rho^{-1}(y) = \inf\{x > 0; H_\rho(x) \leq y\}$.

Note that H_ρ is a non-increasing function and $H_\rho^{-1}(y) \leq x$ if and only if $y \geq H_\rho(x)$. Then $U_i \leq U_{i-1}, \forall i$ and

$$\begin{aligned} &P(U_i \leq x_i | U_1 = x_1, \ldots, U_{i-1} = x_{i-1}) = P(\Gamma_i \geq H_\rho(x_i) | \Gamma_1 = H_\rho(x_1), \ldots, \Gamma_{i-1} = H_\rho(x_{i-1})) \\ &= P(E_i \geq H_\rho(x_i) - H_\rho(x_{i-1}) | \Gamma_1 = H_\rho(x_1), \ldots, \Gamma_{i-1} = H_\rho(x_{i-1})) \\ &= P(E_i \geq H_\rho(x_i) - H_\rho(x_{i-1})) = e^{-(H_\rho(x_i) - H_\rho(x_{i-1}))} \quad \text{for all } x_i \leq x_{i-1} \leq \ldots \leq x_1 \end{aligned} \tag{16}$$

Relation (16) allows us to invoke a powerful (and highly non-trivial) construction, due to Ferguson and Klass (see [15]). More precisely, let $(V_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with values in $[0,1]$ and common distribution G, which is independent of $(U_i)_{i\geq 1}$, and define $Y_t = \sum_{i\geq 1} U_i 1_{\{V_i \leq t\}}, t \in [0,1]$. Then, Ferguson and Klass showed that $(Y_t)_{t\in[0,1]}$ is a Lévy process with characteristic function $E(e^{iuY_t}) = \exp\left\{G(t)\int_0^\infty (e^{iux} - 1)\rho(dx)\right\}$, $\forall u \in \mathbb{R}$. The proof is complete by observing that $X = Y_1 = \sum_{i\geq 1} U_i$. $\Box$

Example 3.4 $\rho(dx) = \alpha x^{-1} e^{-x} 1_{\{x>0\}} dx$ with $\alpha > 0$. In this case, X has a Gamma(α) distribution.

Example 3.5 $\rho(dx) = c_\alpha x^{-\alpha-1} 1_{\{x>0\}} dx$ with $\alpha \in (0,1)$. In this case, X has a stable distribution of index α.

As a by-product of the previous lemma, we obtain a representation of an infinitely divisible distribution, similar to the LePage-Woodroofe-Zinn representation of the stable law (Theorem 2, [24]). The proof of this corollary is based on a representation of a Poisson process, which is included in Appendix A.

Corollary 3.6 *Let ρ be a measure on $(0,\infty)$, which is given by the following "product-convolution" type formula:*

$$\rho(A) = \int_0^\infty \int_0^\infty 1_A(wy) F(dw) \nu(dy), \quad \forall A \in \mathcal{B}((0,\infty)), \tag{17}$$

where ν is an arbitrary Radon measure ν on $(0,\infty)$ and F is an arbitrary probability measure on $(0,\infty)$.

If the measure ρ is Lévy and satisfies (14), then any infinitely divisible random variable X with characteristic function (15) admits the representation $X \stackrel{d}{=} \sum_{i\geq 1} P_i W_i$, where $(P_i)_{i\geq 1}$ are the points of a Poisson process of intensity ν, and $(W_i)_{i\geq 1}$ is an independent i.i.d. sequence with distribution F.

Proof: Let N^* be a Poisson process on $(0,\infty)$, of intensity ρ. Using the definition (17) of ρ, and by invoking Proposition A.1 (Appendix A) , it follows that N^* admits the representation $N^* \stackrel{d}{=} \sum_{i\geq 1} \delta_{P_i W_i}$, where $(P_i)_{i\geq 1}$ and $(W_i)_{i\geq 1}$ are as in the statement of the corollary. By Lemma 3.3, it follows that the random variable $X := \sum_{i\geq 1} P_i W_i$ has an infinitely divisible distribution with characteristic function (15). $\square$

Remark 3.7 If we let $\nu(dx) = \alpha x^{-\alpha-1} 1_{\{x>0\}} dx$ and F be an arbitrary probability measure F on $(0,\infty)$, then the measure ρ given by (17) satisfies:

$$\rho(x,\infty) = \int_0^\infty \int_0^\infty 1_{(x,\infty)}(wy) F(dw)\nu(dy) = \int_0^\infty \int_0^\infty \nu\left(\frac{x}{w},\infty\right) F(dw) = x^{-\alpha} \int_0^\infty w^\alpha F(dw) = \gamma_\alpha x^{-\alpha},$$

where $\gamma_\alpha = \int_0^\infty w^\alpha F(dw)$. Hence $\rho(dx) = \alpha \gamma_\alpha x^{-\alpha-1} 1_{\{x>0\}} dx$.

To obtain the convergence of the partial sum sequence, we introduce a new asymptotic negligibility condition.

Definition 3.8 $(X_{j,n})_{1\leq j\leq n, n\geq 1}$ *satisfies* **condition (AN')** *if* $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} nE[X_{1,n} 1_{\{X_{1,n}\leq \varepsilon\}}] = 0$.

The next theorem is a generalization of Theorem 3.1, [6], to the case of an arbitrary infinitely divisible law (without Gaussian component, and whose Lévy measure ρ satisfies (14)), as the limiting distribution of $(S_n)_n$.

Theorem 3.9 *For each $n \geq 1$, let $(X_{j,n})_{1\leq j\leq n}$ be a strictly stationary sequence of positive random variables. Suppose that the array $(X_{j,n})_{1\leq j\leq n, n\geq 1}$ satisfies condition (AN'). Let $N_n = \sum_{j=1}^n \delta_{X_{j,n}}$ and $S_n = \sum_{j=1}^n X_{j,n}$.*

If

(i) $N_n \stackrel{d}{\to} N$, where N is an infinitely divisible point process, whose canonical measure λ satisfies Assumption 3.1; and

(ii) $\rho := \lambda \circ T^{-1}$ is a Lévy measure and satisfies (14),

then $(S_n)_n$ converges in distribution to an infinitely divisible random variable with characteristic function (15).

