Some linear SPDEs driven by a fractional noise with Hurst index greater than 1/2

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Abstract

In this article, we identify the necessary and sufficient conditions for the existence of a random field solution for some linear s.p.d.e.’s of parabolic and hyperbolic type. These equations rely on a spatial operator $\mathcal{L}$ given by the $L^2$-generator of a $d$-dimensional Lévy process $X = (X_t)_{t \geq 0}$, and are driven by a spatially-homogeneous Gaussian noise, which is fractional in time with Hurst index $H > 1/2$. As an application, we consider the case when $X$ is a $\beta$-stable process, with $\beta \in (0, 2]$. In the parabolic case, we develop a connection with the potential theory of the Markov process $\tilde{X}$ (defined as the symmetrization of $X$), and we show that the existence of the solution is related to the existence of a “weighted” intersection local time of two independent copies of $\tilde{X}$.

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1 Introduction

In 1944, in his seminal article [19], Itô introduced the stochastic integral with respect to the Brownian motion, which turned out to be one of the most fruitful ideas in mathematics in the 20th century. This lead to the theory of diffusions (whose origins can be traced back to [20]), and the development of the stochastic calculus with respect to martingales (initiated in [25]). These ideas have grown into a solid branch of probability theory called stochastic analysis, which includes the study of stochastic partial differential equations (s.p.d.e.’s)

Traditionally, there have been several approaches for the study of s.p.d.e.’s. The most important are: the Da Prato and Zabczyk approach which uses

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stochastic integrals with respect to Hilbert-space-valued Wiener processes (see [10]), the Krylov approach which uses the concept of function-space-valued solution (see [24]), and the Walsh approach which relies on stochastic integrals with respect to martingale-measures (see [30]). These approaches have been developed at the same time, and nowadays a lot of effort is dedicated to unify them (see the recent article [8] and the references therein).

The fractional Brownian motion (fBm) was introduced by Kolmogorov in [23], who called it the “Wiener spiral”, and is defined as a zero-mean Gaussian process \( \{B_t\}_{t \geq 0} \) with covariance:

\[
R_H(t, s) = E(B_t B_s) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).
\]

The parameter \( H \) lies in \((0, 1)\), and is called the Hurst index (due to [18]). The case \( H = 1/2 \) corresponds to the Brownian motion, whereas the cases \( H > 1/2 \) and \( H < 1/2 \) have many contrasting properties and cannot be handled simultaneously. The representation of the fBm as a stochastic integral with respect to the Brownian motion on \( \mathbb{R} \) was obtained as early as 1968 (see [26]), but the fBm began to be used intensively in stochastic analysis only in the late 1990’s. It is the flexibility which stems from the choice of the parameter \( H \) that makes the fBm a much more attractive model for the noise than the Brownian motion (and its infinite-dimensional counterparts).

Among the fBm’s remarkable properties is the fact that it is not a semi-martingale. Consequently, Itô calculus cannot be used in this case. A stochastic calculus with respect to the fBm was developed for the first time in [11]. Subsequent important contributions were made in [1], [5] and [12]. The stochastic integral used by these authors is an extension of the Itô integral introduced by Hitsuda in [16] (and refined in [21] and [29]), and coincides with the divergence operator. These techniques are based on Malliavin calculus. Alternative methods for defining a stochastic integral with respect to the fBm exploit the Hölder continuity property of its sample paths and are based on Young-type integrals.

In the present article, we consider the parabolic Cauchy problem

\[
\frac{\partial u}{\partial t}(t, x) = \mathcal{L}u(t, x) + \dot{W}(t, x), \quad t > 0, x \in \mathbb{R}^d \quad (1)
\]

\[
u(0, x) = 0, \quad x \in \mathbb{R}^d,
\]

and the hyperbolic Cauchy problem

\[
\frac{\partial^2 u}{\partial t^2}(t, x) = \mathcal{L}u(t, x) + \dot{W}(t, x), \quad t > 0, x \in \mathbb{R}^d \quad (2)
\]

\[
u(0, x) = 0, \quad x \in \mathbb{R}^d
\]

\[
\frac{\partial u}{\partial t}(0, x) = 0, \quad x \in \mathbb{R}^d,
\]

where \( \mathcal{L} \) is a “spatial operator” (i.e. it acts only on the \( x \) variable) given by the \( L^2(\mathbb{R}^d) \)-generator of a \( d \)-dimensional Lévy process \( X = (X_t)_{t \geq 0} \), and \( W \) is
a Gaussian noise whose covariance is written formally as:

$$E[W(t,x)W(s,y)] = |t - s|^{2H - 2} f(x - y),$$

for some index $H > 1/2$ and some kernel $f$ (to be defined below).

The rigorous definition of the noise $\dot{W}$ is given in Section 2. At this point, we should just mention that the covariance structure of the noise has two components: a spatially-homogeneous component specified by the kernel $f$ (the example that we have in mind being the Riesz kernel $f(x) = |x|^{-(d-\alpha)}$, with $0 < \alpha < d$), and a temporal component inherited from the fBm. This becomes clear once we realize that if $H > 1/2$, $R_H(t,s)$ can be written as:

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |u - v|^{2H - 2} du dv,$$

with $\alpha_H = H(2H - 1)$.

(The case $H < 1/2$ has to be treated differently and is not discussed here.)

The solution to problem (1) (or (2)) is understood in the mild-sense, and one of the goals of the present article is to give a necessary and sufficient condition for the existence of this solution, in terms of the parameters $(H,f)$ of the noise, and the spatial operator $\mathcal{L}$. A similar problem has been considered in [14] and [15] in the case $H = 1/2$ (which corresponds to the white noise in time). This motivated us to examine the case $H > 1/2$.

The case of the hyperbolic equation with spatial operator $\mathcal{L} = -(-\Delta)^{\beta/2}$, $\beta > 0$, driven by a white noise in time was examined in [7] and [9]. In fact, these authors consider the much more difficult case of the non-linear equation $\partial_t u = \mathcal{L}u + \sigma(u)\dot{W} + b(u)$ with arbitrary initial conditions, and Lipschitz continuous functions $\sigma$ and $b$. For the linear equation, it turns out that the necessary and sufficient condition for the existence of the solution is:

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^\beta} \mu(d\xi) < \infty,$$

where the measure $\mu$ is the inverse Fourier transform of $f$ in $\mathcal{S}'(\mathbb{R}^d)$.

In the case of equations (1) and (2) driven by a space-time white noise (i.e. $H = 1/2$ and $f = \delta_0$), the authors of [14] have shown that the necessary and sufficient condition for the existence of a random field solution is:

$$\int_{\mathbb{R}^d} \frac{1}{1 + \text{Re} \Psi(\xi)} \mu(d\xi) < \infty,$$

where $\Psi(\xi)$ is the characteristic exponent of the underlying Lévy process $X$. An important observation of [14] is that condition (4) can be extrapolated in a different context, being the condition which guarantees the existence of a local time $\int_0^t \delta_0(\bar{X}_s) ds$ of the symmetrization $\bar{X}$ of $X$. This line of investigation was continued in [15] in the case of the parabolic equation (1) with white noise in time, but covariance kernel $f$ in space. Surprisingly, it is shown there that condition (4) is related not only to the existence of the “occupation” local time $L_t(f) = \int_0^t f(\bar{X}_s) ds$, but also to the potential theory of the process $\bar{X}$, when viewed as a Markov process.
In the present article, we carry out a similar program in the case of the fractional noise in time. More precisely, after the introduction of some background material in Section 2, the article is split between the two problems: Section 3 is dedicated to the parabolic problem (1), while Section 4 treats the hyperbolic problem (2). For the parabolic problem, we discuss three things: the existence of a random field solution, the connection with the potential theory of Markov processes, and the relationship with the “weighted” intersection local time

\[ L_{t,H}(f) = \alpha_H \int_0^t \int_0^t |r - s|^{2H-2} f(\hat{X}_r^1 - \hat{X}_s^2) dr ds, \]

where \( \hat{X}_r^1 \) and \( \hat{X}_s^2 \) are two independent copies of \( \hat{X} \). For the hyperbolic problem, we only discuss the existence of a random field solution in the case when \( \Psi(\xi) \) is real-valued (i.e. \( X \) is symmetric).

Unlike the case of the white noise in time, it turns out that for the fractional noise, the conditions for the existence of the solution are different for the parabolic and hyperbolic problems. These conditions are:

\[ \int_{\mathbb{R}^d} \left( \frac{1}{1 + \text{Re}\Psi(\xi)} \right)^{2H} \mu(d\xi) < \infty, \] (5)

in the parabolic case, respectively,

\[ \int_{\mathbb{R}^d} \left( \frac{1}{1 + \text{Re}\Psi(\xi)} \right)^{H+1/2} \mu(d\xi) < \infty, \] (6)

in the hyperbolic case. This phenomenon was observed for the first time in [4] for the wave and heat equations. As an application, we discuss the case when \( X \) is a \( \beta \)-stable process with \( \beta \in (0, 2] \), and hence \( \Psi(\xi) = |\xi|^\beta \). In this case, conditions (5) and (6) turn out to be generalizations of (3).

The fact that the fractional noise induces a connection with the weighted intersection local time was also used in [17], when \( f = \delta_0 \). In [3], it is shown that the existence of the exponential moment of \( L_{t,H}(f) \) is closely related to the existence of the (mild) solution of the heat equation with multiplicative noise.

We now introduce the notation used in the present article. The Fourier transform of a function \( \varphi \in L^1(\mathbb{R}^d) \) is defined by:

\[ \mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \varphi(x) dx. \]

It is known that the Fourier transform can be extended to \( L^2(\mathbb{R}^d) \) (see e.g. [13]).

Let \( \mathcal{S}(\mathbb{R}^d) \) be the Schwartz space of rapidly decreasing infinitely differentiable functions on \( \mathbb{R}^d \). A continuous linear functional on \( \mathcal{S}(\mathbb{R}^d) \) is called a tempered distribution. Let \( \mathcal{S}'(\mathbb{R}^d) \) be the space of tempered distributions. The Fourier transform \( \mathcal{F}S \) of a functional \( S \in \mathcal{S}'(\mathbb{R}^d) \) is defined by:

\[ (\mathcal{F}S, \varphi) = (S, \mathcal{F}\varphi), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d). \]
2 Preliminaries

In this section, we introduce some background material about the Gaussian noise \( W \) and the Lévy process \( X \).

