The Stochastic Wave Equation with Multiplicative Fractional Noise: a Malliavin calculus approach

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Abstract

We consider the stochastic wave equation with multiplicative noise, which is fractional in time with index $H > 1/2$, and has a homogeneous spatial covariance structure given by the Riesz kernel of order α. The solution is interpreted using the Skorohod integral. We show that the sufficient condition for the existence of the solution is $\alpha > d-2$, which coincides with the condition obtained in [4], when the noise is white in time. Under this condition, we obtain estimates for the p-th moments of the solution, we deduce its Hölder continuity, and we show that the solution is Malliavin differentiable of any order. When $d \leq 2$, we prove that the first-order Malliavin derivative of the solution satisfies a certain integral equation.

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1 Introduction

In the present article, we consider the following Cauchy problem:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x) + u\dot{W}(t,x), \quad t>0, x\in\mathbb{R}^d \\ u(0,x) &= u_0(x), \quad x\in\mathbb{R}^d \\ \frac{\partial u}{\partial t}(0,x) &= u_1(x), \quad x\in\mathbb{R}^d, \end{aligned} \tag{1}$$

where u_0 and u_1 are deterministic functions, and W is a zero-mean Gaussian process which is fractional in time, with Hurst index $H > 1/2$, and homogeneous

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in space, with covariance kernel f (to be specified below). In other words, $W = \{W(\varphi); \varphi \in \mathcal{HP}\}$ is an isonormal Gaussian process, defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with covariance: $E[W(\varphi)W(\psi)] = \langle \varphi, \psi \rangle_{\mathcal{HP}}$.

Throughout this article, $\mathcal{HP}$ denotes a Hilbert space (which may contain distributions in $\mathcal{S}'(\mathbb{R}^{d+1})$), defined as the closure of the set $\mathcal{E}$ of linear combinations of elementary functions $1_{[0,t]\times A}, t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)$, with respect to the inner product:

$$\langle 1_{[0,t]\times A}, 1_{[0,s]\times B} \rangle_{\mathcal{HP}} := R_H(t,s) \int_A \int_B f(x-y) dx dy. \tag{2}$$

(Here, $\mathcal{B}_b(\mathbb{R}^d)$ denotes the class of bounded Borel sets in $\mathbb{R}^d$.)

The notation appearing in (2) needs some explanation. $R_H(t,s)$ denotes the covariance of the fractional Brownian motion of index H, and since we assume that $H > 1/2$, we have:

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |u-v|^{2H-2} du dv,$$

where $\alpha_H = H(2H-1)$. On the other hand, $f : \mathbb{R}^d \to \mathbb{R}_+$ is the Fourier transform in $\mathcal{S}'(\mathbb{R}^d)$, of a tempered measure μ on $\mathbb{R}^d$, i.e.

$$\int_{\mathbb{R}^d} f(x)\phi(x) dx = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi) \mu(d\xi), \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d),$$

where $\mathcal{F}\phi(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \phi(x) dx$ is the Fourier transform of ϕ, and $\mathcal{S}(\mathbb{R}^d)$ is the space of rapidly decreasing C^∞-functions on $\mathbb{R}^d$.

In the present article, a solution of (1) is an adapted square-integrable process $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ which satisfies the following integral equation:

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) u(s,y) W(\delta s, \delta y), \tag{3}$$

where G is the fundamental solution of the wave operator, the stochastic integral is interpreted in the Skorohod sense, and w is the solution of the equation $w_{tt} = \Delta w$, with initial conditions $w(0,\cdot) = u_0$, $w_t(0,\cdot) = u_1$.

We are interested in the case $u_0 = 1$ and $u_1 = 0$, and hence $w(t,x) = 1$ for all $t \geq 0, x \in \mathbb{R}^d$. (In the case $H = 1/2$, this corresponds to the equation $u_{tt} = \Delta u + (u+1)\dot{W}$ with zero initial conditions.)

We note that the stochastic wave equation with additive noise W and zero initial conditions was considered in [2]. For this equation, the solution is given by:

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) W(ds, dy).$$

Since the integrand above is deterministic, the Malliavin calculus techniques are not needed for defining the solution. The question of existence of the solution in the additive case is much simpler than in the multiplicative case considered in the present article, the main difficulty being that the integrand of (3) is random.

