

SPECIAL ISSUE - LIMIT THEOREMS AND TIME SERIES

A Cluster Limit Theorem for Infinitely Divisible Point Processes

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Abstract

In this article, we consider a sequence $(N_n)_{n \geq 1}$ of point processes, whose points lie in a subset E of $\mathbb{R} \setminus \{0\}$, and satisfy an asymptotic independence condition. Our main result gives some necessary and sufficient conditions for the convergence in distribution of $(N_n)_{n \geq 1}$ to an infinitely divisible point process N . As applications, we discuss the exceedance processes and point processes based on regularly varying sequences.

MSC 2000 subject classification: Primary 60G55; secondary 60G70

Keywords: point process, infinite divisibility, weak dependence, exceedance process, extremal index

1 Introduction, notation and main assumptions

Let E be a locally compact Hausdorff space with a countable basis (abbreviated LCCB) and $M_p(E)$ be the set of Radon measures on E with values in \mathbb{Z}_+ . (Recall that a measure μ is Radon if $\mu(B) < \infty$ for any $B \in \mathcal{B}$, where \mathcal{B} is the class of relatively compact Borel sets in E .) The space $M_p(E)$ is endowed with the topology of vague convergence. (Recall that $\mu_n \xrightarrow{v} \mu$ if $\mu_n(f) \rightarrow \mu(f)$ for any $f \in C_K^+(E)$, where $\mu(f) = \int_E f d\mu$ and $C_K^+(E)$ is the class of continuous functions $f : E \rightarrow [0, \infty)$ with compact support.)

A measurable map $N : \Omega \rightarrow M_p(E)$ defined on a probability space (Ω, \mathcal{K}, P) is called a point process. The law of N is uniquely determined by its Laplace functional: $L_N(f) = E(e^{-N(f)})$, $f \in \mathcal{F}$, where \mathcal{F} is the class of measurable functions $f : E \rightarrow [0, \infty)$ (see e.g. [13]).

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[§]Partially supported by the ANR grant ANR-08-BLAN-0314-02.

A point process N is infinitely divisible (ID) if for any integer $n \geq 1$, there exist some i.i.d. point processes $(N_{i,n})_{1 \leq i \leq n}$ such that $N \stackrel{d}{=} \sum_{i=1}^n N_{i,n}$. An ID point process enjoys many properties similar to those of ID random vectors. In particular, a point process N is ID if and only if there exists a (unique) measure λ on $M_p(E)$ (called the canonical measure), satisfying $\lambda(\{o\}) = 0$, (o being the null measure on E), and

$$\int_{M_p(E)} (1 - e^{-\mu(B)}) \lambda(d\mu) < \infty \quad \forall B \in \mathcal{B}, \quad (1)$$

such that:

$$L_N(f) = \exp \left\{ - \int_{M_p(E)} (1 - e^{-\mu(f)}) \lambda(d\mu) \right\}, \quad \forall f \in \mathcal{F}. \quad (2)$$

A sequence $(N_n)_{n \geq 1}$ of point processes converges in distribution to N if their laws converge weakly (in $M_p(E)$) to the law of N . We write $N_n \xrightarrow{d} N$.

The study of point process convergence is important from the theoretical point of view (by applying the continuous mapping theorem, one can obtain various limit theorems for some functionals of the points see for instance [18] and [4]) and practical (see for instance [15], [9], [14] and the references therein). Point processes of exceedances play an important role in extreme value theory and their limiting behavior has been extensively studied (we refer for instance to [2], [6], [11], [12], [16] and the references therein).

The purpose of the present article is to establish *minimal* conditions for the convergence $N_n \xrightarrow{d} N$, when N is an ID point process and E is a subspace of $\mathbb{R} \setminus \{0\}$. This question has been studied by several authors in different contexts (see e.g. [8], [11]). Our contribution consists in providing a general (unifying) result, which contains the results of [8] and [11], established in two different situations.

The following definition introduces an asymptotic independence condition.

Definition 1.1 A sequence $(N_n)_{n \geq 1}$ of point processes satisfies **condition (AI)** if there exists a sequence $(N_{i,n})_{1 \leq i \leq k_n, n \geq 1}$ of point processes such that $k_n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \max_{i \leq k_n} P(N_{i,n}(B) > 0) = 0, \quad \forall B \in \mathcal{B}, \quad (3)$$

and

$$\lim_{n \rightarrow \infty} |E(e^{-N_n(f)}) - \prod_{i=1}^{k_n} E(e^{-N_{i,n}(f)})| = 0, \quad \forall f \in C_K^+(E).$$

Condition (AI) requires that N_n behaves asymptotically (in distribution) as the superposition of k_n independent point processes. This condition contains various conditions encountered in the literature related to asymptotic results for triangular arrays (see condition $\Delta(\{u_n\})$ of [11], condition $\mathcal{A}(\{a_n\})$ of [8], or condition (AD-1) of [1]). More precisely, if $N_n = \sum_{j=1}^n \delta_{X_{j,n}}$, where

$(X_{j,n})_{1 \leq j \leq n, n \geq 1}$ is a triangular array of random variables, then (AI) holds under suitable dependence conditions imposed on the array (see e.g. [1]).

In the present article, we relax the independence assumption between the components $(N_{i,n})_{i \leq k_n}$, by requiring that (AI) holds. The following result is an immediate consequence of Theorem 6.1 of [13].

Theorem 1.2 *Let $(N_n)_{n \geq 1}$ be a sequence of point processes satisfying (AI). Then $N_n \xrightarrow{d} N$ if and only if there exists a measure λ on $M_p(E)$ satisfying $\lambda(\{o\}) = 0$ and (1), such that*

$$\int_{M_p(E)} (1 - e^{-\mu(f)}) \lambda_n(d\mu) \rightarrow \int_{M_p(E)} (1 - e^{-\mu(f)}) \lambda(d\mu), \quad \forall f \in C_K^+(E), \quad (4)$$

where $\lambda_n = \sum_{i=1}^{k_n} P \circ N_{i,n}^{-1}$. In this case, N is an infinitely divisible point process with canonical measure λ .

This article is organized as follows. Section 2 is dedicated to our main result (Theorem 2.2) and its consequences for triangular arrays of random variables. This result is based on Theorem 1.2 and gives some necessary and sufficient conditions for the convergence of $(N_n)_{n \geq 1}$ to an ID point process N , in terms of the asymptotic behavior of the pair $\{(Y_{i,n}, N_{i,n})\}_{1 \leq i \leq k_n}$, where $(N_{i,n})_{1 \leq i \leq k_n}$ are given by (AI), and $Y_{i,n}$ is the largest point of $N_{i,n}$ (in modulus). We apply our main result to triangular arrays of the form $(\xi_i/a_n)_{i,n}$ (see Proposition 2.4 below). A relationship between the canonical measure λ of N and the extremal index of the sequence $(|\xi_i|)_i$ when it exists, is established in Corollary 2.7. In Section 3, we apply our result to the case of exceedance processes and processes based on regularly varying sequences, and thus recover the results of [11] and [8], respectively.