2.1 The Gaussian noise

As in [6], we let \( f : \mathbb{R}^d \to [0, \infty] \) be a measurable locally integrable function (or a kernel). We assume that \( f \) is the Fourier transform in \( S' (\mathbb{R}^d) \) of a tempered measure \( \mu \), i.e.

\[
\int_{\mathbb{R}^d} f(x) \varphi(x) dx = \int_{\mathbb{R}^d} \mathcal{F} \varphi(\xi) \mu(d\xi), \quad \forall \varphi \in S(\mathbb{R}^d).
\]

(7)

It follows that for any \( \varphi, \psi \in S(\mathbb{R}^d) \),

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x) \psi(y) f(x-y) dx dy = \int_{\mathbb{R}^d} \mathcal{F} \varphi(\xi) \overline{\psi(\xi)} \mu(d\xi).
\]

(8)

Similarly to [4], we let \( \mathcal{E} \) be the set of elementary functions of the form

\[
h(t,x) = \phi(t) \psi(x), \quad t \geq 0, \ x \in \mathbb{R}^d,
\]

where \( \phi \) is a linear combination of indicator functions \( 1_{[0,a]} \) with \( a > 0 \), and \( \psi \in S(\mathbb{R}^d) \). We endow \( \mathcal{E} \) with the inner product:

\[
\langle h_1, h_2 \rangle_{\mathcal{H}^P} = \alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |t-s|^{2H-2} f(x-y) h_1(t,x) h_2(s,y) dx dy ds dt.
\]

Let \( \mathcal{H}^P \) be the completion of \( \mathcal{E} \) with respect to the inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}^P} \). We note that the space \( \mathcal{H}^P \) may contain distributions in both \( t \) and \( x \) variables.

We consider a zero-mean Gaussian process \( \{W(h); h \in \mathcal{E}\} \) with covariance

\[
E(W(h_1)W(h_2)) = \langle h_1, h_2 \rangle_{\mathcal{H}^P}.
\]

The map \( h \mapsto W(h) \) is an isometry between \( \mathcal{E} \) and the Gaussian space of \( W \), which can be extended to \( \mathcal{H}^P \). This extension defines an isonormal Gaussian process \( W = \{W(h); h \in \mathcal{H}^P\} \). We write

\[
W(h) = \int_0^\infty \int_{\mathbb{R}^d} h(t,x) W(dt,dx), \quad \text{for any } h \in \mathcal{H}^P.
\]

This defines the stochastic integral of an element \( h \in \mathcal{H}^P \) with respect to the noise \( W \).

2.2 The Lévy process

As in [15], we let \( X = (X_t)_{t \geq 0} \) be a \( d \)-dimensional Lévy process with characteristic exponent \( \Psi(\xi) \). Hence, \( X_0 = 0 \) and for any \( t > 0 \),

\[
E(e^{-\xi \cdot X_t}) = e^{-t\Psi(\xi)}, \quad \text{for all } \xi \in \mathbb{R}^d.
\]

(9)
By the Lévy-Khintchine formula (see e.g. Theorem 8.1 of [28]),
\[ \Psi(\xi) = i\gamma \cdot \xi + \xi^T A \xi - \int_{\mathbb{R}^d} (e^{-i\xi \cdot x} - 1 + i\xi \cdot x 1_{\{|x| \leq 1\}}) \nu(dx), \]
where \((\gamma, A, \nu)\) is the generating triplet of \(X\). Note that \(\text{Re}\Psi(\xi) \geq 0 \ \forall \xi \in \mathbb{R}^d\).

\(X\) is a homogenous Markov process with transition probabilities:
\[ Q_t(x; B) = P(X_{t+t} \in B | X_t = x) = P(X_t \in B - x), \]
for any \(x \in \mathbb{R}^d\) and Borel set \(B \subset \mathbb{R}^d\). Let \((P_t)_{t \geq 0}\) be the associated semigroup:
\[ (P_t\phi)(x) = \int_{\mathbb{R}^d} \phi(y)Q_t(x; dy) = E[\phi(x + X_t)], \]
for any bounded (or non-negative) measurable function \(\phi: \mathbb{R}^d \to \mathbb{R}\).

Let \(L\) be the \(L^2(\mathbb{R}^d)\)-generator of \((P_t)_{t \geq 0}\), defined by: (see p.16 of [15])
\[ \mathcal{L}\phi = \lim_{t \to 0} \frac{P_t\phi - \phi}{t} \quad \text{in} \quad L^2(\mathbb{R}^d) \quad \text{(if it exists)}. \]
Then \(\mathcal{L}\phi\) exists if and only if \(\phi \in \text{Dom}(\mathcal{L}) = \{\phi \in L^2(\mathbb{R}^d); (F\phi)\Psi \in L^2(\mathbb{R}^d)\}\).

### 3 The Parabolic Equation

In this section, we assume that the law of \(X_t\) has a density denoted by \(p_t\). This assumption allows us to identify the fundamental solution of \(\partial_t u - Lu = 0\).

To see this, note that \(Q_t(x; \cdot)\) has density \(p_t(\cdot - x)\), and \(P_t\phi = \phi * \tilde{p}_t\), where \(\tilde{p}_t(x) = p_t(-x)\). Since the solution of the Kolmogorov’s equation \(\partial_t u(t, x) = Lu(t, x)\) with initial condition \(u(0, x) = u_0(x)\) is
\[ u(t, x) = (P_t u_0)(x) = \int_{\mathbb{R}^d} u_0(y)p_t(y - x)dy, \]
it follows that the fundamental solution of \(\partial_t u - Lu = 0\) is the function:
\[ G(t, x) = p_t(-x), \quad t > 0, x \in \mathbb{R}^d. \]
From (9), we obtain that:
\[ \mathcal{F}G(t, \cdot)(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} p_t(x)dx = E(e^{i\xi \cdot X_t}) = e^{-t\Psi(\xi)}. \quad (10) \]

#### 3.1 Existence of the Random-Field Solution

There are two equivalent ways of defining a random field solution for problem (1). Similarly to [4], one can say that the process \(\{u(t, x); t \geq 0, x \in \mathbb{R}^d\}\) defined by:
\[ u(t, x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)W(ds, dy) \]
is a random field solution of (1), provided that the stochastic integral above is well-defined, i.e. the integrand
\[ \mathbb{R}_+ \times \mathbb{R}^d \ni (s,y) \mapsto g_{tx}(s,y) = 1_{[0,t]}(s)G(t-s,x-y) \] belongs to \( \mathcal{H}P \).

Since \( g_{tx} \) satisfies conditions (i)-(iii) of Theorem 2.1 of [4], to check that \( g_{tx} \in \mathcal{H}P \), it suffices to prove that:
\[ I_t := \alpha_H \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty Fg_{tx}(r,\cdot)(\xi)Fg_{tx}(s,\cdot)(\xi)|r-s|^{2H-2}drds\mu(d\xi) < \infty. \]

In this case,
\[ E|u(t,x)|^2 = E|W(g_{tx})|^2 = \| g_{tx} \|^2_{\mathcal{H}P} = I_t. \]

Note that by (10), \( I_t = \int_{\mathbb{R}^d} N_t(\xi)\mu(d\xi) \), where
\[ N_t(\xi) = \alpha H \int_0^t \int_0^t e^{-r\Psi(\xi)}e^{-s\Psi(\xi)}|r-s|^{2H-2}drds. \]

Therefore, the question about the existence of a random field solution of (1) reduces to finding suitable upper and lower bounds for \( N_t(\xi) \).

Alternatively, the authors of [14] suggest a different method for defining the random field solution of (1), which has the advantage that can be applied also to the hyperbolic problem (2) (for which one cannot identify the fundamental solution \( G \)). Since this is the method that we use in the present article, we explain it below.

We say that \( \{ u(t,\varphi); t \geq 0, \varphi \in S(\mathbb{R}^d) \} \) is a the weak solution of (1) if
\[ u(t,\varphi) = \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\varphi(x)dx \right) W(ds,dy). \]

Note that the stochastic integral above is well-defined (as a random variable in \( L^2(\Omega) \)) if and only if the integrand
\[ \mathbb{R}_+ \times \mathbb{R}^d \ni (s,y) \mapsto h_{t,\varphi}(s,y) = 1_{[0,t]}(s)(\varphi * p_{t-s})(y) \] belongs to \( \mathcal{H}P \).

Since \( h_{t,\varphi} \) satisfies the conditions (i)-(iii) of Theorem 2.1 of [4], to check that \( h_{t,\varphi} \in \mathcal{H}P \) it suffices to show that
\[ I_{t,\varphi} := \alpha_H \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty |r-s|^{2H-2}\mathcal{F}h_{t,\varphi}(r,\cdot)(\xi)\mathcal{F}h_{t,\varphi}(s,\cdot)(\xi)drds\mu(d\xi) < \infty. \]

In this case, we have
\[ E|u(t,\varphi)|^2 = E|W(h_{t,\varphi})|^2 = \| h_{t,\varphi} \|^2_{\mathcal{H}P} = I_{t,\varphi}. \]

Since both \( \varphi \) and \( p_{t-s} \) are in \( L^1(\mathbb{R}^d) \),
\[ \mathcal{F}(\varphi * p_{t-s})(\xi) = \mathcal{F}\varphi(\xi)\mathcal{F}p_{t-s}(\xi) = \mathcal{F}\varphi(\xi)e^{-(t-s)\Psi(\xi)}, \]
and hence

\[ I_{t, \varphi} = \int_{\mathbb{R}^d} N_t(\xi)|\mathcal{F}\varphi(\xi)|^2 \mu(d\xi). \]

Using the trivial bound \( N_t(\xi) \leq t^{2H} \) (which is obtained using the fact that \(|e^{-s\Psi(\xi)}| = e^{-s\text{Re}\Psi(\xi)} \leq 1\) for all \( s > 0 \) and \( \xi \in \mathbb{R}^d \)), we get:

\[ I_{t, \varphi} \leq t^{2H} \int_{\mathbb{R}^d} |\mathcal{F}\varphi(\xi)|^2 \mu(d\xi) < \infty \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^d). \]

This shows that \( u(t, \varphi) \) is a well-defined random variable in \( L^2(\Omega) \).