The case of the heat equation with multiplicative fractional noise was treated in [5], [6] and [1], while the recent article [7] gives a Feynman-Kac representation for the solution. In the case of the heat equation, the key estimate which leads to a sufficient condition for the existence of the solution is:

$$\int_{\mathbb{R}^d} e^{-t|\xi|^2/2}|\xi-\eta|^{-\alpha}d\xi \leq C_{\alpha,d}t^{-(d-\alpha)/2}, \quad \text{for any } t>0, \eta \in \mathbb{R}^d, \tag{4}$$

for any $0<\alpha<d$ (see Lemma 6.1 of [5], or Lemma 3.3 of [1]).

The following estimate lies at the origin of our developments, being the analogue of (4), which is needed for the wave equation:

$$\int_{\mathbb{R}^d} \frac{\sin^2(t|\xi|)}{|\xi|^2}|\xi-\eta|^{-\alpha}d\xi \leq C_{\alpha,d}t^{\alpha-d+2}, \quad \text{for any } t>0, \eta \in \mathbb{R}^d,$$

for any $d-2<\alpha<d$ (see Lemma 3.1 below).

As far as we know, the only other study of the wave equation, in which the fractional noise enters the equation in a multiplicative way is [12], which treats the case $d=1$. The authors of [12] use a pathwise integral for interpreting the solution, instead of the Skorohod integral used in the present article, which makes it difficult to compare the results.

The study of the wave and heat equations driven by a Gaussian noise which is white in time and homogeneous in space was considered by many authors, using martingale methods (see [4], [9], [13], [3], and the references therein). These methods work for more general equations (in which the factor u multiplying the noise may be replaced by $\sigma(u)$, for a Lipschitz function σ), but fail in the fractional case. The method that we use in the present article is specific to the case $\sigma(u)=u$, in which the solution has a Wiener chaos decomposition whose kernels can be written down in closed form.

This article is organized as follows. Section 2 includes the background and preliminaries. In Section 3, we show the existence of the solution, in the case when f is the Riesz kernel of order $\alpha>d-2$. In Section 4, we give some estimates for the moments of order $p>2$ of the solution, and in Section 5 we study its Hölder continuity. In Section 6, we show the Malliavin differentiability of the solution; finally, assuming that $d\leq 2$, we prove that the first-order Malliavin derivative of the solution satisfies an integral equation, using a Hilbert-space-valued Skorohod integral.

2 Preliminary Results

Intuitively, the solution of (1) should be given by a series of iterated integrals:

$$u(t,x)=1+\int_0^t\int_{\mathbb{R}^d} G(t-t_1,x-x_1)W(dt_1,dx_1)+$$

$$\int_0^t\int_{\mathbb{R}^d} G(t-t_2,x-x_2)\left(\int_0^{t_2}\int_{\mathbb{R}^2} G(t_2-t_1,x_2-x_1)W(dt_1,dx_1)\right)W(dt_2,dx_2)+\ldots$$

Since in dimension $d \geq 3$, $G(t, \cdot)$ is a distribution in $\mathbb{R}^d$, the product

$$f_n(t_1, x_1, \ldots, t_n, x_n, t, x) := G(t - t_n, x - x_n) G(t_n - t_{n-1}, x_n - x_{n-1}) \ldots \\ G(t_2 - t_1, x_2 - x_1) 1_{\{0 < t_1 < \ldots < t_n < t\}}, \quad (5)$$

has to be defined as the product of distributions, and one has to be careful with the definition of the iterated integrals above.

The goal of this section is to take care of this difficulty, by tackling the following three problems:

- (Subsection 2.1) For any $t > 0, x \in \mathbb{R}^d$ and for any $0 < t_1 < \ldots < t_n < t$, we give a meaning to $f_n(t_1, \cdot, \ldots, t_n, \cdot, t, x)$ as a distribution in $\mathcal{S}'(\mathbb{R}^{nd})$, and calculate its Fourier transform.
- (Subsection 2.2) For any $t > 0, x \in \mathbb{R}^d$, we give a general criterion which ensures that $f_n(\cdot, t, x) \in \mathcal{HP}^{\otimes n}$.
- (Subsection 2.3) Suppose that for any $t > 0, x \in \mathbb{R}^d$, $f_n(\cdot, t, x) \in \mathcal{HP}^{\otimes n}$, and the series $u(t, x) := 1 + \sum_{n \geq 1} I_n(f_n(\cdot, t, x))$ converges in $L^2(\Omega)$, where I_n denotes the multiple Wiener integral with respect to W. We show that $u = \{u(t, x); t \geq 0, x \in \mathbb{R}^d\}$ is a solution of (1), in a sense which will be described below.