Proof: For each $\varepsilon > 0$ arbitrary, we write

$$S_n = S_n(\varepsilon,\infty) + S_n(0,\varepsilon). \tag{18}$$

where $S_n(\varepsilon,\infty) = \sum_{j=1}^n X_{j,n} 1_{\{X_{j,n}>\varepsilon\}}$ and $S_n(0,\varepsilon) = \sum_{j=1}^n X_{j,n} 1_{\{X_{j,n}\leq\varepsilon\}}$.

Define $T_\varepsilon : \mathcal{M}^* \to (0,\infty)$ by $T_\varepsilon(\mu = \sum_{j\geq 1} \delta_{t_j}) = \sum_{j\geq 1} t_j 1_{\{t_j>\varepsilon\}}$. Note that T_ε is continuous $P \circ N^{-1}$-a.s. By the continuous mapping theorem, we get $T_\varepsilon(N_n) = S_n(\varepsilon,\infty) \stackrel{d}{\to} T_\varepsilon(N) = \sum_{i,j\geq 1} T_{ij} 1_{\{T_{ij}>\varepsilon\}}$, as $n \to \infty$. Since $X = \sum_{i,j\geq 1} T_{ij}$ converges a.s., it follows that $\sum_{i,j\geq 1} T_{ij} 1_{\{T_{ij}>\varepsilon\}} \stackrel{a.s}{\to} X = \sum_{i,j\geq 1} T_{ij}$ as $\varepsilon \to 0$. Hence

$$S_n(\varepsilon,\infty) \stackrel{d}{\to} X \quad \text{as } n\to\infty, \varepsilon\to 0. \tag{19}$$

By Markov's inequality and condition (AN'), we see that for any $\delta > 0$, $P(S_n(0,\varepsilon) > \delta) \leq \delta^{-1}E[S_n(0,\varepsilon)] = \delta^{-1}nE[X_{1,n}1_{\{X_{1,n}\leq\varepsilon\}}] \to 0$, as $n\to\infty, \varepsilon\to 0$. Hence

$$S_n(0,\varepsilon) \xrightarrow{P} 0 \quad \text{as } n\to\infty, \varepsilon\to 0. \tag{20}$$

From (18), (19) and (20), we conclude that $S_n \xrightarrow{d} X$.

By Lemma 3.3, $X = \sum_{i,j\geq 1} T_{ij} = \sum_{i\geq 1} U_i$ has an infinitely divisible distribution and (15) holds. □

Remark 3.10 Lemma 3.3 can be extended to a Poisson process whose intensity ρ is an arbitrary Lévy measure ρ on $(0,\infty)$. More precisely, using the Theorem of [15], one can prove that if $N^* = \sum_{i\geq 1}\delta_{U_i}$ is a Poisson process on $(0,\infty)$, whose intensity ρ is a Lévy measure (i.e. it satisfies the condition $\int_0^\infty x^2/(1+x^2)\rho(dx) < \infty$), then the random variable $Y = \sum_{i\geq 1}(U_i - c_i)$ is finite a.s. and has an infinitely divisible distribution with

$$E(e^{iuY}) = \exp\left\{\int_0^\infty \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)\rho(dx)\right\}, \quad \forall u \in \mathbb{R},$$

where the constants c_i are defined by: $c_i = \int_{H_\rho^{-1}(i)}^{H_\rho^{-1}(i-1)} x/(1+x^2)\rho(dx)$. If $\gamma = \sum_{i\geq 1} c_i = \int_0^\infty x/(1+x^2)\rho(dx)$ is finite, then one can conclude that the random variable $X = \sum_{i\geq 1} U_i = Y + \gamma$ (which appears in Theorem 3.9) has an infinitely divisible distribution with characteristic function

$$E(e^{iuX}) = \exp\left\{iu\gamma + \int_0^\infty \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)\rho(dx)\right\}, \quad \forall u \in \mathbb{R}.$$

Unfortunately, requiring that γ is finite is equivalent to saying that $\int_{(0,1]} x\rho(dx) < \infty$, which is precisely the restriction imposed on ρ in Lemma 3.3. In other words, condition (14) cannot be removed from Theorem 3.9, using the Ferguson and Klass approach.

We finish this section with an example for which the hypothesis of Theorem 3.9 are verified. This example is based on the recent work [23], generalizing the moving average model MA(∞) to the case of random coefficients.

Example 3.11 (Linear processes with random coefficients) Let $X_{i,n} = X_i/a_n$ for all $1 \leq i \leq n$, where

$$X_i = \sum_{j=0}^{\infty} C_{i,j} Z_{i-j} \quad \text{for all } i \geq 1.$$

The objects $(Z_k)_{k\in\mathbb{Z}}$, $(a_n)_{n\geq 1}$ and $(C_{i,j})_{i\geq 1, j\geq 0}$ are defined as follows:

- $(Z_k)_{k\in\mathbb{Z}}$ is a sequence of i.i.d. positive random variables such that $Z_0 \stackrel{d}{=} Z$, where Z has heavy tails, i.e. $P(Z > x) = x^{-\alpha}L(x)$ for $\alpha \in (0,2)$ and L a slowly varying function.
- $(a_n)_{n\geq 1}$ is a non-decreasing sequence of positive numbers such that $P(Z > a_n) \sim n^{-1}$.
- $(C_{i,j})_{i\geq 1, j\geq 0}$ is an array of positive random variables, which are independent of $(Z_k)_{k\in\mathbb{Z}}$. We suppose that the rows $(C_{1,j})_{j\geq 0}, (C_{2,j})_{j\geq 0}, \ldots$ of this array are i.i.d. copies of a sequence $(C_j)_{j\geq 0}$ of positive random variables. Moreover, we suppose that the sequence $(C_j)_{j\geq 0}$ satisfies certain moment conditions, which imply that $c := \sum_{j=0}^\infty E[C_j^\alpha] < \infty$. (We refer the reader to condition (D) of [23] for the exact moment conditions. In fact, we may allow for a mixing-type dependence structure between the rows.)