We continue to explain the method of [14]. We endow \( \mathcal{S}(\mathbb{R}^d) \) with the inner product:

\[ \langle \varphi, \psi \rangle_t = E(u(t, \varphi)u(t, \psi)). \]

We denote by \( \| \cdot \|_t \) the norm induced by the inner product \( \langle \cdot, \cdot \rangle_t \), i.e.

\[ \|\varphi\|_t^2 = E|u(t, \varphi)|^2 = \int_{\mathbb{R}^d} N_t(\xi)|\mathcal{F}\varphi(\xi)|^2 \mu(d\xi). \quad (12) \]

Let \( M_t \) be the completion of \( \mathcal{S}(\mathbb{R}^d) \) with respect to \( \langle \cdot, \cdot \rangle_t \) and \( M = \cap_{t > 0} M_t \). We say that (1) has a random field solution if and only if

\[ \delta_x \in M \quad \text{for all } x \in \mathbb{R}^d. \quad (13) \]

The random field solution is defined by \( \{u(t, x) = u(t, \delta_x); t \geq 0, x \in \mathbb{R}^d\} \).

To prove (13), we introduce the space \( \mathcal{Z} = \cap_{t > 0} \mathcal{Z}_t \), where \( \mathcal{Z}_t \) is the completion of \( \mathcal{S}(\mathbb{R}^d) \) with respect to the inner product \( \langle \cdot, \cdot \rangle_t \) defined by:

\[ \langle \varphi, \psi \rangle_t = \int_{\mathbb{R}^d} \left( \frac{1}{1/|t + \text{Re}\Psi(\xi)|} \right)^{2H} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)} \mu(d\xi) =: \mathcal{E}(t; \varphi, \psi) \]

We denote by \( ||| \cdot |||_t \) the norm induced by the inner product \( \langle \cdot, \cdot \rangle_t \), i.e.

\[ |||\varphi|||_t = \int_{\mathbb{R}^d} \left( \frac{1}{1/|t + \text{Re}\Psi(\xi)|} \right)^{2H} |\mathcal{F}\varphi(\xi)|^2 \mu(d\xi) =: \mathcal{E}(t; \varphi). \quad (14) \]

By Lemma A.2 (Appendix A), for any \( s, t > 0 \), there exist some positive constants \( c_1(s, t) \) and \( c_2(s, t) \) such that for any \( \varphi \in \mathcal{S}(\mathbb{R}^d) \),

\[ c_1(s, t)^{2H} \mathcal{E}(s; \varphi) \leq \mathcal{E}(t; \varphi) \leq c_2(s, t)^{2H} \mathcal{E}(s; \varphi). \]

Therefore, the norms \( ||| \cdot |||_t \) and \( ||| \cdot |||_t \) are equivalent and \( \mathcal{Z}_t = \mathcal{Z}_s = \mathcal{Z} \).

The idea for proving (13) is to show that any norm \( ||| \cdot |||_t \) is equivalent to a norm \( ||| \cdot |||_{\rho(t)} \), for a certain bijective function \( \rho : \mathbb{R}_+ \to \mathbb{R}_+ \). From this, one infers that \( M_t = \mathcal{Z}_{\rho(t)} \) for any \( t > 0 \), and hence \( M = \mathcal{Z} = \mathcal{Z}_1 \). Condition (13) becomes \( \delta_x \in \mathcal{Z}_1 \) for all \( x \in \mathbb{R}^d \), for which one can find a natural necessary and sufficient condition (see Theorem 3.3 below). In the case of the parabolic problem (1), it turns out that \( \rho(t) = t \). (We will see in Section 4 that for the hyperbolic problem (2), \( \rho(t) = t^2 \).)
The next theorem is the main result of the present section, and gives the desired upper and lower bounds for $N_t(\xi)$. Unfortunately, for the lower bound, we had to introduce an additional condition of boundedness on the ratio between the imaginary part and the real part of the characteristic exponent $\Psi(\xi)$. A similar difficulty has been encountered in [22] for obtaining a lower bound for the “sojourn operator”. Our condition (16) is similar to condition (3.3) of [22], and is trivially satisfied when $\Psi$ is real-valued.

We use the following inequality: there exists a constant $b_H > 0$, such that

$$\alpha_H \int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi(r)||\varphi(s)||r-s|^{2H-2}drds \leq b_H^2 \left( \int_{\mathbb{R}} |\varphi(s)|^{1/H}ds \right)^{2H} \quad (15)$$

for any $\varphi \in L^{1/H}(\mathbb{R})$. This inequality was proved in [27] and is a consequence of the Littlewood-Hardy inequality.

For complex-valued functions $\varphi$, we define:

$$\|\varphi\|^2_{(0,t)} := \alpha_H \int_0^t \int_0^t \varphi(r)\overline{\varphi(s)}|r-s|^{2H-2}drds = \|\text{Re}\varphi\|^2_{(0,t)} + \|\text{Im}\varphi\|^2_{(0,t)}.$$

**Theorem 3.1** For any $t > 0$ and $\xi \in \mathbb{R}^d$,

$$N_t(\xi) \leq C_H \left( \frac{1}{1/t + \text{Re}\Psi(\xi)} \right)^{2H},$$

where $C_H = H^{2H}b_H^2 e^2$. If in addition, there exists a constant $K > 0$ such that:

$$|\text{Im}\Psi(\xi)| \leq K\text{Re}\Psi(\xi), \quad \forall \xi \in \mathbb{R}^d. \quad (16)$$

then, for any $t > 0$ and $\xi \in \mathbb{R}^d$,

$$N_t(\xi) \geq C_{H,K} \left( \frac{1}{1/t + \text{Re}\Psi(\xi)} \right)^{2H},$$

where $C_{H,K}$ is a positive constant depending on $H$ and $K$.

**Proof:** For the upper bound, we note that $N_t(\xi)$ can be written as

$$N_t(\xi) = \alpha_H \int_0^t \int_0^t e^{-r\text{Re}\Psi(\xi)}e^{-s\text{Re}\Psi(\xi)}|r-s|^{2H-2}\cos[(r-s)\text{Im}\Psi(\xi)]drds.$$

Using the fact that $|\cos x| \leq 1$ and $e^{-r\text{Re}\Psi(\xi)} \leq e^{t/\lambda}e^{-r(1/\lambda+\text{Re}\Psi(\xi))}$ for any $r \in [0,t]$, we get:

$$N_t(\xi) \leq e^{2t/\lambda} \alpha_H \int_0^t \int_0^t e^{-r(1/\lambda+\text{Re}\Psi(\xi))}e^{-s(1/\lambda+\text{Re}\Psi(\xi))}|r-s|^{2H-2}drds.$$

By (15), it follows that:

$$N_t(\xi) \leq e^{2t/\lambda}b_H^2 \left( \int_0^t e^{-r(1/\lambda+\text{Re}\Psi(\xi))/H}dr \right)^{2H} \leq b_H^2 e^{2t/\lambda} \left( \frac{H}{1/\lambda + \text{Re}\Psi(\xi)} \right)^{2H}.$$
The conclusion follows by taking $\lambda = t$.

For the lower bound, suppose first that $t \text{Re}\Psi(\xi) \leq a$, for some constant $a = aK \in (0,1)$ such that $Ka < \pi/2$. By (16),

$$t |\text{Im}\Psi(\xi)| \leq Kt \text{Re}\Psi(\xi) \leq Ka < \frac{\pi}{2}.$$ 

Using the fact that $e^{-x} \geq 1 - x$ for $x > 0$, we obtain: for any $r \in [0,t],$

$$e^{-r \text{Re}\Psi(\xi)} \geq 1 - r \text{Re}\Psi(\xi) \geq 1 - a.$$ 

Since $\cos$ is decreasing on the interval $[0, \pi/2]$, for any $0 < s < r < t$,

$$\cos[(r - s)|\text{Im}\Psi(\xi)|] \geq \cos[t|\text{Im}\Psi(\xi)|] \geq \cos(Ka) > 0.$$ 

Therefore,

$$N_t(\xi) = 2\alpha H \int_0^t \int_0^r e^{-t \text{Re}\Psi(\xi)} e^{-s \text{Re}\Psi(\xi)} (r - s)^{2H-2} \cos[(r - s)|\text{Im}\Psi(\xi)|] dsdr$$

$$\geq (1 - a)^2 \cos(Ka) 2\alpha H \int_0^t \int_0^r (r - s)^{2H-2} dsdr$$

$$= (1 - a)^2 \cos(Ka)^t t^{2H} \geq (1 - a)^2 \cos(Ka) \left(\frac{1}{1 + \text{Re}\Psi(\xi)}\right)^{2H},$$

where for the last inequality we used the fact that $t \geq \frac{1}{1 - \text{Re}\Psi(\xi)}$.

Suppose next that $t \text{Re}\Psi(\xi) \geq a$. Note that $N_t(\xi) = \|e^{-\Phi(\xi)}\|_H^{0,t)}$. Using Lemma B.1 (Appendix B) for expressing the $H(0,t)$-norm of the exponential function in the spectral domain, we obtain:

$$N_t(\xi) = c_H \int_\mathbb{R} \sin^2[(\tau + \text{Im}\Psi(\xi))t] + [e^{-t \text{Re}\Psi(\xi)} - \cos[(\tau + \text{Im}\Psi(\xi))t]]^2 |\tau|^{-(2H-1)} d\tau.$$

We denote

$$T = t \text{Re}\Psi(\xi) \quad \text{and} \quad b = \frac{\text{Im}\Psi(\xi)}{\text{Re}\Psi(\xi)}.$$

Using the change of variable $\tau' = \tau / \text{Re}\Psi(\xi)$, we obtain that:

$$N_t(\xi) = \frac{c_H}{\text{Re}\Psi(\xi)^{2H}} \int_\mathbb{R} |\tau|^{-(2H-1)} \frac{[f_T^2(\tau) + g_T^2(\tau)]}{1 + (\tau + b)^2} d\tau,$$

where $f_T(\tau) = \sin[(\tau + b)T]$ and $g_T(\tau) = e^{-T} - \cos[(\tau + b)T]$.

From the proof of Lemma B.1 (Appendix B), we know that:

$$\frac{1}{1 + (\tau + b)^2} [f_T^2(\tau) + g_T^2(\tau)] = |F_{0,T}(\tau)|^2,$$

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where \( \varphi(x) = e^{-x(1+b)} \).