2 The result

In the present article, we assume that E is one of the following spaces:

- (i) $E = [-b, -a) \cup (a, b]$, $E = (a, b]$ or $E = [-b, -a)$ for some $0 \leq a < b < \infty$;
- (ii) $E = [-\infty, -a) \cup (a, \infty]$, $E = (a, \infty]$ or $E = [-\infty, -a)$ for some $0 \leq a < \infty$.

Note that in both cases, E is a LCCB subspace of $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, which satisfies the following conditions:

$$\text{for any } x > 0, [-x, x]^c := \{y \in E; |y| > x\} \text{ is relatively compact in } E, \quad (5)$$

$$\text{for any compact set } K \subset E, \text{ there exists } x > 0 \text{ such that } K \subset [-x, x]^c. \quad (6)$$

Let N be an infinitely divisible point process on E , with canonical measure λ . We assume that:

$$\text{if the space } E \text{ contains } \pm\infty, \text{ then } N(\{\pm\infty\}) = 0 \text{ a.s.} \quad (7)$$

Remark 2.1 (a) The Poisson process N on $[0, 1]$ of intensity 1 is included in our framework. In this case, we can exclude 0 from the space, since the points of the process are strictly positive. N is a well-defined point process on $E = (0, 1]$.

(b) The Poisson process on $[0, \infty)$ of intensity 1 is *not* included in our framework. It is known that $N = \sum_{i \geq 1} \delta_{\Gamma_i}$, where $\Gamma_i = \sum_{j=1}^i E_j$ and $(E_j)_{j \geq 1}$ are i.i.d. exponential random variables of mean 1. As in example (a) above, we can exclude 0 from the space. Clearly, we may assume that the points of N lie in the space $E = (0, \infty]$. But the points of N accumulate at ∞ , and so, N is *not a point process* on E : the number of points in the (relatively compact) set (x, ∞) is ∞ , a.s.

(c) The Poisson process on $(0, \infty)$ of intensity $\nu(dx) = \alpha x^{-\alpha-1} dx$ (for some $\alpha > 0$) is included in our framework. In this case, $N = \sum_{i \geq 1} \delta_{\Gamma_i^{-1/\alpha}}$, and the points of N accumulate at 0. This does not contradict the definition of a point process, since a set of the form $(0, \varepsilon)$ is *not* relatively compact in $(0, \infty)$. In this case, N is a (well-defined) point process on the space $E = (0, \infty]$.

Let $M_0 = M_p(E) \setminus \{0\}$. For $\mu = \sum_{j \geq 1} \delta_{t_j} \in M_0$, we let $x_\mu := \sup_{j \geq 1} |t_j|$. Note that $x_\mu < \infty$ for all $\mu \in M_0$. (This is clear if $E = [-b, a) \cup (a, b]$ for some $0 \leq a < b < \infty$. Suppose now that $E = [-\infty, -a) \cup (a, \infty]$ for some $0 \leq a < \infty$. Assume that there exists $\mu = \sum_j \delta_{t_j} \in M_0$ with $\sup_j |t_j| = \infty$. Then, there exists a subsequence $(t_{j_k})_k$ such that $\lim_k |t_{j_k}| = \infty$. It follows that the point measure μ has an infinite number of points in the relatively compact set $[-\infty, c) \cup (c, \infty]$ for some $c > 0$, which is a contradiction.)

Note that for any $\mu \in M_0$, there exists $x \in (0, \infty)$ such that $\mu([-x, x]^c) = 0$. (We may take $x = x_\mu$.) For any $x > 0$, we let

$$M_x = \{\mu \in M_0; \mu([-x, x]^c) > 0\}.$$

Then,

$$\mu \in M_x \quad \text{if and only if} \quad x_\mu > x.$$

Using (1) with $B = [-x, x]^c$, the fact that $\mu([-x, x]^c) \geq 1$ for all $\mu \in M_x$, and the function $f(y) = 1 - e^{-y}$ is non-decreasing, we obtain:

$$\infty > \int (1 - e^{-\mu([-x, x]^c)}) \lambda(d\mu) = \int_{M_x} (1 - e^{-\mu([-x, x]^c)}) \lambda(d\mu) \geq (1 - e^{-1}) \lambda(M_x).$$

Hence $\lambda(M_x) < \infty$ for all $x > 0$.

Recall that x is a *fixed atom* of a point process N if $P(N\{x\} > 0) > 0$. (Consequently, x is not a fixed atom of N , if $N\{x\} = 0$ a.s.) It is known that the set D of all fixed atoms of any point process is countable.

Let $\mathcal{M}_p(E)$ be the class of Borel sets in $M_p(E)$, and $\mathcal{M}_0 = \{M \in \mathcal{M}_p(E); M \subset M_0\}$. We consider the measurable map $\Phi : M_0 \rightarrow (0, \infty)$, defined by $\Phi(\mu) = x_\mu$.

The following theorem is the main result of the present article.

Theorem 2.2 Let $(N_n)_{n \geq 1}$ be a sequence of point processes on E satisfying (AI). Suppose that $N_{i,n} \in M_0$ a.s. for all $n \geq 1$. Let $Y_{i,n} = \Phi(N_{i,n})$.

Let N be an infinitely divisible point process on E which satisfies (7). Let D be the set of fixed atoms of N and $D' = \{x > 0; x \in D \text{ or } -x \in D\}$. Assume that

$$\limsup_{n \rightarrow \infty} \sum_{i=1}^{k_n} P(Y_{i,n} > x) < \infty, \quad \forall x > 0, x \notin D'. \quad (8)$$

Then $N_n \xrightarrow{d} N$ if and only if the following two conditions hold:

- (a) $\sum_{i=1}^{k_n} P(Y_{i,n} > x) \rightarrow \lambda(M_x)$, for any $x > 0, x \notin D'$
- (b) $\sum_{i=1}^{k_n} P(Y_{i,n} > x, N_{i,n} \in M) \rightarrow \lambda(M \cap M_x)$, for any $x > 0, x \notin D'$ for which $\lambda(M_x) > 0$, and for any set $M \in \mathcal{M}_0$ such that $\lambda(\partial M \cap M_x) = 0$.

Proof: The ideas of this proof are borrowed from the proof of Theorem 2.5 of [8]. Suppose that $N_n \xrightarrow{d} N$. We first prove (a). Let $\tilde{N}_n = \sum_{i=1}^{k_n} \tilde{N}_{i,n}$ where $(\tilde{N}_{i,n})_{i \leq k_n}$ are independent point processes, and $\tilde{N}_{i,n} \stackrel{d}{=} N_{i,n}$. Using (AI) and the fact that $N_n \xrightarrow{d} N$, it follows that $\tilde{N}_n \xrightarrow{d} N$.