Proposition 2.1, [23] shows that $P(X_1 > x) \sim cP(Z > x)$ as $x \to \infty$. Since Z has heavy tails, it follows that X_1 has heavy tails too. Assume that $\alpha \in (0,1)$. In this case, one can prove that: (see e.g. (3.6) in [6])

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{n}{a_n} E[X_1 1_{\{X_{1,n} \leq a_n\varepsilon\}}] = 0,$$

i.e. the array $(X_{j,n})_{1\leq j\leq n, n\geq 1}$ satisfies condition (AN').

Let $N_n = \sum_{i=1}^{n} \delta_{X_{i,n}}$. By Theorem 3.1, [23], $N_n \xrightarrow{d} N$, where N is an infinitely divisible point process with the cluster representation $N \stackrel{d}{=} \sum_{i\geq 1}\sum_{j\geq 0} \delta_{P_i C_{i,j}}$. Here $(P_i)_{i\geq 1}$ are the points of a Poisson process of intensity $\nu(dx) = \alpha x^{-\alpha-1} 1_{\{x>0\}} dx$, which is independent of the array $(C_{i,j})_{i\geq 1, j\geq 0}$. Since $\alpha \in (0,1)$, it follows that $W_i := \sum_{j\geq 0} C_{i,j} < \infty$ a.s. for all $i \geq 1$. Hence, the random variables $U_i := \sum_{j\geq 0} P_i C_{i,j} = P_i W_i, i \geq 1$ are finite a.s. and Assumption 3.1 is verified. This proves that condition (i) in Theorem 3.9 is satisfied.

By Lemma 3.2, the process $N^* := \sum_{i\geq 1} \delta_{U_i} = \sum_{i\geq 1} \delta_{P_i W_i}$ is a Poisson process of intensity $\rho := \lambda \circ T^{-1}$, where λ is the canonical measure of N. From Proposition A.1 (Appendix A), it follows that ρ satisfies (17). By Remark 3.7, it follows that $\rho(dx) = \alpha\gamma_\alpha x^{-\alpha-1} 1_{\{x>0\}}$, where $\gamma_\alpha = \int_0^\infty w^\alpha F(dw)$. Clearly, this measure ρ is Lévy; it satisfies condition (14) since $\alpha < 1$. This proves that condition (ii) in Theorem 3.9 is satisfied.

By applying Theorem 3.9, it follows that $(S_n)_{n\geq 1}$ converges in distribution to an infinitely divisible law with characteristic function (15), which is in fact the stable law of index α.

4 Real Valued Observations

In this section, we assume that $E = \mathbb{R}\backslash\{0\}$. By Lemma 2.1, [6], the support of the canonical measure λ of an infinitely divisible point process on E, is contained in the set $M_0(E)$, defined by:

$$M_0(E) = \{\mu = \sum_{j\geq 1} \delta_{t_j} \in M_p(E);\ \exists\ x_\mu \in (0,\infty) \text{ such that } |t_j| \leq x_\mu \forall j \geq 1\}.$$

Let $\tilde{M}(E) = \{\mu \in M_0(E); |t_j| \leq 1, \forall j \geq 1\}$. The following result gives the necessary and sufficient condition for a "product-type" cluster representation of an infinitely divisible process, as in Corollary 2.4, [6].

Proposition 4.1 *Let N be an infinitely divisible point process on $E = \mathbb{R}\backslash\{0\}$, with canonical measure λ. Then $N \stackrel{d}{=} \sum_{i,j\geq 1} \delta_{P_i Q_{ij}}$, where $(P_i)_{i\geq 1}$ are the points of a Poisson process on $(0,\infty)$ of intensity ν and $(Q_{1j})_{j\geq 1}, (Q_{2j})_{j\geq 1}, \ldots$ are i.i.d. sequences, independent of $(P_i)_{i\geq 1}$, if and only if there exists a probability measure $\mathcal{O}$ on $\tilde{M}(E)$ such that, for every measurable non-negative function f on E, we have*

$$\int_{M_0(E)} (1 - e^{-\mu(f)})\lambda(d\mu) = \int_0^\infty \int_{\tilde{M}(E)} (1 - e^{-\mu(f(y\cdot))})\mathcal{O}(d\mu)\nu(dy).$$

In this case, $\mathcal{O}$ is the distribution of $\sum_{j\geq 1} \delta_{Q_{1j}}$.

Proof: Let $N' = \sum_{i,j\geq 1} \delta_{P_i Q_{ij}}$. Clearly, $L_N(f) = \exp\left\{-\int_{M_0(E)} (1 - e^{-\mu(f)})\lambda(d\mu)\right\}$. Following the same lines as for the proof of (41) (Appendix A), one can show that $L_{N'}(f) = \exp\left\{-\int_0^\infty \int_{\tilde{M}(E)} (1 - e^{-\mu(f(y\cdot))})\mathcal{O}(d\mu)\nu(dy)\right\}$. The result follows since $N \stackrel{d}{=} N'$ if and only if $L_N(f) = L_{N'}(f)$ for every measurable function f. $\Box$

In the light of Proposition 4.1, the following result becomes an immediate consequence of Theorem 2.6.

Corollary 4.2 *For each $n \geq 1$, let $(X_{j,n})_{1\leq j\leq n}$ be a strictly stationary sequence of random variables with values in $E = \mathbb{R}\backslash\{0\}$. Suppose that the array $(X_{j,n})_{1\leq j\leq n, n\geq 1}$ satisfies conditions (AD-1), (AD-2), and (AN).*

If there exists a Radon measure ν on $(0,\infty)$ and a probability measure $\mathcal{O}$ on $\tilde{M}(E)$, such that

$$\lim_{m\to m_0} \limsup_{n\to\infty} \left| n\left(L_{m-1,n}(f) - L_{m,n}(f)\right) - \int_0^\infty \int_{\tilde{M}(E)} (1 - e^{-\mu f(y\cdot)})\mathcal{O}(d\mu)\nu(dy) \right| = 0, \quad \forall f \in C_K^+(E), \qquad (21)$$

then $N_n \xrightarrow{d} N$, where $N = \sum_{i,j\geq 1} \delta_{P_i Q_{ij}}$, $(P_i)_{i\geq 1}$ are the points of a Poisson process on $(0,\infty)$ of intensity ν, and $(Q_{1j})_{j\geq 1}, (Q_{2j})_{j\geq 1}, \ldots$ are i.i.d. sequences with distribution $\mathcal{O}$, independent of $(P_i)_{i\geq 1}$.