2.1 The definition of the kernels $f_n(\cdot, t, x)$

In this subsection, we give a rigorous meaning to the kernels $f_n(\cdot, t, x)$. We let $C_0^\infty(\mathbb{R}^d)$ be the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact support, and $\mathcal{D}'(\mathbb{R}^d)$ be the space of (Schwartz) distributions on $\mathbb{R}^d$.

Assume first that $n = 2$ and let $0 < t_1 < t_2 < t$ be arbitrary. We proceed to the formal calculation of the action of $f_2(t_1, \cdot, t_2, \cdot, t, x)$ on a test function $\phi = \phi_1 \otimes \phi_2$ with $\phi_1, \phi_2 \in C_0^\infty(\mathbb{R}^d)$:

$$\begin{aligned} (f_2(t_1, \cdot, t_2, \cdot, t, x), \phi) &= \int_{\mathbb{R}^d} G(t - t_2, x - x_2) \phi_2(x_2) \int_{\mathbb{R}^d} G(t_2 - t_1, x_2 - x_1) \phi_1(x_1) dx_1 dx_2 \\ &= \int_{\mathbb{R}^d} G(t - t_2, x - x_2) \phi_2(x_2) \varphi_1(t_2 - t_1, x_2) dx_2 \\ &= \int_{\mathbb{R}^d} G(t - t_2, x - x_2) \psi_2(t_2 - t_1, x_2) dx_2 \\ &= \varphi_2(t_2 - t_1, t - t_2, x) \end{aligned}$$

where

$$\begin{aligned} &\psi_1(\cdot) = \phi_1(\cdot), &&\text{and} \quad \psi_2(s, \cdot) = \phi_2(\cdot) \varphi_1(s, \cdot) \\ &\varphi_1(s, \cdot) = \psi_1(\cdot) * G(s, \cdot), && \qquad \varphi_2(s_1, s_2, \cdot) = \psi_2(s_1, \cdot) * G(s_2, \cdot), \end{aligned}$$

and $*$ denotes the convolution with respect to the space variable. Similar formal calculations can be done for any n.

Based on these calculations, for any $0 < t_1 < \ldots < t_n < t$ fixed, we let $f_n(t_1, \cdot, \ldots, t_n, \cdot, t, x)$ be the element of $\mathcal{D}'(\mathbb{R}^{nd})$ whose action on a test function $\phi = \phi_1 \otimes \ldots \otimes \phi_n$ with $\phi_i \in C_0^\infty(\mathbb{R}^d)$, is given by:

$$(f_n(t_1, \cdot, \ldots, t_n, \cdot, t, x), \phi) := \varphi_n(t_2 - t_1, t_3 - t_2, \ldots, t - t_n, x), \tag{6}$$

where the pairs (ψ_k, φ_k) are defined recursively for $k = 1, \ldots, n$ by the following relations:

$$\psi_k(s_1, \ldots, s_{k-1}, \cdot) = \phi_k(\cdot)\varphi_{k-1}(s_1, \ldots, s_{k-1}, \cdot) \tag{7}$$
$$\varphi_k(s_1, \ldots, s_k, \cdot) = \psi_k(s_1, \ldots, s_{k-1}, \cdot) * G(s_k, \cdot). \tag{8}$$

Note that $\psi_k(s_1, \ldots, s_{k-1}, \cdot) \in C_0^\infty(\mathbb{R}^d)$ and $\varphi_k(s_1, \ldots, s_k, \cdot) \in \mathcal{S}(\mathbb{R}^d)$, since $G(s, \cdot)$ is a distribution with rapid decrease in $\mathbb{R}^d$ (see p. 245 of [14]). The previous definition is extended to $\phi = \phi_1 \otimes \ldots \otimes \phi_n$ with $\phi_i \in \mathcal{S}(\mathbb{R}^d)$.