Let $x > 0, x \notin D'$ be arbitrary. We prove that (a) holds. By Lemma 4.4 of [13]), $\tilde{N}_n([-x, x]^c) \xrightarrow{d} N([-x, x]^c)$, since $N\{x\} = N\{-x\} = 0$ a.s. Hence,

$$-\log P(\tilde{N}_n([-x, x]^c) = 0) \rightarrow -\log P(N([-x, x]^c) = 0). \quad (9)$$

Note that

$$E(e^{-uN([-x, x]^c)}) = \exp \left\{ - \int_{M_0} (1 - e^{-u\mu([-x, x]^c)}) \lambda(d\mu) \right\}, \quad \forall u \in \mathbb{R}_+,$$

and hence

$$-\log P(N([-x, x]^c) = 0) = \lambda(\{\mu \in M_0; \mu([-x, x]^c) > 0\}) = \lambda(M_x). \quad (10)$$

From (9) and (10), we obtain that:

$$P(\tilde{N}_n([-x, x]^c) = 0) \rightarrow e^{-\lambda(M_x)}. \quad (11)$$

Let $a_{i,n} := P(Y_{i,n} > x) = P(N_{i,n}([-x, x]^c) > 0)$. Using the independence of $(\tilde{N}_{i,n})_{i \leq k_n}$, and the fact that $N_{i,n} \stackrel{d}{=} \tilde{N}_{i,n}$, we obtain:

$$\begin{aligned} P(\tilde{N}_n([-x, x]^c) = 0) &= \prod_{i=1}^{k_n} P(\tilde{N}_{i,n}([-x, x]^c) = 0) = \prod_{i=1}^{k_n} P(N_{i,n}([-x, x]^c) = 0) \\ &= \prod_{i=1}^{k_n} (1 - a_{i,n}). \end{aligned} \quad (12)$$

From (11) and (12), we infer that $\prod_{i=1}^{k_n} (1 - a_{i,n}) \rightarrow e^{-\lambda(M_x)}$, and hence $\sum_{i=1}^{k_n} -\log(1 - a_{i,n}) \rightarrow \lambda(M_x)$. By (3), $\max_{i \leq k_n} a_{i,n} \rightarrow 0$. Using (8) and the fact that $-\log(1 - x) = x + O(x^2)$ as $x \rightarrow 0$, we infer that $\sum_{i=1}^{k_n} a_{i,n} \rightarrow \lambda(M_x)$, i.e. condition (a) holds.

We now prove (b). Let $x > 0, x \notin D'$ be such that $\lambda(M_x) > 0$. Let $P_{n,x}$ and P_x be probability measures on M_0 , defined by:

$$P_{n,x}(M) = \frac{\lambda_n(M \cap M_x)}{\lambda_n(M_x)} \quad \text{and} \quad P_x(M) = \frac{\lambda(M \cap M_x)}{\lambda(M_x)},$$

where $\lambda_n = \sum_{i=1}^{k_n} P \circ N_{i,n}^{-1}$. By (a),

$$\lambda_n(M_x) = \sum_{i=1}^{k_n} P(N_{i,n} \in M_x) = \sum_{i=1}^{k_n} P(Y_{i,n} > x) \rightarrow \lambda(M_x).$$

As in the proof of Theorem 2.5 of [8], using (4), one can prove that: $P_{n,x} \xrightarrow{w} P_x$. This is equivalent to saying that

$$P_{n,x}(M) \rightarrow P_x(M), \tag{13}$$

for any $M \in \mathcal{M}_0$ with $\lambda(\partial M \cap M_x) = 0$. Condition (b) follows, since

$$P_{n,x}(M) = \frac{\sum_{i=1}^{k_n} P(N_{i,n} \in M \cap M_x)}{\sum_{i=1}^{k_n} P(N_{i,n} \in M_x)} = \frac{\sum_{i=1}^{k_n} P(Y_{i,n} > x, N_{i,n} \in M)}{\sum_{i=1}^{k_n} P(Y_{i,n} > x)}.$$

Suppose that (a) and (b) hold. Using Theorem 1.2, it suffices to prove that (4) holds. Let $f \in C_K^+(E)$ be arbitrary. Pick $x > 0, x \notin D'$ such that $\lambda(M_x) > 0$ and the support of f is contained in $[-x, x]^c$. (This choice is possible due to (6).)

Let $P_{n,x}$ and P_x be defined as above. Let $M \in \mathcal{M}_0$ be such that $P_x(\partial M) = 0$. Due to (a) and (b), (13) holds. Hence $P_{n,x} \xrightarrow{w} P_x$. It follows that for any bounded continuous function $h : M_0 \rightarrow \mathbb{R}$

$$\frac{1}{\lambda_n(M_x)} \int_{M_x} h(\mu) \lambda_n(d\mu) \rightarrow \frac{1}{\lambda(M_x)} \int_{M_x} h(\mu) \lambda(d\mu).$$

Taking $h(\mu) = e^{-\mu(f)}$ and using the fact that $\lambda_n(M_x) \rightarrow \lambda(M_x)$ (which is condition (a)), we obtain that:

$$\int_{M_x} (1 - e^{-\mu(f)}) \lambda_n(d\mu) \rightarrow \int_{M_x} (1 - e^{-\mu(f)}) \lambda(d\mu).$$

Note that if $\mu \notin M_x$, then $\mu([-x, x]^c) = 0$, $\mu(f) = 0$ (since the support of f is contained in $[-x, x]^c$), and hence

$$\int_{M_0} (1 - e^{-\mu(f)}) \lambda_n(d\mu) = \int_{M_x} (1 - e^{-\mu(f)}) \lambda_n(d\mu)$$

$$\int_{M_0} (1 - e^{-\mu(f)})\lambda(d\mu) = \int_{M_x} (1 - e^{-\mu(f)})\lambda(d\mu).$$

Relation (4) follows using the fact that $\lambda_n(M_0^c) = \lambda(M_0^c) = 0$. \square

Remark 2.3 As it is seen from the proof, in condition (b), one may replace M by $M' = M \cap M_x$, where $M \in \mathcal{M}_0$ and $\lambda(\partial M \cap M_x) = 0$.

The following result illustrates a typical application of the Theorem 2.2.

Proposition 2.4 For each $n \geq 1$, let $(X_{j,n})_{1 \leq j \leq n}$ be a strictly stationary sequence of random variables with values in E , such that

$$\limsup_{n \rightarrow \infty} nP(|X_{1,n}| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0.$$

Suppose that there exists a sequence $(r_n)_n$ of positive integers such that $k_n = \lfloor n/r_n \rfloor \rightarrow \infty$, such that

$$\lim_{n \rightarrow \infty} |E(e^{-\sum_{j=1}^n f(X_{j,n})}) - \{E(e^{-\sum_{j=1}^{r_n} f(X_{j,n})})\}^{k_n}| = 0, \quad \forall f \in C_K^+(E). \quad (14)$$

Let $N_n = \sum_{j=1}^n \delta_{X_{j,n}}$ and $N_{r_n,n} = \sum_{j=1}^{r_n} \delta_{X_{j,n}}$.