Let \( \rho > K \) be a positive constant whose value will be specified later. Since the integrand of (17) is non-negative, the integral can be bounded below by the integral over the region \( |\tau| \leq \rho \). In this region, \( |\tau|^{-(2H-1)} \geq \rho^{-(2H-1)} \). We obtain:

\[
N(t) \geq e^{-x(1+b)} \left( I(T) - \int_{|\tau| \geq \rho} \frac{1}{1 + (\tau + b)^2} [f^2_T(\tau) + g^2_T(\tau)] d\tau \right),
\]

where

\[
I(T) := \int \frac{1}{1 + (\tau + b)^2} [f^2_T(\tau) + g^2_T(\tau)] d\tau = 2\pi \int_0^T |e^{-x(1+b)}|^2 dx = \pi(1 - e^{-2T}),
\]

by Plancherel’s theorem.

Using (19), we obtain the lower bound:

\[
I(T) \geq \pi(1 - e^{-2a}), \quad \text{since} \quad T \geq a.
\]

To find an upper bound for the second integral on the right-hand side of (18), we use the fact that:

\[
f^2_T(\tau) + g^2_T(\tau) \leq 5, \quad \forall \tau \in \mathbb{R}.
\]

It follows that:

\[
\int_{|\tau| \geq \rho} \frac{f^2_T(\tau) + g^2_T(\tau)}{1 + (\tau + b)^2} d\tau \leq \int_{|\tau| \geq \rho} \frac{5}{(\tau + b)^2} d\tau = \frac{10\rho}{\rho^2 - b^2} \leq \frac{10\rho}{\rho^2 - K^2},
\]

since \( |b| \leq K \) (by (16)). We choose \( \rho = \rho_K \) large enough such that

\[
C_K := \pi(1 - e^{-2a}) - \frac{10\rho}{\rho^2 - K^2} > 0.
\]

Using (18), (20) and (21), we obtain:

\[
N(t) \geq C_K \frac{c_H \rho^{-(2H-1)}}{|\text{Re}\Psi(\xi)|^{2H}} \geq C_K c_H \rho^{-(2H-1)} \left( \frac{1}{1/t + \text{Re}\Psi(\xi)} \right)^{2H}.
\]

The conclusion follows, letting

\[
C_{H,K} = \min \left\{ (1-a)^2 \cos(Ka), C_K c_H \rho^{-(2H-1)} \right\}.
\]

\( \square \)

The following result is an immediate consequence of Theorem 3.1.

**Corollary 3.2** a) For any \( t > 0, \varphi \in S(\mathbb{R}^d) \),

\[
E|u(t, \varphi)|^2 \leq C_H \mathcal{E}(t; \varphi),
\]
where \( C_H = H^{2H} \beta_H e^2 \). Hence, \( M_t \supseteq Z_t \) for all \( t > 0 \), and \( M \supseteq Z \).

b) If (16) holds, then for any \( t > 0, \varphi \in \mathcal{S}(\mathbb{R}^d) \),

\[
E|u(t, \varphi)|^2 \geq C_{H, K} E(t; \varphi),
\]

where \( C_{H, K} \) is a positive constant depending on \( H \) and \( K \). Hence, \( M_t = Z_t \) for all \( t > 0 \), and \( M = Z \).

**Proof:** We use Theorem 3.1 and the definitions (12) and (14) of the norms \( \| \cdot \|_t \), respectively. \( \square \)

The next result gives the necessary and sufficient condition for \( \delta_x \in Z_1 \) for all \( x \in \mathbb{R}^d \).

**Theorem 3.3** In order that \( \delta_x \in Z_1 \) for all \( x \in \mathbb{R}^d \), it is necessary and sufficient that condition (5) holds.

**Proof:** Suppose first that (5) holds. To show that \( \delta_x \in Z_1 \) for all \( x \in \mathbb{R}^d \), we use an argument similar to the proof of Theorem 2 of [6].

Let \( Z_0 \) be the set Schwartz distributions \( \varphi \) such that \( F\varphi \) is a function and

\[
\|\varphi\|_1 := \int_{\mathbb{R}^d} \left( \frac{1}{1 + \text{Re}\Psi(\xi)} \right)^{2H} |F\varphi(\xi)|^2 \mu(d\xi) < \infty.
\]

Note that \( \mathcal{S}(\mathbb{R}^d) \subset Z_0 \) and the definition of \( \| \cdot \|_1 \) agrees on \( \mathcal{S}(\mathbb{R}^d) \) with the one given by (14). Therefore, to show that a distribution \( \varphi \in Z_0 \) is in \( Z_1 \), it suffices to show that there exists a sequence \( (\varphi_n)_{n \geq 1} \subset \mathcal{S}(\mathbb{R}^d) \) such that

\[
\|\varphi_n - \varphi\|_1 \to 0.
\]

We apply this to \( \varphi = \delta_x \). In this case, \( F\varphi(\xi) = e^{-i\xi \cdot x} \), \( |F\varphi(\xi)| = 1 \) for all \( \xi \in \mathbb{R}^d \), and \( \|\varphi\|_1 \) coincides with the integral of (5).

Let \( \varphi_n = \varphi \ast \phi_n \in \mathcal{S}(\mathbb{R}^d) \), where \( \phi_n(x) = n^d \phi(nx) \) and \( \phi \in \mathcal{S}(\mathbb{R}^d) \) is such that \( \phi \geq 0 \) and \( \int_{\mathbb{R}^d} \phi(x)dx = 1 \). Then \( F\varphi_n(\xi) = F\varphi(\xi)F\phi_n(\xi) \) and

\[
\|\varphi_n - \varphi\|_1 = \int_{\mathbb{R}^d} \left( \frac{1}{1 + \text{Re}\Psi(\xi)} \right)^{2H} |F\varphi_n(\xi) - F\varphi(\xi)|^2 \mu(d\xi)
\]

\[
= \int_{\mathbb{R}^d} \left( \frac{1}{1 + \text{Re}\Psi(\xi)} \right)^{2H} |F\phi_n(\xi) - 1|^2 \mu(d\xi) \to 0,
\]

by the Dominated Convergence Theorem, since \( |F\phi_n(\xi)| \leq 1 \) for all \( \xi \in \mathbb{R}^d \).

For the reverse implication, suppose that \( \delta_x \in Z_1 \) for all \( x \in \mathbb{R}^d \). To show that (5) holds, one can use the same argument as in the proof of Lemma 4.2 of [15]. We omit the details. \( \square \)

The following result concludes our discussion about the existence of a random-field solution.

**Theorem 3.4** (Existence of Solution in the Parabolic Case)

a) If (5) holds, then equation (1) has a random-field solution.

b) Suppose that (16) holds. If (1) has a random-field solution, then (5) holds.

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**Proof:** a) Suppose that (5) holds. By Theorem 3.3, $\delta_x \in Z_1$ for all $x \in \mathbb{R}^d$. By Corollary 3.2.a), $Z_1 = Z \subset M$. Hence (13) holds.

b) Suppose that (13) holds. By Corollary 3.2.b), $M = Z = Z_1$. Hence $\delta_x \in Z_1$ for all $x \in \mathbb{R}^d$. By Theorem 3.3, (5) holds. □

**Example 3.5** (Stable processes) Suppose that $L = -(-\Delta)^{\beta/2}$ for $\beta \in (0, 2]$. Then $(X_t)_{t \geq 0}$ is a rotation invariant strictly $\beta$-stable process on $\mathbb{R}^d$, and $\Psi(\xi) = |\xi|^{\beta}$ (see Theorem 14.14 in [28] and Example 30.6 in [28]). It can be shown that $(X_t)_{t \geq 0}$ is subordinate to the Brownian motion on $\mathbb{R}^d$ by a strictly $(\beta/2)$-stable subordinator (see Example 32.7 of [28]).

In this case, condition (5) becomes:

$$
\int_{\mathbb{R}^d} \left( \frac{1}{1 + |\xi|^2} \right)^{2H} \mu(d\xi) < \infty. \quad (22)
$$

We consider two kernels:

(i) $f(x) = c_{\alpha,d} |x|^{-(d-\alpha)}$ for $0 < \alpha < d$. In this case, $\mu(\xi) = |\xi|^{-\alpha} d\xi$, and condition (22) is equivalent to $2H\beta > d - \alpha$.

(ii) $f(x) = \prod_{i=1}^d (\alpha_{H_i} |x_i|^{2H_i - 2})$. In this case, $\mu(\xi) = \prod_{i=1}^n c_{H_i} |\xi_i|^{-(2H_i - 1)} d\xi$, and condition (22) is equivalent to $2H\beta > d - \sum_{i=1}^d (2H_i - 1)$.

**Remark 3.6** (Fractional Powers of the Laplacian) As in [7] and [9], we can consider also the case $L = -(-\Delta)^{\beta/2}$ for arbitrary $\beta > 0$, even if there is no corresponding Lévy process whose generator is $L$. Note that the fundamental solution $G$ of $\partial_t u - Lu = 0$ exists and satisfies:

$$
\mathcal{F}G(t, \xi) = \exp(-t|\xi|^\beta).
$$

Using Theorem 2.1 of [4] and estimates similar to those given by Theorem 3.1 above, one can show that a random field solution of (1) (in the sense of [4]) exists if and only if (22) holds.

**3.2 A Maximum Principle**

Throughout this section, we assume that $p_t \in L^2(\mathbb{R}^d)$ for all $t > 0$, and

$$
\mu \text{ has a (non-negative) density } g, \quad (23)
$$

i.e. $f$ is a kernel of positive type (see Definition 5.1 of [22]).

We consider the symmetric Lévy process $\bar{X} = (\bar{X}_t)_{t \geq 0}$ defined by:

$$
\bar{X}_t := X_t - \tilde{X}_t,
$$

where $(\tilde{X}_t)_{t \geq 0}$ is an independent copy of $(X_t)_{t \geq 0}$. We denote by $(\bar{P}_t)_{t \geq 0}$ the semigroup of $(\bar{X}_t)_{t \geq 0}$, i.e.

$$
(\bar{P}_t \phi)(x) = \int_{\mathbb{R}^d} \phi(y) \bar{p}_t(x - y) dy,
$$

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where $\tilde{p}_t = p_t * \tilde{p}_t$. From (10), it follows that $\mathcal{F}\tilde{p}_t(\xi) = e^{-2t\text{Re}\Psi(\xi)}$.