Remark 4.3 In particular, one may restate Corollary 4.2, in the case $X_{j,n} = X_j/a_n$, where $(X_j)_{j\geq 1}$ is a strictly stationary sequence of random variables with values in $\mathbb{R}\backslash\{0\}$ such that X_1 has heavy tails, and $(a_n)_{n\geq 1}$ satisfies $P(X_1 > a_n) \sim n^{-1}$. The result obtained in this manner can be viewed as a complement to Theorem 2.3, [6].

Recall that a bounded Borel set in $\mathbb{R}\backslash\{0\}$ is bounded away from 0. Therefore, condition (AN) holds if $\limsup_{n\to\infty} nP(|X_{1,n}| \geq \eta) < \infty, \forall \eta > 0$. Note also that condition (AD-2') holds if

$$\lim_{m\to m_0} \limsup_{n\to\infty} n \sum_{j=m+1}^{r_n} P(|X_{1,n}| \geq \eta, |X_{j,n}| \geq \eta) = 0, \quad \forall \eta > 0. \tag{22}$$

Condition (22) can be viewed as an asymptotic"anti-clustering" condition for the process $S_n(t) = \sum_{j=1}^{[nt]} X_{j,n}, t \in [0,1]$. To see this, note that this cadlag process jumps at times $t_j = j/n$ with $1 \leq j \leq n$, the respective jump heights being $\Delta S_n(t_j) = X_{j,n}$. By (5), we have

$$P(\exists i < k \leq r_n \text{ with } k - i \geq m \text{ such that } |\Delta S_n(t_i)| \geq \eta, |\Delta S_n(t_k)| \geq \eta) \leq r_n \sum_{j=m+1}^{r_n} P(|X_{1,n}| \geq \eta, |X_{j,n}| \geq \eta).$$

Therefore, we can explain intuitively condition (22) by saying that the chance that the process $(S_n(t))_{t\in[0,1]}$ has at least two jumps that exceed η in the time interval $[0, r_n/n]$ (and are located at a minimum distance of m/n of each other) is asymptotically zero. (See also p. 213, [13].)

5 Examples of arrays satisfying (AD-1)

In this section, we examine condition (AD-1) in the case of arrays which possess a known dependence structure on each row.

5.1 m-dependent or strongly mixing sequences

Recall that the m-th order mixing coefficient of a sequence $(X_j)_{j\geq 1}$ of random variables is defined by:

$$\alpha(m) = \sup_{k\geq 1} \sup\{|P(A \cap B) - P(A)P(B)|; A \in \sigma(X_1, \ldots, X_k), B \in \sigma(X_{k+m}, X_{k+m+1}, \ldots)\}.$$

The random variables $(X_j)_{j\geq 1}$ are called *strongly mixing* if $\lim_{m\to\infty} \alpha(m) = 0$.

If X is a $\sigma(X_1, \ldots, X_k)$-measurable bounded random variable and Y is a $\sigma(X_{k+m}, X_{k+m+1}, \ldots)$-measurable bounded random variable, then: (see e.g. [17])

$$|E(XY) - E(X)E(Y)| \leq 4\alpha(m)\|X\|_\infty \|Y\|_\infty. \tag{23}$$

Lemma 5.1 *For each $n \geq 1$, let $(X_{j,n})_{1\leq j\leq n}$ be a strictly stationary sequence of random variables and $\alpha_n(m)$ be its m-th order mixing coefficient, for $m < n$. Suppose that either*
(i) $\alpha_n(m') = 0$ for all $m' \geq m$, $n \geq 1$; or
(ii) $\alpha_n(m) = \alpha(m) \quad \forall n > m, \forall m \geq 1$ and $\lim_{m\to\infty} \alpha(m) = 0$.
If the triangular array $(X_{j,n})_{1\leq j\leq n, n\geq 1}$ satisfies condition (AN), then it also satisfies condition (AD-1).

Remark 5.2 a) Condition (i) requires that the sequence $(X_{j,n})_{1\leq j\leq n}$ is m-dependent, for any $n \geq 1$.
b) Condition (ii) is satisfied if $X_{j,n} = X_j/a_n$ and $(X_j)_{j\geq 1}$ a strictly stationary strongly mixing sequence.

Proof: We want to prove that there exists a sequence $(r_n)_n \to \infty$ with $k_n = [n/r_n] \to \infty$ such that

$$E(e^{-N_n(f)}) - \{E(e^{-N_{r_n,n}(f)})\}^{k_n} \to 0, \quad \forall f \in C_K^+(E). \tag{24}$$

Note that $e^{-N_{k_n r_n,n}(f)} - e^{-N_n(f)} = e^{-N_{k_n r_n,n}(f)}(1 - e^{-\sum_{j=k_n r_n+1}^{n} f(X_{j,n})}) \leq \sum_{j=k_n r_n+1}^{n} f(X_{j,n})$, using the fact that $1 - e^{-x} \leq x$ for any $x \geq 0$. By stationarity and condition (AN), we obtain that:

$$\left|E(e^{-N_n(f)}) - E(e^{-N_{k_n r_n,n}(f)})\right| \leq (n - r_n k_n)E[f(X_{1,n})] \leq r_n \|f\|_\infty P(X_{1,n} \in K) \leq \frac{1}{k_n}\|f\|_\infty C \to 0,$$

where K is the compact support of f. Therefore, in order to prove (24), it is enough to show that

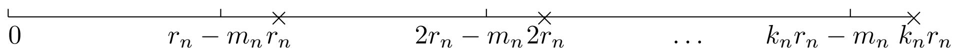
$$E(e^{-N_{k_n r_n,n}(f)}) - \{E(e^{-N_{r_n,n}(f)})\}^{k_n} \to 0. \tag{25}$$

To prove (25), we will implement Jakubowski's "block separation" technique (see the proof of Proposition 5.2, [18], for a variant of this technique). Let $(m_n)_n$ be a sequence of positive integers such that

$$m_n \to \infty, \quad m_n/r_n \to 0, \quad \text{and} \quad k_n \alpha(m_n) \to 0. \tag{26}$$

(The construction of sequences $(r_n)_n$ and $(m_n)_n$ which satisfy (26) is given in Appendix B.)