Let N be an infinitely divisible point process on E which satisfies (7). Let D be the set of fixed atoms of N and $D' = \{x > 0; x \in D \text{ or } -x \in D\}$.

Then $N_n \xrightarrow{d} N$ if and only if the following two conditions hold:

- (a) $k_n P(\max_{j \leq r_n} |X_{j,n}| > x) \rightarrow \lambda(M_x)$, for any $x > 0, x \notin D'$
- (b) $k_n P(\max_{j \leq r_n} |X_{j,n}| > x, N_{r_n,n} \in M) \rightarrow \lambda(M \cap M_x)$, for any $x > 0, x \notin D'$

for which $\lambda(M_x) > 0$, and for any set $M \in \mathcal{M}_0$ such that $\lambda(\partial M \cap M_x) = 0$.

Remark 2.5 For each $n \geq 1$, let $(X_{j,n})_{1 \leq j \leq n}$ be a sequence of random variables with values in E fulfilling all the requirements of Proposition 2.4. We claim that if (a) holds with $r_n = 1$ and $k_n = n$ then the limit point process N is a Poisson process. To see this, let μ be a measure on E such that $\lambda(M) = \mu(\{y \in E; \delta_y \in M\})$. Condition (a) of Proposition 2.4 becomes $nP(|X_{1,n}| > x) \rightarrow \mu([-x, x]^c)$ for all $x > 0, x \notin D'$, which is equivalent to

$$nP(X_{1,n} \in \cdot) \xrightarrow{v} \mu(\cdot). \quad (15)$$

Let $N_n^* = \sum_{j=1}^n \delta_{X_{j,n}^*}$ where $(X_{j,n}^*)_{j \leq n}$ are i.i.d. copies of $X_{1,n}$. By Proposition 3.21 of [17], $N_n^* \xrightarrow{d} N$ where N is a Poisson process on E of intensity μ , such that $N(\{\pm\infty\}) = 0$ a.s. Combining this with (14), we infer that $N_n \xrightarrow{d} N$.

The next result gives an expression for the limits which appear in conditions (a) and (b) of Theorem 2.2, using a pair (ν, K) , where ν is a measure on $(0, \infty)$ and K is a transition kernel from $(0, \infty)$ to $M_p(E)$ (i.e. $K(x, \cdot)$ is a probability measure on $M_p(E)$ for all $x > 0$, and $K(\cdot, M)$ is Borel measurable for any $M \in \mathcal{M}_p(E)$). In Section 3, we will identify the pair (ν, K) in some particular cases.

Lemma 2.6 *Let λ be the canonical measure of an infinitely divisible point process on E . Then there exists a measure ν on $(0, \infty)$ and a transition kernel K from $(0, \infty)$ to $M_p(E)$, such that for any $M \in \mathcal{M}_p(E)$ and for any $x > 0$,*

$$\lambda(M) = \int_0^\infty K(y, M)\nu(dy), \quad (16)$$

$$\lambda(M_x) = \nu(x, \infty) \quad \text{and} \quad \lambda(M \cap M_x) = \int_x^\infty K(y, M)\nu(dy). \quad (17)$$

Proof: Let $\Psi : M_0 \rightarrow (0, \infty) \times M_0$ be the measurable map defined by:

$$\Psi(\mu) = (\Phi(\mu), \mu) = (x_\mu, \mu).$$

Let $\Lambda = \lambda \circ \Psi^{-1}$. Since λ is σ -finite, the measure Λ is σ -finite on $(0, \infty) \times M_p(E)$. The space $M_p(E)$ is Polish (by 15.7.7 of [13]), and hence it has the disintegration property (see e.g. 15.3.3 of [13]). More precisely, for any Borel set $B \subset (0, \infty)$ and for any set $M \in \mathcal{M}_p(E)$,

$$\Lambda(B \times M) = \int_B K(y, M)\nu(dy),$$

where $\nu = \lambda \circ \Phi^{-1}$ is the first marginal of Λ , and K is a transition kernel from $(0, \infty)$ to $M_p(E)$. In particular, (16) holds. We have:

$$\begin{aligned} \lambda(M_x) &= \lambda(\{\mu; x_\mu > x\}) = (\lambda \circ \Phi^{-1})(x, \infty) = \nu(x, \infty) \\ \lambda(M \cap M_x) &= \lambda(\{\mu; x_\mu > x, \mu \in M\}) = (\lambda \circ \Psi^{-1})((x, \infty) \times M) \\ &= \Lambda((x, \infty) \times M) = \int_x^\infty K(y, M)\nu(dy). \quad \square \end{aligned}$$

The next result is an application of Proposition 2.4 to triangular arrays of the form $X_{j,n} = \xi_j/a_n$, where $(\xi_j)_{j \geq 1}$ is a stationary sequence (combined with Proposition 0.4.(ii) of [17]). This result gives the relationship between the canonical measure λ of the limit process N , and the extremal index of the sequence $(|\xi_j|)_{j \geq 1}$, when it exists.

Corollary 2.7 *Let $(\xi_j)_{j \geq 1}$ be a stationary sequence of real-valued random variables. Suppose that the extremal index θ of the sequence $(|\xi_j|)_{j \geq 1}$ exists and is positive. Let $(a_n)_{n \geq 1}$ be a sequence of positive real numbers such that $nP(|\xi_1| > xa_n) \rightarrow \mu(x, \infty)$, where μ is a measure on $(0, \infty)$, such that $\mu \neq 0$ and $\mu(x, \infty) < \infty$ for all $x > 0$. Suppose that the triangular array $X_{j,n} = \xi_j/a_n$ satisfies condition (14). Let $N_n = \sum_{j=1}^n \delta_{\xi_j/a_n}$ and $N_{r_n,n} = \sum_{j=1}^{r_n} \delta_{\xi_j/a_n}$. Let N and D be as in Proposition 2.4.*

Then $N_n \xrightarrow{d} N$ if and only if the following two conditions hold:

- (a) $k_n P(\max_{j \leq r_n} |\xi_j| > xa_n) \rightarrow \theta \mu(x, \infty)$, for any $x > 0, x \notin D'$
- (b) $k_n P(\max_{j \leq r_n} |\xi_j| > xa_n, N_{r_n,n} \in M) \rightarrow \lambda(M \cap M_x)$, for any $x > 0$ with $\mu(x, \infty) > 0$, and for any set $M \in \mathcal{M}_0$ such that $\lambda(\partial M \cap M_x) = 0$.

In this case, $(|\xi_j|)_j$ is regularly varying and $\mu(x, \infty) = x^{-\alpha}$ for some $\alpha > 0$.

Remark 2.8 Under the hypothesis of Corollary 2.7, by Lemma 2.6, there exists a transition probability K from $(0, \infty)$ to $M_p(E)$ such that $\lambda(M) = \theta \int_0^\infty K(y, M) \mu(dy)$. Condition (b) of Corollary 2.7 becomes:

$$k_n \int_x^\infty K_n(y, M) P(\max_{j \leq r_n} |\xi_j| / a_n \in dy) \rightarrow \theta \int_x^\infty K(y, \cdot) \mu(dy),$$

where $K_n(y, M) = P(N_{r_n, n} \in M | \max_{j \leq r_n} |\xi_j| / a_n = y)$.