Let $(\tilde{R}_\alpha)_{\alpha>0}$ be the resolvent of $(\tilde{P}_t)_{t \geq 0}$, i.e. $(\tilde{R}_\alpha \phi)(x) = \int_0^\infty e^{-\alpha \tau}(\tilde{P}_\tau \phi)(x) d\tau$.

The following maximum principle has been obtained recently in [15]:

\[(\tilde{R}_\alpha f)(0) = \sup_{x \in \mathbb{R}^d} (\tilde{R}_\alpha f)(x) = \Upsilon(\alpha) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |r-s|^{2H-2}e^{-\alpha(r+s)}(\tilde{P}_r \phi)(x) dr ds du(\xi). \tag{24}\]

An important consequence of (24) (combined with the results of [6]) is that $(\tilde{R}_\alpha f)(0) < \infty$ for all $\alpha > 0$ is necessary and sufficient for the existence of a random field solution of (1), when the Gaussian noise $W$ is white in time (i.e. $H = 1/2$).

In the present article, we develop a maximal principle similar to (24), which has a connection with the existence of a random field solution of (1), when the noise $W$ is fractional in time.

We define the following “fractional analogue” of the resolvent operator:

\[(\tilde{R}_{\alpha,H} f)(x) = \alpha H \int_0^\infty \int_0^\infty |r-s|^{2H-2}e^{-\alpha(r+s)}(\tilde{P}_{r+s} \phi)(x) dr ds, \]

and we let

\[\Upsilon_H(\alpha) := \alpha H \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |r-s|^{2H-2}e^{-\alpha(r+s)+(2H-2)\text{Re}\Psi(\xi)}(r+s) dr ds du(\xi). \tag{24}\]

As in [22], we assume that $f$ satisfies the following condition:

\[f(x) < \infty \text{ if and only if } x \neq 0. \tag{25}\]

Under this condition, the following harmonic-analysis result holds:

\[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x) \psi(y) f(x-y) dx dy = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi) \mathcal{F}\psi(\xi) g(\xi) d\xi, \tag{26}\]

for any non-negative functions $\varphi, \psi \in L^1(\mathbb{R}^d)$ (see Lemma 5.6 of [22]).

**Theorem 3.7** (A maximum principle) If (25) holds, then for any $\alpha > 0$,

\[(\tilde{R}_{\alpha,H} f)(0) = \sup_{x \in \mathbb{R}^d} (\tilde{R}_{\alpha,H} f)(x) = \Upsilon_H(\alpha). \]

The proof of Theorem 3.7 follows from Lemma 3.10 and Lemma 3.11 below. Before this, we need some intermediate results. Let $C_0(\mathbb{R}^d)$ be the space of continuous functions which vanish at infinity.
By the Fourier inversion formula in Lemma, \( \phi \) where (Lemma 3.9, for any \( x \))

Proof: The proof is similar to Proposition 3.5 of [15]. By Fatou’s lemma and Lemma 3.10, for any \( x \in \mathbb{R}^d \),

\[
(\tilde{R}_{a,H}f)(x) \leq \liminf_{n \to \infty} (\tilde{R}_{a,H}(f \ast \phi_n))(x) \leq \Upsilon_H(\alpha),
\]

where \((\phi_n)_{n \geq 1}\) is a sequence of approximations to the identity, consisting of probability density functions in \( S(\mathbb{R}^d) \). Hence,

\[
\sup_{x \in \mathbb{R}^d} (\tilde{R}_{a,H}f)(x) \leq \Upsilon_H(\alpha).
\]
For the reverse inequality, we let \( \phi_n(x) = (2\pi)^{-d/2}n^{d/2}\exp(-n|x|^2/2) \). By Lemma 3.9,

\[
\left( \hat{R}_{\alpha,H}(f * \phi_n) \right)(0) = \alpha_H \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{2n}} \int_{\mathbb{R}^d_+} |r-s|^{2H-2} e^{-(\alpha+2\Re\Psi(\xi))(r+s)} dr ds \mu(d\xi),
\]
and therefore, by applying the monotone convergence theorem,

\[
\lim_{n \to \infty} \left( \hat{R}_{\alpha,H}(f * \phi_n) \right)(0) = \Upsilon_H(\alpha).
\] (29)

Using (28) and the symmetry of the function \( \phi_n \), we obtain:

\[
\left( \hat{P}_{r+s}(f * \phi_n) \right)(0) = \int_{\mathbb{R}^d} \left( \hat{P}_{r+s}f \right)(x) \phi_n(x) dx.
\]

Therefore,

\[
\left( \hat{R}_{\alpha,H}(f * \phi_n) \right)(0) = \int_{\mathbb{R}^d} \left( \hat{R}_{\alpha,H}f \right)(x) \phi_n(x) dx \leq \sup_{x \in \mathbb{R}^d} \left( \hat{R}_{\alpha,H}f \right)(x). \] (30)

From (29) and (30), we infer that \( \Upsilon_H(\alpha) \leq \sup_{x \in \mathbb{R}^d} \left( \hat{R}_{\alpha,H}f \right)(x) \).

The last assertion follows by taking \( \phi_n \) with the support in the ball of radius \( 1/n \) and center 0. \( \square \)

**Lemma 3.11** If \( f \) satisfies (25), then for any \( \alpha > 0 \),

\[
(\hat{R}_{\alpha,H}f)(0) = \Upsilon_H(\alpha).
\]

**Proof:** Using (26), we have:

\[
(\hat{P}_{r+s}f)(0) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{p}_r(x) \tilde{p}_s(y) f(x-y) dx dy = \int_{\mathbb{R}^d} e^{-2(r+s)\Re\Psi(\xi)} g(\xi) d\xi.
\]

The conclusion follows from the definitions of \( \left( \hat{R}_{\alpha,H}f \right)(0) \) and \( \Upsilon_H(\alpha) \). \( \square \)

To investigate the connection with the parabolic problem (1), we let

\[
\Upsilon_H^+(\alpha) = \int_{\mathbb{R}^d} \left( \frac{1}{\alpha + 2\Re\Psi(\xi)} \right)^{2H} \mu(d\xi).
\]

By Lemma A.1 (Appendix A), \( \Upsilon_H^+(\alpha) < \infty \) for all \( \alpha > 0 \) if and only if \( \Upsilon_H(\alpha) < \infty \) for some \( \alpha > 0 \).

The following result gives the relationship between \( \Upsilon_H(\alpha) \) and \( \Upsilon_H^+(\alpha) \).

**Lemma 3.12** For any \( \alpha > 0 \),

\[
c_{\alpha,H} \Upsilon_H^+(\alpha) \leq \Upsilon_H(\alpha) \leq b_{2H}^2H H^{2H} \Upsilon_H^+(\alpha),
\]

where \( c_{\alpha,H} = 2^{-(2H+2)}((\alpha \wedge 1)/(\alpha + 3/2))^{2H} \).
Proposition 4.3 of [4], for any $H$, note that, since the integrand from the definition of $\Upsilon_H(\alpha)$ is non-negative, the integral $drds$ over $[0, \infty)^2$ can be bounded below by the integral over $[0, 1]^2$. By Proposition 4.3 of [4], for any $t > 0$ and $\lambda > 0$

$$\alpha_H \int_0^t \int_0^t |r-s|^{2H-2} e^{-\lambda(r+s)} drds \geq \frac{1}{4} (t^{2H} \wedge 1) \left( \frac{1}{2} \right)^{2H} \left( \frac{1}{1/2 + \lambda} \right)^{2H}.$$  

Applying this result for $t = 1$ and $\lambda = \alpha + 2\text{Re}\Psi(\xi)$, we obtain:

$$\int_0^1 \int_0^1 |r-s|^{2H-2} e^{-(\alpha+2\text{Re}\Psi(\xi))(r+s)} drds \geq \left( \frac{1}{2} \right)^{2H+2} \left( \frac{1}{1/2 + \alpha + 2\text{Re}\Psi(\xi)} \right)^{2H}.$$  

Hence,

$$\Upsilon_H(\alpha) \geq \left( \frac{1}{2} \right)^{2H+2} \Upsilon_H(\alpha + 1/2) \geq c_{\alpha,H} \Upsilon_H(\alpha),$$

where we used Lemma A.1 (Appendix A) for the second inequality. □

Recall that by Theorem 3.4, condition (5) is the necessary and sufficient for problem (1) to have a random field solution. As a consequence of the maximum principle, we obtain the following result.

**Corollary 3.13** Suppose that $f$ satisfies (25). Then (5) is equivalent to

$$(\tilde{R}_{\alpha,H})f(0) < \infty \quad \text{for any } \alpha > 0. \quad (31)$$

**Proof:** The result follow by Theorem 3.7 and Lemma 3.12. □

### 3.3 Connection with the Intersection Local Time

In this section, we develop a connection between condition (31) and the existence of the “weighted” intersection local time $L_{x,H}(f)$.

Clearly for any fixed $t > 0$, $E[L_{x,H}(f)] < \infty$ is a sufficient condition for $L_{x,H}(f) < \infty$ a.s., but the negligible set depends on $t$. Our result will show that under condition (31), $L_{x,H}(f) < \infty$ for all $t > 0$ a.s. To motivate this result, we consider first an example.

Note that $\bar{X}_r - \bar{X}_s^2 \overset{d}{=} \bar{X}_{r+s}$ for any $r, s \in [0,t]$, and therefore,

$$E[L_{x,H}(f)] = \alpha_H \int_0^t \int_0^t |r-s|^{2H-2} E[f(\bar{X}_{r+s})] drds. \quad (32)$$

**Example 3.14** (Stable processes) Refer to Example 3.5. Since $(X_t)_{t \geq 0}$ is self-similar with exponent $1/\beta$ (see Theorem 13.5 of [28]),

$$\bar{X}_{r+s} \overset{d}{=} (r+s)^{1/\beta} X_1.$$  

Suppose in addition that $f(x) = |x|^{-(d-\alpha)}$ for $0 < \alpha < d$. Then $E[f(\bar{X}_{r+s})] = E[|\bar{X}_{r+s}|^{-(d-\alpha)}] = c_{\alpha,d}(r+s)^{-(d-\alpha)/\beta}$, where $c_{\alpha,d} = E[|\bar{X}_1|^{-(d-\alpha)}]$. By (32),

$$E[L_{x,H}(f)] = \alpha_H c_{\alpha,d} \int_0^t \int_0^t |r-s|^{2H-2} (r+s)^{-(d-\alpha)/\beta} drds.$$  

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One can see that $E[L_{t,H}(f)] < \infty$ if and only if $2H\beta > d - \alpha$.