For each $n \geq 1$, we consider k_n blocks of consecutive integers of length $r_n - m_n$, separated by "small" blocks of length m_n:



More precisely, for each $1 \leq i \leq k_n$, let $H_{i,n}$ be the (big) block of consecutive integers between $(i-1)r_n + 1$ and $ir_n - m_n$ and $I_{i,n}$ be the (small) block of size m_n, consisting of the integers between $ir_n - m_n$ and ir_n. Let

$$U_{i,n} = \sum_{j \in H_{i,n}} f(X_{j,n}) = N_{ir_n - m_n,n}(f) - N_{(i-1)r_n,n}(f).$$

By the stationarity of the array, $(U_{i,n})_{1 \leq i \leq k_n}$ are identically distributed. Clearly, $U_{1,n} = N_{r_n - m_n,n}(f)$.

On the other hand, since the separation blocks have size m_n, which is "relatively small" compared to r_n,

$$\lim_{n\to\infty} |E(e^{-N_{k_n r_n,n}(f)}) - E(e^{-\sum_{i=1}^{k_n} U_{i,n}})| = 0 \tag{27}$$

$$\lim_{n\to\infty} |\{E(e^{-N_{r_n,n}(f)})\}^{k_n} - \{E(e^{-U_{1,n}})\}^{k_n}| = 0. \tag{28}$$

(To prove (27), note that $e^{-\sum_{i=1}^{k_n} U_{i,n}} - e^{-N_{k_n r_n,n}(f)} = e^{-\sum_{i=1}^{k_n} U_{i,n}}(1 - e^{-\sum_{i=1}^{k_n}\sum_{j \in I_{i,n}} f(X_{j,n})}) \leq \sum_{i=1}^{k_n}\sum_{j\in I_{i,n}} f(X_{j,n})$, using the fact that $1 - e^{-x} \leq x$ for any $x \geq 0$. Hence $|E(e^{-N_{k_n r_n,n}(f)}) - E(e^{-\sum_{i=1}^{k_n} U_{i,n}^{(m)}})| \leq m_n k_n E[f(X_{1,n})] \leq m_n k_n \|f\|_\infty P(X_{1,n} \in K) \leq (m_n/r_n)\|f\|_\infty C \to 0$, where we used condition (AN) and (26). Relation (28) follows by a similar argument, using the fact that $|x^k - y^k| \leq k|x-y|$ for any $x, y \geq 0$.)

Therefore, in order to prove (25), it suffices to show that:

$$\lim_{n\to\infty} |E(e^{-\sum_{i=1}^{k_n} U_{i,n}}) - \{E(e^{-U_{1,n}})\}^{k_n}| = 0. \tag{29}$$

In case (i), this follows immediately since the random variables $(U_{i,n})_{1 \leq i \leq k_n}$ are independent, for n large.

In case (ii), we claim that, for any $1 \leq k \leq k_n$ we have

$$|E(e^{-\sum_{i=1}^{k} U_{i,n}}) - \{E(e^{-U_{1,n}})\}^k| \leq 4(k-1)\alpha(m_n). \tag{30}$$

(Relation (30) can be proved by induction on the number k of terms. If $k = 2$, then (30) follows from inequality (23). If relation (30) holds for $k-1$, then $|E(e^{-\sum_{i=1}^{k} U_{i,n}}) - \{E(e^{-U_{1,n}})\}^k| \leq |E(e^{-\sum_{i=1}^{k} U_{i,n}}) - E(e^{-\sum_{i=1}^{k-1} U_{i,n}})E(e^{-U_{k,n}})| + |E(e^{-\sum_{i=1}^{k-1} U_{i,n}}) - \{E(e^{-U_{1,n}})\}^{k-1}|$. For the first term we use (23), since $\sum_{i=1}^{k-1} U_{i,n}$ and $U_{k,n}$ are separated by a block of length m_n. For the second term we use the induction hypothesis.)

From (30) and (26), we get:

$$|E(e^{-\sum_{i=1}^{k_n} U_{i,n}}) - \{E(e^{-U_{1,n}})\}^{k_n}| \leq 4k_n\alpha(m_n) \to 0.$$

□

5.2 Associated sequences

In this subsection, we assume that $E = \mathbb{R}\backslash\{0\}$. Recall that the random variables $(X_j)_{j\geq 1}$ are called *associated* if for any finite disjoint sets A, B in $\{1, 2, \ldots\}$ and for any coordinate-wise non-decreasing functions $h : \mathbb{R}^{\#A} \to \mathbb{R}$ and $k : \mathbb{R}^{\#B} \to \mathbb{R}$

$$\mathrm{Cov}(h(X_j, j \in A), k(X_j, j \in B)) \geq 0,$$

where $\#A$ denotes the cardinality of the set A. (See e.g. [1], [14] for more details about the association.)