3 Applications

3.1 Exceedance processes

In this subsection, we take $E = (0, 1]$. Let $(\xi_j)_{j \geq 1}$ be a stationary sequence of random variables, and $(u_n)_n$ be a sequence of real numbers such that:

$$\lim_{n \rightarrow \infty} nP(\xi_1 > u_n) = 1. \quad (18)$$

Supposing that ξ_j represents a measurement made at time j , we define the process N_n which counts the “normalized” times j/n when the measurement ξ_j exceeds the level u_n , i.e.

$$N_n(\cdot) = \sum_{j=1}^n \delta_{j/n}(\cdot) 1_{\{\xi_j > u_n\}}. \quad (19)$$

N_n is a point process on $(0, 1]$, called the (time normalized) *exceedance process*.

The following mixing-type condition was introduced in [11]. We say that $(\xi_j)_{j \geq 1}$ satisfies **condition** $\Delta(\{u_n\})$ if there exists a sequence $(m_n)_{n \geq 1} \subset \mathbb{Z}_+$ such that:

$$m_n = o(n) \quad \text{and} \quad \alpha_n(m_n) \rightarrow 0,$$

where

$$\alpha_n(m) := \sup\{|P(A \cap B) - P(A)P(B)|; A \in \mathcal{F}_1^k(u_n), \mathcal{F}_{k+m}^n(u_n), k + m \leq n\},$$

and $\mathcal{F}_i^j(u_n) = \sigma(\{\{\xi_s \leq u_n\}; i \leq s \leq j\})$. Lemma 2.2 of [11] shows that if $(\xi_j)_j$ satisfies $\Delta(\{u_n\})$, then $(N_n)_{n \geq 1}$ satisfies (AI) with

$$N_{i,n}(\cdot) = \sum_{j \in J_{i,n}} \delta_{j/n}(\cdot) 1_{\{\xi_j > u_n\}}, \quad (20)$$

where $J_{i,n} = ((i-1)r_n, ir_n]$, $r_n = [n/k_n]$ and $k_n \rightarrow \infty$.

Theorem 4.1 and Theorem 4.2 of [11] show that under $\Delta(\{u_n\})$, $N_n \xrightarrow{d} N$ if and only if the following two conditions hold:

$$k_n P(\max_{j \in J_{1,n}} \xi_j > u_n) \rightarrow a, \quad \text{for some } a > 0 \quad (21)$$

$$\pi_n(k) := P\left(\sum_{j \in J_{1,n}} 1_{\{\xi_j > u_n\}} = k \mid \max_{j \in J_{1,n}} \xi_j > u_n\right) \rightarrow \pi_k, \quad \forall k \geq 1. \quad (22)$$

In this case, N is a compound Poisson process on $(0, 1]$ with Poisson rate a and the distribution of multiplicities $(\pi_k)_{k \geq 1}$, i.e.

$$L_N(f) = \exp \left\{ -a \int_0^1 \left(1 - \sum_{k \geq 1} e^{-kf(x)} \pi_k \right) dx \right\}.$$

Note that condition $\Delta(\{u_n\})$ is a mixing-type condition which is sufficient for (AI) (by Lemma 2.2 in [4]), but may not be necessary. However, as our next result shows, condition $\Delta(\{u_n\})$ is much stronger than needed for the convergence of $(N_n)_{n \geq 1}$. In fact, this convergence can be obtained under (AI) alone.

Proposition 3.1 *Suppose that $(N_n)_{n \geq 1}$ satisfies (AI) with $(N_{i,n})_{i \leq k_n}$ given by (20). If (21) and (22) hold, then conditions (a) and (b) of Theorem 2.2 are satisfied, and (17) holds with*

$$\nu(dy) = a 1_{(0,1]}(y) dy \quad \text{and} \quad K(y, M) = \sum_{k \geq 1} \pi_k 1_M(k \delta_y).$$

Before proving Proposition 3.1, we shall discuss an application of Proposition 3.1 to the case of associated random variables $(\xi_j)_{j \geq 1}$, for which the dependence condition $\Delta(\{u_n\})$ is not appropriate. Recall that the random variables $(\xi_j)_{j \geq 1}$ are *associated* if

$$\text{Cov}(g(\xi_1, \dots, \xi_n), h(\xi_1, \dots, \xi_n)) \geq 0,$$

for any $n \geq 1$, and any coordinate-wise non-decreasing functions $g : \mathbb{R}^n \rightarrow \mathbb{R}$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}$ for which the covariance is well-defined (see e.g. [3], [10]). This notion is very different from mixing. To see this, let $(\varepsilon_i)_{i \in \mathbb{Z}}$ be a sequence of i.i.d. Bernoulli random variables with parameter $1/2$. The linear process

$$\xi_j = \sum_{i \geq 0} 2^{-i} \varepsilon_{j-i}, \quad j \in \mathbb{Z}$$

is associated (by \mathcal{P}_2 and \mathcal{P}_4 of [10]), but fails to be mixing (see [7] and the references therein).

The next result identifies a condition under which the sequence $(N_n)_{n \geq 1}$ of exceedance processes satisfies (AI), when $(\xi_j)_{j \geq 1}$ are associated.

Lemma 3.2 *Let $(\xi_j)_{j \geq 1}$ be a stationary sequence of associated random variables, and $(u_n)_{n \geq 1}$ be a sequence of real numbers such that (18) holds. Suppose that*

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} n \sum_{j=m+1}^n \text{Cov}(1_{\{\xi_1 > u_n\}}, 1_{\{\xi_j > u_n\}}) = 0. \quad (23)$$

Then the sequence $(N_n)_{n \geq 1}$ of exceedance processes defined by (19) satisfies (AI) (with $(N_{i,n})_{i \leq k_n}$ defined by (20)), assuming that $\lim_{n \rightarrow \infty} r_n = \infty$.