The previous example shows that the existence of $L_{t,H}(f)$ is related to the potential-theoretic condition (31). We show below that this is a general phenomenon. For this, suppose that $X^0_1 = x_1$ and, $X^0_2 = x_2$. Let $P_{x_i}$ be the law of $X^i$ for $i = 1, 2$. We denote by $E_{x_1,x_2}$ the expectation under $P_{x_1} \times P_{x_2}$.

**Theorem 3.15** (Connection with the Local Time) Suppose that $f$ satisfies (25). If (31) holds, then for any $x_1, x_2 \in \mathbb{R}^d$,

$$P_{x_1,x_2}(L_{t,H}(f) < \infty \text{ for all } t > 0) = 1$$

$$P_{x_1,x_2}\left(\limsup_{t \to \infty} \frac{\log L_{t,H}(f)}{t} \leq 0\right) = 1.$$ 

**Proof:** We follow the lines of the proof of Theorem 3.13 of [15]. Since $f$ is non-negative, it follows that for any $t > 0$,

$$e^{-2\alpha t}L_{t,H}(f) \leq \alpha H \int_0^\infty \int_0^\infty e^{-\alpha(r+s)}|r-s|^{2H-2}f(X^1_t - X^2_s)drds. \quad (33)$$

Note that

$$E_{x_1,x_2}[f(X^1_t - X^2_s)] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y-z)\hat{p}_r(x_1-y)\hat{p}_s(x_2-z)dydz = (\hat{P}_{r+s}f)(x_1 - x_2).$$

Taking supremum over $t$, and expectation with respect to $P_{x_1,x_2}$ in (33), we obtain:

$$E_{x_1,x_2}\left[\sup_{t > 0} e^{-2\alpha t}L_{t,H}(f)\right] \leq \hat{R}_{\alpha,H}f(x_1 - x_2).$$

From here, using Theorem 3.7 and condition (31), we infer that:

$$\sup_{x_1,x_2 \in \mathbb{R}^d} E_{x_1,x_2}\left[\sup_{t > 0} e^{-2\alpha t}L_{t,H}(f)\right] \leq \sup_{x \in \mathbb{R}^d} (\hat{R}_{\alpha,H}f)(x) = (\hat{R}_{\alpha,H}f)(0) < \infty.$$ 

The result follows. \(\square\)

### 4 The Hyperbolic Equation

In this section we consider the hyperbolic problem (2). Throughout this section, we assume that $X$ is symmetric, i.e.

$$\text{Im}\Psi(\xi) = 0, \quad \text{for all } \xi \in \mathbb{R}^d. \quad (34)$$

To define the weak solution, we cannot use the same method as in the parabolic case, since in general, we may not be able to identify the fundamental solution $G$ of $\partial_t u - Lu = 0$. Note that in some particular cases, we are able to identify $G$ (see Remark 4.7 below).
To circumvent this difficulty, we use the method of [14], whose salient features we recall briefly below. Consider first the deterministic equation:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = Lu(t,x) + F(t,x), \quad (35)$$

with zero initial conditions, where $F$ is a smooth function. By taking formally the Fourier transform in the $x$ variable, and using Duhamel’s principle, we arrive to the following (formal) definition of a weak solution of (35):

$$u(t,\varphi) = \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbb{R}^d} \sin(\sqrt{\Psi(\xi)}(t-s)) \frac{\mathcal{F}\varphi(\xi)}{\sqrt{\Psi(\xi)}} \mathcal{F}F(s,\xi) d\xi ds. \quad (36)$$

If instead of $F$ we consider the random noise $\dot W$, the integral above is replaced by a stochastic integral $\mathcal{F}W(ds,d\xi)$ (whose meaning is defined below).

We let $\mathcal{P}(\mathbb{R}^d)$ be the completion of $S(\mathbb{R}^d)$ with respect to the inner product:

$$\langle \psi_1, \psi_2 \rangle_{\mathcal{P}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi_1(x)\overline{\psi_2(y)} f(x-y) dxdy.$$

Let $\hat{\mathcal{P}}(\mathbb{R}^d)$ be the completion of $S(\mathbb{R}^d)$ with respect to the inner product:

$$\langle \psi_1, \psi_2 \rangle_{\hat{\mathcal{P}}(\mathbb{R}^d)} := \langle \mathcal{F}\psi_1, \mathcal{F}\psi_2 \rangle_{\mathcal{P}(\mathbb{R}^d)}. \quad (37)$$

Note that if the noise $W$ is white in space, then by Plancherel theorem, $\langle \psi_1, \psi_2 \rangle_{\hat{\mathcal{P}}(\mathbb{R}^d)} = (2\pi)^d \langle \psi_1, \psi_2 \rangle_{L^2(\mathbb{R}^d)}$ and $\hat{\mathcal{P}}(\mathbb{R}^d) = \mathcal{P}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$.

The following lemma gives a more direct way of calculating $\langle \psi_1, \psi_2 \rangle_{\hat{\mathcal{P}}(\mathbb{R}^d)}$.

**Lemma 4.1** For any $\psi_1, \psi_2 \in S(\mathbb{R}^d)$,

$$\langle \psi_1, \psi_2 \rangle_{\hat{\mathcal{P}}(\mathbb{R}^d)} = (2\pi)^d \int_{\mathbb{R}^d} \psi_1(\xi)\overline{\psi_2(\xi)} \mu(d\xi).$$

**Proof:** Note that for any $\varphi \in L^1(\mathbb{R}^d)$, $\mathcal{F}\varphi(\xi) = \mathcal{F}^{-1}\mathcal{F}\varphi(\xi)$, where

$$\mathcal{F}^{-1}\varphi(\xi) := \int_{\mathbb{R}^d} e^{i\xi \cdot x} \varphi(x) dx, \quad \forall \xi \in \mathbb{R}^d.$$

We denote $\varphi_i := \mathcal{F}\psi_i \in S(\mathbb{R}^d)$ for $i = 1, 2$. We obtain:

$$\langle \psi_1, \psi_2 \rangle_{\hat{\mathcal{P}}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi_1(x)\varphi_2(y) f(x-y) dxdy = \int_{\mathbb{R}^d} \mathcal{F}\varphi_1(\xi)\mathcal{F}\varphi_2(\xi) \mu(d\xi)$$

$$= \int_{\mathbb{R}^d} \mathcal{F}^{-1}(\mathcal{F}\psi_1)(\xi)\mathcal{F}^{-1}(\mathcal{F}\psi_2)(\xi) \mu(d\xi).$$

By the Fourier inversion theorem, $\mathcal{F}^{-1}(\mathcal{F}\psi_i) = (2\pi)^d \psi_i$ for $i = 1, 2$. The result follows. □

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We endow the space $\mathcal{E}$ with the inner product:

$$\langle h_1, h_2 \rangle_{\mathcal{H}^P} := \langle \mathcal{F} h_1, \mathcal{F} h_2 \rangle_{\mathcal{H}^P},$$

where $\mathcal{F}$ denotes the Fourier transform in the $x$ variable. By Lemma 4.1,

$$\langle h_1, h_2 \rangle_{\mathcal{H}^P} = \alpha_H(2\pi)^{2d} \int_{\mathbb{R}^d} \int_0^T \int_0^T |r - s|^{2H-2} h_1(r, \xi) \overline{h_2(s, \xi)} \, dr \, ds \, d\mu(d\xi),$$

for any $h_1, h_2 \in \mathcal{E}$. For any $h \in \mathcal{E}$, we define: $\mathcal{F} W(h) := W(\mathcal{F} h)$. By the isometry property of $W$, for any $h \in \mathcal{E}$,

$$E |\mathcal{F} W(h)|^2 = E |W(\mathcal{F} h)|^2 = \|\mathcal{F} h\|_{\mathcal{H}^P}^2 = \|h\|_{\mathcal{H}^P}^2.$$

Let $\mathcal{H}^P$ be the completion of $\mathcal{E}$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}^P}$. The map $\mathcal{E} \ni h \mapsto \mathcal{F} W(h) \in L^2(\Omega)$ is an isometry which can be extended to $\mathcal{H}^P$. We write $\mathcal{F} W(h) = \int_0^T \int_{\mathbb{R}^d} h(t, \xi) \mathcal{F} W(dt, d\xi)$ for any $h \in \mathcal{H}^P$.

The following result gives a criterion for a function $h$ to be in $\mathcal{H}^P$.

**Lemma 4.2** Let $h : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{C}$ be a deterministic function such that $h(t, \cdot) = 0$ if $t > T$. Suppose that $h$ satisfies the following conditions:

(i) $h(t, \cdot) \in L^2(\mathbb{R}^d)$ for all $t \in [0, T]$;

(ii) $\mathcal{F} h \in \mathcal{H}^P$, where $\mathcal{F} h$ denotes the Fourier transform in the $x$ variable.

Then $h \in \mathcal{H}^P$ and

$$\|h\|_{\mathcal{H}^P}^2 = \alpha_H(2\pi)^{2d} \int_{\mathbb{R}^d} \int_0^T \int_0^T |r - s|^{2H-2} h(r, \xi) \overline{h(s, \xi)} \, dr \, ds \, d\mu(d\xi).$$

In particular, the stochastic integral of $h$ with respect to the noise $\mathcal{F} W$ is well-defined.

**Proof:** Let $g = \mathcal{F} h$. By the Fourier inversion formula on $L^2(\mathbb{R}^d)$, $h(s, \xi) = (2\pi)^{-d} \mathcal{F} g(s, \xi)$, and hence,

$$\mathcal{F} g(s, \xi) = (2\pi)^d \overline{h(s, \xi)}. \quad (38)$$

Since $g \in \mathcal{H}^P$, there exists a sequence $(g_n)_{n \geq 1}$ of the form $g_n(s, x) = \phi_n(s) \gamma_n(x)$, where $\phi_n$ is a linear combination of indicator functions $1_{[0, a]}$, $a \in [0, T]$ and $\gamma_n \in \mathcal{S}(\mathbb{R}^d)$, such that $\|g_n - g\|_{\mathcal{H}^P} \to 0$ (see [4]).