If $(X_j)_{j\geq 1}$ is a sequence of associated random variables, then for any finite disjoint sets A, B in $\{1, 2, \ldots\}$ and for any functions $h : \mathbb{R}^{\#A} \to \mathbb{R}$ and $k : \mathbb{R}^{\#B} \to \mathbb{R}$ (not necessarily coordinate-wise non-decreasing), which are partially differentiable and have bounded partial derivatives, we have: (see Lemma 3.1.(i), [2])

$$|\mathrm{Cov}(h(X_j, j \in A), k(X_j, j \in B))| \leq \sum_{i\in A}\sum_{j\in B} \left\|\frac{\partial h}{\partial x_i}\right\|_\infty \left\|\frac{\partial k}{\partial x_j}\right\|_\infty \mathrm{Cov}(X_i, X_j). \tag{31}$$

Let $\mathcal{C}$ be the class of all bounded nondecreasing functions g, for which there exists a compact subset K of E such that $g(x) = x$ for all $x \in K$. Let $\mathcal{S}_1$ be the set of all $m \in \mathbb{Z}_+$ for which $\limsup_{n\to\infty} n \sum_{j=m+1}^{n} \mathrm{Cov}(g(X_{1,n}), g(X_{j,n})) = 0$ for any $g \in \mathcal{C}$. Let m_1 be the smallest integer in $\mathcal{S}_1$. By convention, we let $m_1 = \infty$ if $\mathcal{S}_1 = \emptyset$.

We introduce a new asymptotic dependence condition.

Definition 5.3 *We say that the triangular array* $(X_{j,n})_{1\leq j\leq n, n\geq 1}$ *satisfies* **condition (AD-3)** *if*

$$\lim_{m\to m_1} \limsup_{n\to\infty} n \sum_{j=m+1}^{n} \mathrm{Cov}(g(X_{1,n}), g(X_{j,n})) = 0, \quad \forall g \in \mathcal{C}.$$

Lemma 5.4 *For each $n \geq 1$, let $(X_{j,n})_{1\leq j\leq n}$ be a strictly stationary sequence of associated random variables with values in $\mathbb{R}\backslash\{0\}$. If the triangular array $(X_{j,n})_{1\leq j\leq n,n\geq 1}$ satisfies conditions (AN) and (AD-3), then it also satisfies condition (AD-1).*

Proof: As in the proof of Lemma 5.1, it suffices to show that (25) holds. For this, we use the same "block" technique as in the proof of Lemma 5.1, except that now the separation blocks have size m (instead of m_n).

For each $1 \leq i \leq k_n$, let $H_{i,n}^{(m)}$ be the (big) block of consecutive integers between $(i-1)r_n + 1$ and $ir_n - m$ and $I_{i,n}^{(m)}$ be the (small) block of size m, consisting of consecutive integers between $ir_n - m$ and ir_n. Let

$$U_{i,n}^{(m)} = \sum_{j\in H_{i,n}^{(m)}} f(X_{j,n}) = N_{ir_n-m,n}(f) - N_{(i-1)r_n,n}(f).$$

Similarly to (27) and (28), one can prove that:

$$\lim_{m\to m_1} \limsup_{n\to\infty} |E(e^{-N_{k_n r_n,n}(f)}) - E(e^{-\sum_{i=1}^{k_n} U_{i,n}^{(m)}})| = 0 \tag{32}$$

$$\lim_{m\to m_1} \limsup_{n\to\infty} |\{E(e^{-N_{r_n,n}(f)})\}^{k_n} - \{E(e^{-U_{1,n}^{(m)}})\}^{k_n}| = 0. \tag{33}$$

Therefore, in order to prove that relation (25) holds, it suffices to show that

$$\lim_{m\to m_1} \limsup_{n\to\infty} |E(e^{-\sum_{i=1}^{k_n} U_{i,n}^{(m)}}) - \{E(e^{-U_{1,n}^{(m)}})\}^{k_n}| = 0. \tag{34}$$

Without loss of generality, we suppose that the random variables $(X_{j,n})_{1\leq j\leq k_n}$ are uniformly bounded. (Otherwise, we replace them the random variables $Y_{j,n} = g(X_{j,n}), 1 \leq j \leq k_n$, where g is a bounded non-decreasing function such that $g(x) = x$ on the support of f. The new sequence $(Y_{j,n})_{1\leq j\leq k_n}$ consists of uniformly bounded associated random variables. Moreover, $f(X_{j,n}) = f(Y_{j,n})$ for all $1 \leq j \leq k_n$.)

Moreover, we suppose that the function f satisfies the following condition: there exists $L_f > 0$ such that

$$|f(x) - f(y)| \leq L_f |x - y|, \quad \forall x, y \in \mathbb{R}\backslash\{0\}. \tag{35}$$

(Note that any function $f \in C_K^+(\mathbb{R}\backslash\{0\})$ can be approximated a bounded sequence of step functions, which in turn can be approximated by a sequence of functions which satisfy (35).)

Using an induction argument and (31), one can show that:

$$|E(e^{-\sum_{i=1}^{k} U_{i,n}^{(m)}}) - \{E(e^{-U_{1,n}^{(m)}})\}^k| \le L_f^2 \sum_{1\le i<l\le k} \sum_{j\in H_{i,n}^{(m)}} \sum_{j'\in H_{l,n}^{(m)}} \mathrm{Cov}(X_{j,n}, X_{j',n}). \tag{36}$$

By stationarity, we have

$$\sum_{1\le i<l\le k_n} \sum_{j\in H_{i,n}^{(m)}} \sum_{j'\in H_{l,n}^{(m)}} \mathrm{Cov}(X_{j,n}, X_{j',n}) = \sum_{i=1}^{k_n}(k_n - i)\mathrm{Cov}(\sum_{j\in H_{1,n}^{(m)}} X_{j,n}, \sum_{j'\in H_{i+1,n}^{(m)}} X_{j',n}) =$$

$$\sum_{i=1}^{k_n}(k_n - i)(r_n - m) \sum_{l=(i-1)r_n+m+1}^{(i+1)r_n-m} \mathrm{Cov}(X_{1,n}, X_{l,n}) \le 2k_n r_n \sum_{l=m+1}^{n} \mathrm{Cov}(X_{1,n}, X_{l,n}). \tag{37}$$