Proof of Lemma 3.2: Let $f \in C_K^+(E)$ be arbitrary. Without loss of generality, we may assume that $f(x) \leq 1$ for all $x \in E$. For each $n \geq 1$, define

$$Y_{j,n} = f(j/n)1_{\{\xi_j > u_n\}}, \quad 1 \leq j \leq n.$$

Note that the random variables $(Y_{j,n})_{j \leq n}$ are associated. Clearly,

$$|E(e^{-N_n(f)}) - \prod_{i=1}^{k_n} E(e^{-N_{i,n}(f)})| = |E(e^{-\sum_{j=1}^n Y_{j,n}}) - \prod_{i=1}^{k_n} E(e^{-\sum_{j \in J_{i,n}} Y_{j,n}})|.$$

We follow the lines of proof of Lemma 5.4 of [1]. For each $1 \leq i \leq k_n$, let $H_{i,n}^{(m)}$ be the (big) block of consecutive integers between $(i-1)r_n + 1$ and $ir_n - m$ and $I_{i,n}^{(m)}$ be the (small) block of size m , consisting of consecutive integers between $ir_n - m$ and ir_n . Let

$$U_{i,n}^{(m)} = \sum_{j \in H_{i,n}^{(m)}} Y_{j,n}.$$

Similarly to (32) and (33) of [1], in order to prove (AI), it suffices to show that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} |E(e^{-\sum_{i=1}^{k_n} U_{i,n}^{(m)}}) - \prod_{i=1}^{k_n} E(e^{-U_{i,n}^{(m)}})| = 0. \quad (24)$$

For this we argue as for (36) and (37) in [1] to get,

$$\begin{aligned} & |E(e^{-\sum_{i=1}^{k_n} U_{i,n}^{(m)}}) - \prod_{i=1}^{k_n} E(e^{-U_{i,n}^{(m)}})| \leq \sum_{1 \leq i < l \leq k_n} \sum_{j \in H_{i,n}^{(m)}} \sum_{j' \in H_{l,n}^{(m)}} \text{Cov}(Y_{j,n}, Y_{j',n}) \\ &= \sum_{1 \leq i < l \leq k_n} \sum_{j \in H_{i,n}^{(m)}} \sum_{j' \in H_{l,n}^{(m)}} f(j/n)f(j'/n)\text{Cov}(1_{\xi_j > u_n}, 1_{\xi_{j'} > u_n}) \\ &\leq 2n \sum_{l=m+1}^n \text{Cov}(1_{\xi_1 > u_n}, 1_{\xi_l > u_n}), \end{aligned}$$

and relation (24) follows from (23). \square

Proof of Proposition 3.1: In this case, $D' = \emptyset$ (since N does not have fixed atoms), and

$$Y_{i,n} = \max\{j/n; j \in J_{i,n}, \xi_j > u_n\}, \quad \text{for all } i = 1, \dots, k_n.$$

To verify condition (a) of Theorem 2.2, we show that for any $x \in (0, 1]$:

$$\sum_{i=1}^{k_n} P(Y_{i,n} > x) \rightarrow a(1-x) = \nu(x, \infty). \quad (25)$$

Let $x \in (0, 1]$ be arbitrary. Then $x \leq k_n r_n / n$ for n large enough (since $k_n r_n / n \rightarrow 1$). It follows that for n large enough, $n x \in (0, k_n r_n] = \cup_{i=1}^{k_n} J_{i,n}$, and there exists $i_n \leq k_n$ such that $n x \in J_{i_n,n}$, i.e. $(i_n - 1)r_n / n + 1/n \leq x < i_n r_n / n$. We write

$$\begin{aligned} \sum_{i=1}^{k_n} P(Y_{i,n} > x) &= \sum_{i=1}^{i_n-1} P(Y_{i,n} > x) + P(Y_{i_n,n} > x) + \sum_{i=i_n+1}^{k_n} P(Y_{i,n} > x) \\ &=: I_1(n) + I_2(n) + I_3(n). \end{aligned}$$

Note that $\{Y_{i,n} > x\} = \emptyset$ for all $i \leq i_n - 1$, and hence $I_1(n) = 0$.

We have $\{Y_{i_n,n} > x\} \subset \cup_{j \in J_{i_n,n}} \{\xi_j > u_n\}$ and hence, by (18), $I_2(n) \leq \sum_{j \in J_{i_n,n}} P(\xi_j > u_n) = r_n P(\xi_1 > u_n) \leq \frac{1}{k_n} \cdot n P(\xi_1 > u_n) \rightarrow 0$.

Finally, we claim that $\{Y_{i,n} > x\} = \{\max_{j \in J_{i,n}} \xi_j > u_n\}$ for any $i \geq i_n + 1$. This is true since for any $j \in J_{i,n}$ and $i \geq i_n + 1$, we have:

$$\frac{j}{n} \geq \frac{(i-1)r_n + 1}{n} \geq \frac{i_n r_n}{n} + \frac{1}{n} > x + \frac{1}{n} > x.$$

Using the stationarity of the sequence $(\xi_j)_j$, the fact that $i_n/k_n \rightarrow x$, and (21), we obtain:

$$I_3(n) = \sum_{i=i_n+1}^{k_n} P(\max_{j \in J_{i,n}} \xi_j > u_n) = \left(1 - \frac{i_n}{k_n}\right) k_n P(\max_{j \in J_{1,n}} \xi_j > u_n) \rightarrow (1-x)a.$$

Relation (25) follows To verify condition (b) of Theorem 2.2, we will show that for any $x \in (0, 1]$ and for any $M \in \mathcal{M}_0$,

$$\sum_{i=1}^{k_n} P(N_{i,n} \in M, Y_{i,n} > x) \rightarrow a \sum_{k \geq 1} \pi_k \int_x^1 1_M(k\delta_y) dy = \int_x^\infty K(y, M) \nu(dy).$$

Let $x \in (0, 1]$ and $M \in \mathcal{M}_0$ be arbitrary. Then $n x \in J_{i_n,n}$ for some $i_n \leq k_n$, and n large enough. We write $\sum_{i=1}^{k_n} P(N_{i,n} \in M, Y_{i,n} > x) = J_1(n) + J_2(n) + J_3(n)$, where

$$\begin{aligned} J_1(n) &:= \sum_{i=1}^{i_n-1} P(N_{i,n} \in M, Y_{i,n} > x) \leq I_1(n) = 0 \\ J_2(n) &:= P(N_{i_n,n} \in M, Y_{i_n,n} > x) \leq I_2(n) \rightarrow 0 \\ J_3(n) &:= \sum_{i=i_n+1}^{k_n} P(N_{i,n} \in M, Y_{i,n} > x) = \sum_{i=i_n+1}^{k_n} P(N_{i,n} \in M, \max_{j \in J_{i,n}} \xi_j > u_n) \end{aligned}$$

Hence, it suffices to show that:

$$J'_3(n) := \sum_{i=i_n+1}^{k_n} P(N_{i,n} \in M, \max_{j \in J_{i,n}} \xi_j > u_n) \rightarrow a \sum_{k \geq 1} \pi_k \int_x^1 1_M(k\delta_y) dy. \quad (26)$$

By the stationary of the sequence $(\xi_j)_j$

$$\begin{aligned}
J'_3(n) &= \sum_{i=i_n+1}^{k_n} \sum_{k=1}^{r_n} P(N_{i,n} \in M, \max_{j \in J_{i,n}} \xi_j > u_n, \sum_{j \in J_{i,n}} 1_{\{\xi_j > u_n\}} = k) \\
&= \sum_{k=1}^{r_n} \sum_{i=i_n+1}^{k_n} P(N_{i,n} \in M \mid \sum_{j \in J_{i,n}} 1_{\{\xi_j > u_n\}} = k) P(\max_{j \in J_{i,n}} \xi_j > u_n) \pi_n(k) \\
&= \frac{n}{r_n} P(\max_{j \leq r_n} \xi_j > u_n) \sum_{k=1}^{r_n} p_n(k, M) \pi_n(k), \tag{27}
\end{aligned}$$

where

$$p_n(k, M) := \frac{r_n}{n} \sum_{i=i_n+1}^{k_n} P(N_{i,n} \in M \mid \sum_{j \in J_{i,n}} 1_{\{\xi_j > u_n\}} = k).$$