Let $\psi_n := (2\pi)^{-d} \overline{\mathcal{F} \gamma_n} \in \mathcal{S}(\mathbb{R}^d)$ and $h_n(s, \xi) = \phi_n(s) \psi_n(\xi)$. Then

$$\mathcal{F} g_n(s, \xi) = \phi_n(s) \mathcal{F} \gamma_n(\xi) = (2\pi)^d \phi_n(s) \overline{\psi_n(\xi)} = (2\pi)^d h_n(s, \xi) \quad (39)$$

Using (38) and (39), we obtain that

$$\begin{align*}
\alpha_H(2\pi)^{2d} \int_{\mathbb{R}^d} \int_0^T \int_0^T |r - s|^{2H-2} (h_n - h)(r, \xi) (\overline{h_n - h})(s, \xi) \, dr \, ds \, d\mu(d\xi) \\
= \alpha_H \int_{\mathbb{R}^d} \int_0^T \int_0^T |r - s|^{2H-2} (\mathcal{F} g_n - \mathcal{F} g)(r, \xi) (\mathcal{F} g_n - \mathcal{F} g)(s, \xi) \, dr \, ds \, d\mu(d\xi) \\
= \|g_n - g\|_{\mathcal{H}^P}^2 \to 0.
\end{align*}$$

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The conclusion follows. □

We now return to equation (2). By analogy with (36), we say that the process \( \{u(t, \varphi); t \geq 0, \varphi \in \mathcal{S}(\mathbb{R}^d)\} \) defined by:

\[ u(t, \varphi) = \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbb{R}^d} \frac{\sin(\sqrt{\Psi(t-s)}(t-s))}{\sqrt{\Psi(t)}} F \varphi(\xi) FW(ds, d\xi), \]

is a *weak solution* of (2). The stochastic integral above is well-defined if and only if the integrand

\[ (s, \xi) \mapsto h_{t, \varphi}(s, \xi) = \frac{1}{(2\pi)^d} 1_{[0,t]}(s) \frac{\sin(\sqrt{\Psi(t-s)}(t-s))}{\sqrt{\Psi(t)}} F \varphi(\xi) \]

belongs to \( \dot{\mathcal{H}} \mathcal{P} \).

To check that \( h_{t, \varphi} \in \dot{\mathcal{H}} \mathcal{P} \), it suffices to show that \( h_{t, \varphi} \) satisfies conditions (i) and (ii) of Lemma 4.2. Condition (i) holds since \( |x| \leq |x| \) for any \( x \). For (ii), we have to show that \( g_{t, \varphi} := F h_{t, \varphi} \in \mathcal{H} \mathcal{P} \). For this we apply Theorem 2.1 of [4]. Note that the function \( (s, \xi) \mapsto F g_{t, \varphi}(s, \xi) = (2\pi)^d h_{t, \varphi}(s, \xi) \) satisfies conditions (i)-(iii) of this theorem. So, if suffices to show that:

\[ I_{t, \varphi} := \alpha_H (2\pi)^d \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty |r - s|^{2H-2} h_{t, \varphi}(r, \xi) h_{t, \varphi}(s, \xi) dr ds \mu(d\xi) < \infty. \]

Note that \( I_{t, \varphi} = \int_{\mathbb{R}^d} N_t(\xi)|F \varphi(\xi)|^2 \mu(d\xi) \), where

\[ N_t(\xi) = \frac{\alpha_H}{\Psi(\xi)} \int_0^t \int_0^r \sin(r \sqrt{\Psi(\xi)}) \sin(s \sqrt{\Psi(\xi)}) |r - s|^{2H-2} dr ds. \]

Using the fact that \( |\sin x| \leq |x| \) for any \( x \), it follows that

\[ N_t(\xi) \leq \alpha_H \int_0^t \int_0^t rs |r - s|^{2H-2} dr ds \leq t^{2H+2}, \]

and

\[ I_{t, \varphi} \leq t^{2H+2} \int_{\mathbb{R}^d} |F \varphi(\xi)|^2 \mu(d\xi) < \infty. \]

Hence, \( u(t, \varphi) \) is well-defined and

\[ E|u(t, \varphi)|^2 = E|FW(h_{t, \varphi})|^2 = \|h_{t, \varphi}\|^2_{\dot{\mathcal{H}} \mathcal{P}} = I_{t, \varphi}. \]

To define the random field solution of (2), we proceed as in the case of the parabolic equation. We define the norms:

\[ \|\varphi\|^2 := E|u(t, \varphi)|^2 = \int_{\mathbb{R}^d} N_t(\xi)|F \varphi(\xi)|^2 \mu(d\xi) \]

and

\[ ||\varphi||^2 := E(t; \varphi) = \int_{\mathbb{R}^d} \left( \frac{1}{t + \Psi(\xi)} \right)^{H+1/2} |F \varphi(\xi)|^2 \mu(d\xi). \]

Let \( M_t \) and \( Z_t \) be the completions of \( \mathcal{S}(\mathbb{R}^d) \) with respect to the norms \( \| \cdot \|_t \), respectively \( || \cdot ||_t \). Let \( M = \cap_{t>0} M_t \) and \( Z = \cap_{t>0} Z_t \). By Lemma A.1 (Appendix A), \( Z_t = Z_s = Z \) for any \( s, t > 0 \).
We say that equation (2) has a random field solution if $\delta_x \in M$ for any $x \in \mathbb{R}^d$. In this case, the random field solution is defined by $\{u(t, x) = u(t, \delta_x); t \geq 0, x \in \mathbb{R}^d\}$.

The following result gives some upper and lower bounds for $N_t(\xi)$. For the upper bound, we use an argument similar to Proposition 3.7 of [4]. For the lower bound, we use a new argument.

**Theorem 4.3** For any $t > 0$ and $\xi \in \mathbb{R}^d$,

$$D_H^{(2)} t \left( \frac{1}{\ln t^2 + \Psi(\xi)} \right)^{H+1/2} \leq N_t(\xi) \leq D_H^{(1)} t \left( \frac{1}{\ln t^2 + \Psi(\xi)} \right)^{H+1/2},$$

where $D_H^{(1)}$ and $D_H^{(2)}$ are some positive constants depending only on $H$.

**Proof:** We first prove the upper bound. Suppose that $t^2 \Psi(\xi) \leq 1$. Using (15), the fact that $\|\varphi\|^2_{L^2(0, t)} \leq t^{2H-1} \|\varphi\|^2_{L^2(0, t)}$ and $|\sin x| \leq x$ for all $x > 0$, we obtain:

$$N_t(\xi) \leq b_H^2 t^{2H-1} \frac{1}{\Psi(\xi)} \int_0^t \sin^2(\sqrt{\Psi(\xi)}) \, dr \leq b_H^2 t^{2H-1} \int_0^t r^2 \, dr$$

$$= \frac{1}{3} b_H^2 t^{2H+2} \leq \frac{1}{3} b_H^2 t^{2H+1/2} \left( \frac{1}{\ln t^2 + \Psi(\xi)} \right)^{H+1/2},$$

where for the last inequality, we used the fact that $t^2 \leq \frac{1}{\ln t^2 + \Psi(\xi)}$ if $t^2 \Psi(\xi) \leq 1$.

Suppose next that $t^2 \Psi(\xi) \geq 1$. We denote

$$T = t \sqrt{\Psi(\xi)}.$$

Using the change of variable $r' = r \sqrt{\Psi(\xi)}$ and $s' = s \sqrt{\Psi(\xi)}$, we obtain:

$$N_t(\xi) = \frac{1}{\Psi(\xi)^{H+1}} \|\sin(\cdot)\|^2_{\mathcal{H}(0, T)}.$$

We now use Lemma B.1 of [4] for expressing the $\mathcal{H}(0, T)$-norm of the function $\varphi(x) = \sin x$ in the spectral domain. We obtain that:

$$N_t(\xi) = \frac{c_H}{\Psi(\xi)^{H+1}} \int_{\mathbb{R}} |\tau|^{-2H-1} \left[ f_T^2(\tau) + g_T^2(\tau) \right] d\tau,$$

where $f_T(\tau) = \sin(\tau T) - \tau \sin T$ and $g_T(\tau) = \cos(\tau T) - \cos T$. In fact,

$$\frac{1}{(\tau^2 - 1)^2} \left[ f_T^2(\tau) + g_T^2(\tau) \right] = |\mathcal{F}_{0,T} \varphi(\tau)|^2.$$

We split the integral in (40) into the regions $|\tau| \leq 1/2$ and $|\tau| \geq 1/2$, and denote the two integrals by $N_t^{(1)}(\xi)$ and $N_t^{(2)}(\xi)$. Using the same argument as in the proof of Proposition 3.7 of [4], we get:

$$N_t^{(1)}(\xi) \leq C \frac{c_H}{\Psi(\xi)^{H+1}} \frac{2^{2H-2}}{1 - H} \leq C \frac{c_H}{1 - H} 2^{2H-2} t \left( \frac{2}{\ln t^2 + \Psi(\xi)} \right)^{H+1/2}.$$
On the other hand,

\[ N_t(\xi) \leq c_H 2^{2H-1} \frac{1}{\Psi(\xi)^{H+1}} I(T), \]

where

\[ I(T) := \int_{\mathbb{R}} \left( \frac{f_{\xi}^2(\tau) + g_{\xi}^2(\tau)}{(\tau^2 - 1)^2} \right) d\tau = 2\pi \int_0^T \sin^2 x dx = \pi T \left[ 1 - \frac{\sin(2T)}{2T} \right]. \] (42)

by Plancherel’s theorem. This yields the estimate \( I(T) \leq 2\pi T \). We obtain:

\[ N_t(\xi) \leq c_H 2^{2H-1} \frac{1}{\Psi(\xi)^{H+1}} \left( \frac{1}{I(T)} \right)^{H+1/2}. \]

Combining (41) and (43), we conclude that:

\[ N_t(\xi) \leq C c_H 2^{2H-1} \frac{1}{1 - H^2} \left( \frac{1}{I(T)} + \Psi(\xi) \right)^{H+1/2}. \]

The upper bound follows, letting

\[ D_H^{(1)} = \max \left\{ \frac{1}{3} b_H^2 2^{H+1/2}, C c_H 2^{2H-1} \right\}. \]

We now treat the lower bound. Suppose first that \( t^2 \Psi(\xi) \leq 1 \). Using the fact that \( \sin x \geq x \sin 1 \) for all \( x \in [0, 1] \), we obtain:

\[ N_t(\xi) \geq \alpha_H \sin^2 1 \int_0^t \int_0^t r s |r-s|^{2H-2} dr ds = \alpha_H \sin^2 1 \frac{\beta(2,2H-1)}{H+1} t^{2H+2} \]

\[ \geq \alpha_H \sin^2 1 \frac{\beta(2,2H-1)}{H+1} t \left( \frac{1}{I(T)} + \Psi(\xi) \right)^{H+1/2}, \]

where \( \beta \) denotes the Beta function and we used the fact that \( t^2 \geq \frac{1}{1 + \Psi(\xi)} \).