The next result shows that the Fourier transform of $f_n(t_1, \cdot, \ldots, t_n, \cdot, t, x)$ is a function in $\mathbb{R}^{nd}$, given by:

$$\mathcal{F}f_n(t_1, \cdot, \ldots, t_n, \cdot, t, x)(\xi_1, \ldots, \xi_n) = e^{-i(\xi_1 + \ldots + \xi_n) \cdot x}\overline{\mathcal{F}G(t_2 - t_1, \cdot)(\xi_1)}$$
$$\overline{\mathcal{F}G(t_3 - t_2, \cdot)(\xi_1 + \xi_2)} \ldots \overline{\mathcal{F}G(t - t_n, \cdot)(\xi_1 + \ldots + \xi_n)}. \tag{9}$$

Proposition 2.1 *For any $0 < t_1 < \ldots < t_n < t$ and for any $h = h_1 \otimes \ldots \otimes h_n$ with $h_i \in C_0^\infty(\mathbb{R}^d)$, we have:*

$$(f_n(t_1, \cdot, \ldots, t_n, \cdot, t, x), h) = \int_{\mathbb{R}^{nd}} h(\xi_1, \ldots, \xi_n) e^{-i(\xi_1 + \ldots + \xi_n) \cdot x}\overline{\mathcal{F}G(t_2 - t_1, \cdot)(\xi_1)}$$
$$\overline{\mathcal{F}G(t_3 - t_2, \cdot)(\xi_1 + \xi_2)} \ldots \overline{\mathcal{F}G(t - t_n, \cdot)(\xi_1 + \ldots + \xi_n)} d\xi_1 \ldots d\xi_n.$$

Proof: Note that $\phi := \mathcal{F}h = \phi_1 \otimes \ldots \otimes \phi_n$, where $\phi_i := \mathcal{F}h_i \in \mathcal{S}(\mathbb{R}^d)$. By the definition of the Fourier transform in $\mathcal{S}'(\mathbb{R}^d)$ and (6), we have:

$$\begin{aligned}(\mathcal{F}f_n(t_1, \cdot, \ldots, t_n, \cdot, t, x), h) &= (f_n(t_1, \cdot, \ldots, t_n, \cdot, t, x), \phi) \\ &= \varphi_n(t_2 - t_1, t_3 - t_2, \ldots, t - t_n, x),\end{aligned} \tag{10}$$

where $(\psi_k, \varphi_k), 1 \leq k \leq n$ are defined recursively by (7)-(8).

We proceed to the evaluation of $\varphi_n(t_2 - t_1, t_3 - t_2, \ldots, t - t_n, x)$.

Step 1. For any $s_1, \ldots, s_{n-1} \in [0, t]$, we define recursively the following functions: $g_1 = h_1$,

$$g_k(s_1, \ldots, s_{k-1}, \cdot) = h_k * (g_{k-1}(s_1, \ldots, s_{k-2}, \cdot)\overline{\mathcal{F}G(s_{k-1}, \cdot)}), \quad 2 \leq k \leq n.$$

By induction on k, $2 \leq k \leq n$, it follows that $g_k(s_1, \ldots, s_{k-1}, \cdot) \in C_0^\infty(\mathbb{R}^d)$ (since $\mathcal{F}G(s, \cdot)$ is a C^∞-function on $\mathbb{R}^d$), and

$$g_k(s_1, \ldots, s_{k-1}, \eta_k) = \int_{\mathbb{R}^{(k-1)d}} h_1(\eta_1)h_2(\eta_2 - \eta_1) \ldots h_k(\eta_k - \eta_{k-1})\overline{\mathcal{F}G(s_1, \cdot)(\eta_1)} \ldots$$
$$\overline{\mathcal{F}G(s_{k-1}, \cdot)(\eta_{k-1})} d\eta_1 \ldots d\eta_{k-1}. \tag{11}$$

Step 2. We prove by induction on k, $1 \leq k \leq n$, that:

$$\varphi_k(s_1,\ldots,s_k,\cdot) = \mathcal{F}[g_k(s_1,\ldots,s_{k-1},\cdot)\overline{\mathcal{F}G(s_k,\cdot)}]. \tag{12}$$

Before proving (12), note that:

$$\mathcal{F}g * G(s,\cdot) = \mathcal{F}(g\overline{\mathcal{F}G(s,\cdot)}), \quad \forall g \in C_0^\infty(\mathbb{R}^d), \forall s > 0, \tag{13}$$

since $(\mathcal{F}g) * G(s,\cdot) = \mathcal{F}[\mathcal{F}^{-1}(\mathcal{F}g * G(s,\cdot))] = \mathcal{F}(g\overline{\mathcal{F}G(s,\cdot)})$.

For $k = 1$, we have $g_1 = h_1 = \mathcal{F}^{-1}\phi_1$ and

$$\varphi_1(s,\cdot) = \phi_1 * G(s,\cdot) = \mathcal{F}g_1 * G(s,\cdot) = \mathcal{F}(g_1\overline{\mathcal{F}G(s,\cdot)}),$$

where we used (13) for the last equality. This proves (12) for $k = 1$.