Let \mathcal{D} be the class of sets $M \in \mathcal{M}_0$, for which there exists a set $I_M \subset \{1, \dots, k_n\}$ (which does *not* depend on n) such that for any $n \geq 1$, for any $i \in \{1, \dots, k_n\} \setminus I_M$ and for any $k \in \{1, \dots, r_n\}$, we have:

$$\begin{aligned}
P\left(\sum_{j \in J_{i,n}} \delta_{j/n}(\cdot) 1_{\{\xi_j > u_n\}} \in M \mid \sum_{j \in J_{i,n}} 1_{\{\xi_j > u_n\}} = k\right) = \\
P\left(\sum_{j \in J_{i,n}} \delta_y(\cdot) 1_{\{\xi_j > u_n\}} \in M \mid \sum_{j \in J_{i,n}} 1_{\{\xi_j > u_n\}} = k\right), \quad \forall y \in J_{i,n}/n. \tag{28}
\end{aligned}$$

Using a monotone class argument, one can prove that:

$$\mathcal{D} = \mathcal{M}_0. \tag{29}$$

This shows that if $i \in \{1, \dots, k_n\} \setminus I_M$, then

$$\begin{aligned}
f_{i,n}(y) &:= P\left(\sum_{j \in J_{i,n}} \delta_y(\cdot) 1_{\{\xi_j > u_n\}} \in M \mid \sum_{j \in J_{i,n}} 1_{\{\xi_j > u_n\}} = k\right) \\
&= P\left(\sum_{j \in J_{i,n}} \delta_{j/n}(\cdot) 1_{\{\xi_j > u_n\}} \in M \mid \sum_{j \in J_{i,n}} 1_{\{\xi_j > u_n\}} = k\right) \\
&= P(N_{i,n} \in M \mid \sum_{j \in J_{i,n}} 1_{\{\xi_j > u_n\}} = k), \quad \text{for all } y \in J_{i,n}/n,
\end{aligned}$$

and hence

$$\int_{J_{i,n}/n} f_{i,n}(y) dy = \frac{r_n}{n} P(N_{i,n} \in M \mid \sum_{j \in J_{i,n}} 1_{\{\xi_j > u_n\}} = k).$$

On the other hand, $f_{i,n}(y) = 1_M(k\delta_y)$ and hence

$$\int_{J_{i,n}/n} f_{i,n}(y) dy = \int_{J_{i,n}/n} 1_M(k\delta_y) dy.$$

It follows that if $i \in \{1, \dots, k_n\} \setminus I_M$,

$$\frac{r_n}{n} P(N_{i,n} \in M \mid \sum_{j \in J_{i,n}} 1_{\{\xi_j > u_n\}} = k) = \int_{J_{i,n}/n} 1_M(k\delta_y) dy.$$

We return now to the calculation of $p_n(k, M)$. We split the sum over $i = i_n + 1, \dots, k_n$ into two sums, which contain the terms corresponding to indices $i \in I_M$, respectively $i \notin I_M$. The second sum is bounded by $\text{card}(I_M)r_n/n$. More precisely, we have:

$$\begin{aligned} p_n(k, M) &= \sum_{i \notin I_M} \frac{r_n}{n} P(N_{i,n} \in M \mid \sum_{j \in J_{i,n}} 1_{\{\xi_j > u_n\}} = k) + O\left(\frac{r_n}{n}\right) \\ &= \sum_{i \notin I_M} \int_{J_{i,n}/n} 1_M(k\delta_y) dy + O\left(\frac{r_n}{n}\right) \end{aligned} \quad (30)$$

$$\begin{aligned} &= \sum_{i=i_n+1}^{k_n} \int_{J_{i,n}/n} 1_M(k\delta_y) dy - \sum_{i_n+1 \leq i \leq k_n, i \in I_M} \int_{J_{i,n}/n} 1_M(k\delta_y) dy + O\left(\frac{r_n}{n}\right) \\ &= \int_{i_n r_n/n}^{k_n r_n/n} 1_M(k\delta_y) dy + O\left(\frac{r_n}{n}\right) = \int_x^1 1_M(k\delta_y) dy + O\left(\frac{r_n}{n}\right). \end{aligned} \quad (31)$$

Using (27) and (31), we obtain:

$$J'_3(n) = \frac{n}{r_n} P(\max_{j \leq r_n} \xi_j > u_n) \left\{ \sum_{k=1}^{r_n} \pi_n(k) \int_x^1 1_M(k\delta_y) dy + O\left(\frac{r_n}{n}\right) \right\}$$

Using (21) and (22), it follows that: $J'_3(n) \rightarrow a \sum_{k \geq 1} \pi_k \int_x^1 1_M(k\delta_y) dy$. Here, we used the fact that

$$\left| \sum_{k=1}^{r_n} \pi_n(k) \int_x^1 1_M(k\delta_y) dy - \sum_{k \geq 1} \pi_k \int_x^1 1_M(k\delta_y) dy \right| \leq (1-x) \sum_{k \geq 1} |\pi_n(k) - \pi_k| \rightarrow 0,$$

where the last convergence is justified by Scheffé's theorem (see e.g. Theorem 16.12 of [5]), since $\sum_{k \geq 1} \pi_n(k) = \sum_{k \geq 1} \pi(k) = 1$. This concludes the proof of (26). \square

3.2 Processes based on regularly varying sequences

In this subsection, we take $E = \bar{\mathbb{R}} \setminus \{0\}$. Let $(\xi_j)_{j \geq 1}$ be a stationary sequence of random variables with values in $\mathbb{R} \setminus \{0\}$, and $(a_n)_n$ be a sequence of positive real numbers such that:

$$\lim_{n \rightarrow \infty} nP(|\xi_1| > a_n) = 1. \quad (32)$$

We consider the following point process on $\bar{\mathbb{R}} \setminus \{0\}$: $N_n = \sum_{j=1}^n \delta_{\xi_j/a_n}$. The mixing condition introduced in [8] is the following. We say that $(\xi_j)_{j \geq 1}$ satisfies

condition $\mathcal{A}(\{a_n\})$ if there exists a sequence $k_n \rightarrow \infty$ with $r_n = \lfloor n/k_n \rfloor \rightarrow \infty$ such that:

$$\lim_{n \rightarrow \infty} |E(e^{-\sum_{j=1}^n f(\xi_j/a_n)}) - \{E(e^{-\sum_{j=1}^{r_n} f(\xi_j/a_n)}\}^{k_n}| = 0, \quad \forall f \in C_K^+(E).$$

Note that $\mathcal{A}(\{a_n\})$ is equivalent to saying that $(N_n)_{n \geq 1}$ satisfies (AI), with

$$N_{i,n} = \sum_{j \in J_{i,n}} \delta_{\xi_j/a_n}, \quad (33)$$

and $J_{i,n} = ((i-1)r_n, ir_n]$. In particular, if $(\xi_j)_j$ is strongly mixing, then $\mathcal{A}(\{a_n\})$ holds (see e.g. Lemma 5.1 of [1]). In addition, suppose that ξ_1 has *regularly varying tail probabilities* of order $\alpha > 0$, i.e.

$$P(|\xi_1| > x) = x^{-\alpha}L(x), \quad \lim_{x \rightarrow \infty} \frac{P(\xi_1 > x)}{P(|\xi_1| > x)} = p, \quad \lim_{x \rightarrow \infty} \frac{P(\xi_1 < -x)}{P(|\xi_1| > x)} = q, \quad (34)$$

where L is a slowly varying function, $p \in [0, 1]$ and $q = 1 - p$.