Suppose next that \( T^2 = t^2 \Psi(\xi) \geq 1 \). Let \( \rho > 1 \) be a constant which will be specified below. We use (40). Since the integrand is non-negative, \( N_t(\xi) \) is bounded below by the integral over the region \( |\tau| \leq \rho \). In that region, \( |\tau|^{-(2H-1)} \geq \rho^{-(2H-1)} \), and hence

\[ N_t(\xi) \geq \frac{c_H \rho^{-(2H-1)}}{\Psi(\xi)^{H+1}} \left( I(T) - \int_{|\tau| \geq \rho} \frac{f_{\xi}^2(\tau) + g_{\xi}^2(\tau)}{(\tau^2 - 1)^2} d\tau \right). \] (44)

Using (42) and the inequality \( 1 - (\sin x)/x \geq 1/2 \) for any \( x \geq 2 \), we get:

\[ I(T) \geq \frac{\pi}{2} T, \quad \text{since} \quad T \geq 1. \] (45)
To find an upper bound for the second integral in the right-hand side of (44), we use the fact that $f_2^2(\tau) + g_2^2(\tau) \leq 2T(1 + |\tau|)^2$ for any $\tau \in \mathbb{R}$. It follows that:

$$\int_{|\tau| \geq \rho} \frac{f_2^2(\tau) + g_2^2(\tau)}{(\tau^2 - 1)^2} d\tau \leq C_\rho T, \quad (46)$$

where $C_\rho = 2 \int_{|\tau| \geq \rho} \frac{(1 + |\tau|)^2}{(\tau^2 - 1)^2} d\tau$. Using (44), (45) and (46), we obtain that:

$$N_t(\xi) \geq c_H \rho \frac{f_2^2(\tau) + g_2^2(\tau)}{(\tau^2 - 1)} \left(\tau^2 - 1\right)^{-1/2} d\tau \leq C_\rho T,$$

where $C_\rho = 2 \int_{|\tau| \geq \rho} \frac{(1 + |\tau|)^2}{(\tau^2 - 1)^2} d\tau$. Using (44), (45) and (46), we obtain that:

$$N_t(\xi) \geq c_H \rho \frac{f_2^2(\tau) + g_2^2(\tau)}{(\tau^2 - 1)} \left(\tau^2 - 1\right)^{-1/2} d\tau \leq C_\rho T.$$

Choose $\rho$ large enough such that $C_\rho < \pi/2$, e.g. $\rho = 4$, for which $C_\rho < 4/3$. Using the fact that $1/\Psi(\xi) \geq 1/t - 2 + \Psi(\xi)$, we get

$$N_t(\xi) \geq c_H 4^{-(2H-1)} \left(\frac{\pi}{2} - \frac{4}{3}\right) t \left(\frac{1}{1/t^2 + \Psi(\xi)}\right)^{H+1/2}.$$

The lower bound follows, letting

$$D_\rho^{(2)} = \min \left\{ \alpha_H \sin \frac{1}{2} \beta(2H - 1) \frac{\beta(2H - 1)}{H + 1}, c_H 4^{-(2H-1)} \left(\frac{\pi}{2} - \frac{4}{3}\right) \right\}.$$

□

A consequence of the previous result is that the norms $\|\cdot\|_t$ and $\|\cdot\|_{t_2}$ are equivalent, for any $t > 0$.

**Corollary 4.4** For any $t > 0, \phi \in S(\mathbb{R}^d)$,

$$d_H t E(t^2; \phi) \leq E|u(t, \phi)|^2 \leq D_H t E(t^2; \phi).$$

Hence $M_t = Z_{t_2}$ for any $t > 0$, and $M = Z$.

Below is the main result of this section.

**Theorem 4.5** (Existence of Solution in the Hyperbolic Case) Assume that (34) holds. Then (2) has a random field solution if and only if (6) holds.

**Proof:** The proof is similar to Theorem 3.3 and is omitted. □

**Example 4.6** As in Example 3.5, let $\mathcal{L} = -(-\Delta)^{\beta/2}$ for $\beta \in (0, 2]$. Then $\Psi(\xi) = c_\beta |\xi|^{\beta}$ and condition (6) becomes:

$$\int_{\mathbb{R}^d} \left(\frac{1}{1 + |\xi|^{\beta}}\right)^{H+1/2} \mu(d\xi) < \infty, \quad (47)$$

When $f(x) = c_{\alpha, d} |x|^{-(d-\alpha)}$ with $0 < \alpha < d$, (47) is equivalent to $(H + \frac{1}{2}) \beta > d - \alpha$, whereas for $f(x) = \prod_{i=1}^d (\alpha_H |x_i|^{2H_i-2})$, (47) is equivalent to $(H + \frac{1}{2}) \beta > d - \sum_{i=1}^d (2H_i - 1)$. 

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Remark 4.7 As in Remark 3.6, we can consider the case $L = -(-\Delta)^{\beta/2}$ for arbitrary $\beta > 0$. Note that the fundamental solution $G$ of $\partial_{tt}u - Lu = 0$ satisfies:

$$FG(t, \xi) = \frac{\sin(t|\xi|^{\beta/2})}{|\xi|^{\beta/2}}.$$ (see p.11 of [9]). Using Theorem 2.1 of [4] and estimates similar to those given by Theorem 4.3 above, one can show that a random field solution of (2) (in the sense of [4]) exists if and only if (47) holds.

A Some elementary inequalities

Lemma A.1 For any $\alpha, \beta > 0$,

$$c_1(\alpha, \beta)2H \mathbf{Y}_H^* (\beta) \leq \mathbf{Y}_H^* (\alpha) \leq c_2(\alpha, \beta)2H \mathbf{Y}_H^* (\beta),$$

where $c_1(\alpha, \beta) = (\beta \wedge 1)/(\alpha + 1)$ and $c_2(\alpha, \beta) = (\beta + 1)[(1/\alpha) \lor 1]$.

Proof: The inequalities follow by considering separately the integrals over the regions $\{2\text{Re}\Psi(\xi) \leq 1\}$ and $\{2\text{Re}\Psi(\xi) \geq 1\}$. We omit the details. □

Let

$$E_{\text{par}}(t; \varphi) = \int_{\mathbb{R}^d} \left( \frac{1}{1/t + \text{Re}\Psi(\xi)} \right)^{2H} |F\varphi(\xi)|^2 \mu(d\xi)$$

$$E_{\text{hyp}}(t; \varphi) = \int_{\mathbb{R}^d} \left( \frac{1}{1/t + \text{Re}\Psi(\xi)} \right)^{H+1/2} |F\varphi(\xi)|^2 \mu(d\xi).$$

(“par” stands for parabolic, and “hyp” for hyperbolic.)

Lemma A.2 For any $s > 0, t > 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$c_1(s, t)^{2H}E_{\text{par}}(s; \varphi) \leq E_{\text{par}}(t; \varphi) \leq c_2(s, t)^{2H}E_{\text{par}}(s; \varphi)$$

$$c_1(s, t)^{H+1/2}E_{\text{hyp}}(s; \varphi) \leq E_{\text{hyp}}(t; \varphi) \leq c_2(s, t)^{H+1/2}E_{\text{hyp}}(s; \varphi),$$

where $c_1(s, t) = (s^{-1} \land 1)/(t^{-1} + 1)$ and $c_2(t) = (s^{-1} + 1)(t \lor 1)$.

Proof: The argument is similar to the proof of Lemma A.1. □

B The $\mathcal{H}(0, T)$-norm of the exponential

The next result is needed in the proof of Theorem 3.1.

Lemma B.1 Let $\varphi(x) = e^{-x(a+ib)}$ for $x \in (0, T)$, where $a, b \in \mathbb{R}$. Then

$$\|\varphi\|^2_{\mathcal{H}(0, T)} = c_H \int_{\mathbb{R}} \frac{\sin^2[(\tau + b)T] + [e^{-at} - \cos((\tau + b)T)]^2}{a^2 + (\tau + b)^2} |\tau|^{-(2H-1)} d\tau,$$

where $c_H = \Gamma(2H + 1) \sin(\pi H)/(2H)$. 25
Proof: For complex-valued functions $\varphi \in L^2(0,t)$, we can apply Lemma A.1 of [4] to Re$\varphi$ and Im$\varphi$ to obtain that:

$$\|\varphi\|^2_{H(0,T)} = c_H \int_0^T |F_{0,T}\varphi(\tau)|^2 |\tau|^{-(2H-1)} d\tau,$$

(48)

where $F_{0,T}\varphi(\tau) := \int_0^T e^{-i\tau x} \varphi(x) dx$. When $\varphi(x) = e^{-x(a+ib)}$,

$$|F_{0,T}\varphi(\tau)|^2 = \frac{1}{a^2 + (\tau + b)^2} [f_T^2(\tau) + g_T^2(\tau)],$$

where $f_T(\tau) = \sin[(\tau + b)T]$ and $g_T(\tau) = e^{-at} - \cos[(\tau + b)T]$. □

C A version of Plancherel theorem

The following result is needed in the proof of Lemma 3.9.

Lemma C.1 For any $\varphi \in L^2(\mathbb{R}^d)$, $\psi_1 \in L^2(\mathbb{R}^d)$ and $\psi_2 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi_1(x)\psi_2(y)(\varphi(x-y))dydx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}\psi_1(\xi)\mathcal{F}\psi_2(\xi) \mathcal{F}\varphi(\xi)d\xi.$$

Proof: By Young’s inequality, $\|\varphi \ast \psi_2\|_2 \leq \|\varphi\|_2\|\psi_2\|_1$ and $\varphi \ast \psi_2 \in L^2(\mathbb{R}^d)$.

The result follows by Plancherel theorem, since

$$\int_{\mathbb{R}^d} \psi_1(x)(\varphi \ast \psi_2)(x)dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}\psi_1(\xi)\mathcal{F}(\varphi \ast \psi_2)(\xi)d\xi.$$

□

References