From (36) and (37), we get: $|E(e^{-\sum_{i=1}^{k_n} U_{i,n}^{(m)}}) - \{E(e^{-U_{1,n}^{(m)}})\}^{k_n}| \le 2L_f^2 n \sum_{l=m+1}^{n} \mathrm{Cov}(X_{1,n}, X_{l,n})$, and relation (34) follows from condition (AD-3). □

5.3 Stochastic volatility sequences

In this section, we assume that $E = \mathbb{R}\backslash\{0\}$ and the dependence structure on each row of the array $(X_{j,n})_{1\le j\le n, n\ge 1}$ is that of a *stochastic volatility sequence*. More precisely,

$$X_{j,n} = \sigma_j Z_{j,n}, \quad 1 \le j \le n, \; n \ge 1, \tag{38}$$

where $(Z_{j,n})_{1\le j\le n}$ are an i.i.d. random variables, and $(\sigma_j)_{j\ge 1}$ is a strictly stationary sequence of positive random variables, which is independent of the array $(Z_{j,n})_{1\le j\le n, n\ge 1}$. In this context, $(Z_{j,n})_{1\le j\le n}$ is called the noise sequence, and $(\sigma_j)_{j\ge 1}$ is called a volatility sequence. Such models arise in applications to financial time series (see [8]). The row dependence structure among the variables $(X_{j,n})_{1\le j\le n}$ is inherited from that of the volatility sequence: if $(\sigma_j)_j$ is m-dependent, then so is the sequence $(X_{j,n})_{1\le j\le n}$. This model is different than a GARCH model, in which there is a recurrent dependence between the noise sequence and the volatility sequence.

The dependence structure that we consider for $(\sigma_j)_{j\ge 1}$ is slightly more general than strongly mixing:

(C) there exists a function $\psi : \mathbb{N} \to (0,\infty)$ with $\lim_{m\to\infty} \psi(m) = 0$, such that for any disjoint blocks I, J of consecutive integers, which are separated by a block of at least m integers, and for any $z_j \in \mathbb{R}$

$$\left|\mathrm{Cov}\left(e^{-\sum_{j\in I} f(\sigma_j z_j)}, e^{-\sum_{j\in J} f(\sigma_j z_j)}\right)\right| \le \psi(m), \quad \forall f \in C_K^+(\mathbb{R}\backslash\{0\}). \tag{39}$$

We have the following result.

Lemma 5.5 *Let $(X_{j,n})_{1\le j\le n, n\ge 1}$ be the triangular array given by (38). If the array $(X_{j,n})_{1\le j\le n, n\ge 1}$ satisfies condition (AN) and $(\sigma_j)_{j\ge 1}$ satisfies condition (C), then the array $(X_{j,n})_{1\le j\le n, n\ge 1}$ satisfies condition (AD-1).*

Proof: We use the same argument and notation as in the proof of Lemma 5.1. Let $(m_n)_n, (r_n)_n$ and $(k_n)_n$ be sequences of positive integers such that (26) holds, with the function ψ in the place of α.

It suffices to prove that (29) holds. We now claim that, for any $1 \le k \le k_n$ we have

$$|E(e^{-\sum_{i=1}^{k} U_{i,n}}) - \{E(e^{-U_{1,n}})\}^k| \le (k-1)\psi(m_n). \tag{40}$$

We show this only for $k = 2$, the general induction argument being very similar. Due to the independence between $(\sigma_j)_{j\geq 1}$ and $(Z_{j,n})_{j,n}$, and the independence of the sequence $(Z_{j,n})_j$, we have:

$$E(e^{-(U_{1,n}+U_{2,n})}) = \int E\left(e^{-\sum_{j\in H_{1,n}} f(\sigma_j z_j) - \sum_{j\in H_{2,n}} f(\sigma_j z_j)}\right) dP(z_j)_{j\in H_{1,n}\cup H_{2,n}}$$
$$= \int \left[E\left(e^{-\sum_{j\in H_{1,n}} f(\sigma_j z_j) - \sum_{i\in H_{2,n}} f(\sigma_j z_j)}\right) - E\left(e^{-\sum_{j\in H_{1,n}} f(\sigma_j z_j)}\right) E\left(e^{-\sum_{j\in H_{2,n}} f(\sigma_j z_j)}\right)\right] dP(z_j)_{j\in H_{1,n}\cup H_{2,n}}$$
$$+ \int E\left(e^{-\sum_{j\in H_{1,n}} f(\sigma_j z_j)}\right) dP(z_j)_{j\in H_{1,n}} \int E\left(e^{-\sum_{j\in H_{2,n}} f(\sigma_j z_j)}\right) dP(z_j)_{j\in H_{2,n}}$$
$$= \int \mathrm{Cov}\left(e^{-\sum_{j\in H_{1,n}} f(\sigma_j z_j)}, e^{-\sum_{j\in H_{2,n}} f(\sigma_j z_j)}\right) dP(z_j)_{j\in H_{1,n}\cup H_{2,n}} + E(e^{-U_{1,n}})E(e^{-U_{2,n}})$$

where $dP(z_j)_{j\in H_{1,n}\cup H_{2,n}}$ denotes the law of $(Z_{j,n})_{j\in H_{1,n}\cup H_{2,n}}$, and $dP(z_j)_{j\in H_{l,n}}$ denotes the law of $(Z_{j,n})_{j\in H_{l,n}}$ for $l = 1, 2$. Using condition (39), we obtain: $|E(e^{-(U_{1,n}+U_{2,n})}) - E(e^{-U_{1,n}})E(e^{-U_{2,n}})| \leq \int \left|\mathrm{Cov}\left(e^{-\sum_{j\in H_{1,n}} f(\sigma_j z_j)}, e^{-\sum_{j\in H_{2,n}} f(\sigma_j z_j)}\right)\right| dP(z_j)_{j\in H_{1,n}\cup H_{2,n}} \leq \psi(m_n)$, since the blocks $H_{1,n}$ and $H_{2,n}$ are separated by a block of length m_n. This concludes the proof of (40) in the case $k = 2$.