Let $\tilde{M} = \{\mu \in M_p(E); \mu([-1, 1]^c) = 0, \mu(\{-1, 1\}) > 0\}$, and consider the following point process with values in \tilde{M} , $\xi_{1,n} = \sum_{j=1}^{r_n} \delta_{\xi_j/\max_{i \leq r_n} |\xi_i|}$.

Theorem 2.3 and Theorem 2.5 of [8] show that under (34) and $\mathcal{A}(\{a_n\})$, $N_n \xrightarrow{d} N$ if and only if the following two conditions hold:

$$k_n P(\max_{j \leq r_n} |\xi_j| > a_n x) \rightarrow \theta x^{-\alpha}, \quad \forall x > 0, \quad (35)$$

$$P(\xi_{1,n} \in \cdot \mid \max_{j \leq r_n} |\xi_j| > a_n x) \xrightarrow{w} \mathcal{Q}(\cdot), \quad \forall x > 0, \quad (36)$$

where $\theta \in (0, 1]$ and \mathcal{Q} is a probability measure on \tilde{M} . In this case, θ is the extremal index of $(|\xi_j|)_{j \geq 1}$, and N is an ID point process (without fixed atoms). Since $\lambda \circ \Omega^{-1} = \nu \times \mathcal{Q}$, where

$$\nu(dy) = \theta \alpha y^{-\alpha-1} dy \quad (37)$$

and $\Omega : M_0 \rightarrow (0, \infty) \times \tilde{M}$ is defined by $\Omega(\mu) = (x_\mu, \mu(x_\mu \cdot))$, it follows that:

$$L_N(f) = \exp \left\{ - \int_0^\infty \int_{\tilde{M}} (1 - e^{-\mu(f(y \cdot))}) \mathcal{Q}(d\mu) \nu(dy) \right\}. \quad (38)$$

On the other hand, by Lemma 2.6, the Laplace functional of N is given by:

$$L_N(f) = \exp \left\{ - \int_0^\infty \int_{M_0} (1 - e^{-\mu(f)}) K(y, d\mu) \nu(dy) \right\}$$

This allows us to identify the relationship between the probability measure \mathcal{Q} and the kernel K , namely:

$$K(y, M) = \mathcal{Q}(\pi_y^{-1}(M) \cap \tilde{M}), \quad (39)$$

where $\pi_y : M_0 \rightarrow M_0$ is defined by $\pi_y(\mu) = \mu(y^{-1} \cdot)$, i.e. $\pi_y(\sum_j \delta_{t_j}) = \sum_j \delta_{yt_j}$.

The next result shows that conditions (35) and (36) are equivalent to conditions (a) and (b) of Theorem 2.2, in the setting of the present subsection.

Proposition 3.3 *Suppose that the sequence $(N_n)_n$ satisfies (AI) with $(N_{i,n})_{i \leq k_n}$ given by (33). Then (35) and (36) are equivalent to conditions (a) and (b) of Theorem 2.2, with $\lambda(M_x), \lambda(M \cap M_x)$ given by (17), and ν, K given by (37) and (39).*

Proof: In this case, $D' = \emptyset$ and $Y_{1,n} = \max_{j \leq r_n} |\xi_j|/a_n$. Condition (a) of Theorem 2.2 is in fact (35). Assuming now that (a) holds, we show that condition (b) of Theorem 2.2 is equivalent to (36).

Let $\lambda_n = k_n P \circ N_{1,n}^{-1}$. Define the following probability measures on M_0 :

$$\begin{aligned} P_{n,x}(M) &= \frac{\lambda_n(M \cap M_x)}{\lambda_n(M_x)} = \frac{k_n P(N_{1,n} \in M, Y_{1,n} > x)}{k_n P(Y_{1,n} > x)} \\ P_x(M) &= \frac{\lambda(M \cap M_x)}{\lambda(M_x)} = \frac{1}{\nu(x, \infty)} \int_x^\infty K(y, M) \nu(dy). \end{aligned}$$

By (a), $k_n P(Y_{1,n} > x) \rightarrow \nu(x, \infty)$. Hence, condition (b) of Theorem 2.2 is equivalent to

$$P_{n,x} \xrightarrow{w} P_x, \quad \text{for all } x > 0. \quad (40)$$

On the other hand, for any $M \in \tilde{\mathcal{M}}$,

$$\begin{aligned} k_n P(\xi_{1,n} \in M, Y_{1,n} > x) &= k_n P(N_{1,n} \in \{\mu \in M_0; x_\mu > x, \mu(x_\mu \cdot) \in M\}, N_{1,n} \in M_x) \\ &= k_n (P \circ N_{1,n}^{-1})(\{\mu \in M_0; x_\mu > x, \mu(x_\mu \cdot) \in M\} \cap M_x) = \lambda_n(\Omega^{-1}((x, \infty) \times M) \cap M_x), \end{aligned}$$

and hence

$$P(\xi_{1,n} \in M | Y_{1,n} > x) = \frac{\lambda_n(\Omega^{-1}((x, \infty) \times M) \cap M_x)}{\lambda_n(M_x)} = (P_{n,x} \circ \Omega^{-1})((x, \infty) \times M).$$

Since $\lambda \circ \Omega^{-1} = \nu \times \mathcal{Q}$,

$$\mathcal{Q}(M) = \frac{\lambda(\Omega^{-1}((x, \infty) \times M))}{\nu^*(x, \infty)} = (P_x \circ \Omega^{-1})((x, \infty) \times M), \quad \forall M \in \tilde{\mathcal{M}}.$$

Therefore, relation (36) is equivalent to:

$$P_{n,x} \circ \Omega^{-1}((x, \infty) \times \cdot) \xrightarrow{w} P_x \circ \Omega^{-1}((x, \infty) \times \cdot), \quad \text{for all } x > 0. \quad (41)$$

Using the argument on page 888 of [8], one can show that (40) is equivalent to (41). (This argument uses the continuous mapping theorem, and the fact that both Ω and Ω^{-1} are continuous.) \square

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