Relation (29) follows using (40) with $k = k_n$, and the fact that $k_n\psi(m_n) \to 0$. □

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A Poisson Process Representation

Proposition A.1 *Let N^* be a Poisson process on $(0,\infty)$, with intensity ρ. Then N^* admits the representation $N^* \stackrel{d}{=} \sum_{i\geq 1}\delta_{P_iW_i}$, where $(P_i)_{i\geq 1}$ are the points of a Poisson process of intensity ν and $(W_i)_{i\geq 1}$ is an independent i.i.d. sequence with distribution F, if and only if the measure ρ satisfies:*

$$\int_0^\infty (1-e^{-f(x)})\rho(dx) = \int_0^\infty \int_0^\infty (1-e^{-f(wy)})F(dw)\nu(dy)$$

for every measurable function $f : (0,\infty) \to (0,\infty)$.

Proof: Since N^* is a Poisson process of intensity ρ, for any measurable non-negative function f, we have $L_{N^*}(f) = \exp\left\{-\int_0^\infty (1-e^{-f(x)})\rho(dx)\right\}$. Let $N^{**} = \sum_{i\geq 1}\delta_{P_iW_i}$. Since $N^* \stackrel{d}{=} N^{**}$ if and only if $L_{N^*}(f) = L_{N^{**}}(f)$ for any measurable non-negative function f, the proof will be complete once we show that

$$E\left(e^{-\sum_{i\geq 1} f(P_iW_i)}\right) = \exp\left\{-\int_0^\infty \int_0^\infty (1-e^{-f(wy)})F(dw)\nu(dy)\right\}. \tag{41}$$

We first treat the right hand side of (41). For this, we let $g(y) = -\log\int_0^\infty e^{-f(wy)}F(dw)$. Using the fact that F is a probability measure on $(0,\infty)$ and $M = \sum_{i\geq 1}\delta_{P_i}$ is a Poisson process of intensity ν, we have

$$\exp\left\{-\int_0^\infty \int_0^\infty (1-e^{-f(wy)})F(dw)\nu(dy)\right\} = \exp\left\{-\int_0^\infty (1-e^{-g(y)})\nu(dy)\right\} = E\left(e^{-\sum_{i\geq 1} g(P_i)}\right) = E\left(\prod_{i\geq 1} e^{-g(P_i)}\right)$$

For each $i \geq 1$, let $\phi_i(y) = E(e^{-f(W_iy)}), y \geq 0$. Since $(W_i)_{i\geq 1}$ are i.i.d. random variables with distribution F, for every $y \geq 0$ we have $\phi_1(y) = \phi_i(y) = \int_0^\infty e^{-f(wy)}F(dw) = e^{-g(y)}, \forall i \geq 1$.
By considering the random variable $(P_i)_i : \Omega \to [0,\infty)^N$ whose law is denoted by $dP(p_i)_i$, we get

$$E\left(\prod_{i\geq 1} e^{-g(P_i)}\right) = E\left(\prod_{i\geq 1}\phi_i(P_i)\right) = \int_{[0,\infty)^N} \prod_{i\geq 1}\phi_i(p_i)dP(p_i)_i =$$

$$\int_{[0,\infty)^N}\prod_{i\geq 1}\left(\int_\Omega e^{-f(p_iW_i(\omega_i))}P(d\omega_i)\right)dP(p_i)_i=\int_\Omega\prod_{i\geq 1}\left(\int_\Omega e^{-f(P_i(\omega)W_i(\omega_i))}P(d\omega_i)\right)P(d\omega)=$$
$$\int_\Omega\left(\int_\Omega\prod_{i\geq 1}e^{-f(P_i(\omega)W_i(\omega'))}P(d\omega')\right)P(d\omega)=\int_\Omega\prod_{i\geq 1}e^{-f(P_i(\omega)W_i(\omega))}P(d\omega).$$

For the second last equality above we used the fact that $(W_i)_{i\geq 1}$ are independent, whereas for the last equality above we used the fact that $(P_i)_{i\geq 1}$ and $(W_i)_{i\geq 1}$ are independent. □

B Construction of $(r_n)_n$ and $(m_n)_n$ in the proof of Lemma 5.1

Lemma B.1 *If $\lim_{m\to\infty}\alpha(m)=0$, then there exist some sequences $(r_n)_n$ and $(m_n)_n$ of positive integers such that $r_n\to\infty$, $k_n:=[n/r_n]\to\infty$, $m_n\to\infty$, $m_n/r_n\to 0$ and $k_n\alpha(m_n)\to 0$.*

Proof: Denote $\rho_n:=\alpha([\sqrt{n}])$. Clearly $\rho_n\to 0$. We define

$$\varepsilon_n=\max\{n^{-1/4},\sqrt{\rho_n}\},\quad \delta_n=\frac{n^{-1/2}}{\varepsilon_n},\quad \eta_n=\frac{\rho_n}{2\varepsilon_n},\quad r_n:=[n\varepsilon_n],\quad k_n:=[n/r_n],\quad m_n:=[n\varepsilon_n\delta_n]=[\sqrt{n}].$$

Clearly $\varepsilon_n\to 0$ and $m_n\to\infty$. We will use repeatedly the inequality $x/2\leq[x]\leq x$, for any $x\geq 0$. We have:

$$r_n\geq\frac{n\varepsilon_n}{2}\geq\frac{n^{3/4}}{2}\to\infty\quad\text{and}\quad k_n\geq\frac{1}{2}\cdot\frac{n}{r_n}\geq\frac{1}{2}\cdot\frac{n}{n\varepsilon_n}=\frac{1}{2\varepsilon_n}\to\infty.$$

Finally, since $\delta_n\leq n^{-1/4}\to 0$ and $\eta_n\leq\sqrt{\rho_n}/2\to 0$, we have:

$$\frac{m_n}{r_n}\leq\frac{n\varepsilon_n\delta_n}{n\varepsilon_n/2}=2\delta_n\to 0\quad\text{and}\quad k_n\alpha(m_n)=k_n\rho_n\leq\frac{n}{r_n}\rho_n\leq\frac{2}{\varepsilon_n}\rho_n=4\eta_n\to 0.$$

□

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