Suppose that (12) holds for $k-1$. Then

$$\begin{aligned}
\psi_k(s_1,\ldots,s_{k-1},x) &= \phi_k(x)\varphi_{k-1}(s_1,\ldots,s_{k-1},x) \\
&= \mathcal{F}h_k(x)\mathcal{F}[g_{k-1}(s_1,\ldots,s_{k-2},\cdot)\overline{\mathcal{F}G(s_{k-1},\cdot)}](x) \\
&= \mathcal{F}[h_k * (g_{k-1}(s_1,\ldots,s_{k-2},\cdot)\overline{\mathcal{F}G(s_{k-1},\cdot)})](x) \\
&= \mathcal{F}g_k(s_1,\ldots,s_{k-1},\cdot)(x)
\end{aligned}$$

and

$$\begin{aligned}
\varphi_k(s_1,\ldots,s_k,\cdot) &= \psi_k(s_1,\ldots,s_{k-1},\cdot) * G(s_k,\cdot) = \mathcal{F}g_k(s_1,\ldots,s_{k-1},\cdot) * G(s_k,\cdot) \\
&= \mathcal{F}[g_k(s_1,\ldots,s_{k-1},\cdot)\overline{\mathcal{F}G(s_k,\cdot)}],
\end{aligned}$$

where we used (13) for the last equality. This proves (12).

Step 3. Using (12) and (11), we obtain that for any $1 \leq k \leq n$,

$$\begin{aligned}
\varphi_k(s_1,\ldots,s_k,x) &= \int_{\mathbb{R}^d} e^{-i\eta_k\cdot x} g_k(s_1,\ldots,s_{k-1},\eta_k)\overline{\mathcal{F}G(s_k,\cdot)(\eta_k)}d\eta_k \\
&= \int_{\mathbb{R}^{kd}} e^{-i\eta_k\cdot x} h_1(\eta_1)h_2(\eta_2-\eta_1)\ldots h_k(\eta_k-\eta_{k-1})\overline{\mathcal{F}G(s_1,\cdot)(\eta_1)} \\
&\quad \ldots\overline{\mathcal{F}G(s_{k-1},\cdot)(\eta_{k-1})}\;\overline{\mathcal{F}G(s_k,\cdot)(\eta_k)}d\eta_1\ldots d\eta_k \\
&= \int_{\mathbb{R}^{kd}} e^{-i(\xi_1+\ldots+\xi_k)\cdot x}\prod_{j=1}^k h_j(\xi_j)\prod_{j=1}^k \overline{\mathcal{F}G(s_j,\cdot)(\xi_1+\ldots+\xi_j)}d\xi_1\ldots d\xi_k,
\end{aligned} \tag{14}$$

where for the last equality we used the change of variables $\xi_1 = \eta_1$, $\xi_j = \eta_j - \eta_{j-1}$, for $2 \leq j \leq k$. We now use (10). The conclusion follows using (14) with $k = n$, $s_n = t - t_n$ and $s_i = t_{i+1} - t_i$ for $1 \leq i \leq n-1$. □

2.2 The space $\mathcal{HP}^{\otimes n}$

In this subsection, we give a criterion for checking that an element $\varphi \in \mathcal{HP}^{\otimes n}$, which can be viewed as a multi-dimensional analogue of Theorem 2.1 of [2]. This criterion is then applied to the case $\varphi = f_n(\cdot,t,x)$.

For this purpose, for any $T > 0$ fixed, we define the multi-dimensional transfer operator $K^*_{H,n}$ by:

$$(K^*_{H,n} 1_{[0,t_1]\times \ldots [0,t_n]})(s_1, \ldots, s_n) := \prod_{i=1}^{n} (K^*_H 1_{[0,t_i]})(s_i), \quad s_1, \ldots, s_n \in (0,T).$$

Let $\mathcal{E}_{\mathbb{C}}(0,T)$ be the set of all complex linear combinations of indicator functions $1_{[0,t]}, t \in [0,T]$, and $\mathcal{H}_{\mathbb{C}}(0,T)$ be the closure of $\mathcal{E}_{\mathbb{C}}(0,T)$ with respect to the inner product:

$$\langle \varphi, \psi \rangle_{\mathcal{H}_{\mathbb{C}}(0,T)} = \alpha_H \int_0^T \int_0^T \varphi(u)\overline{\psi(v)}|u-v|^{2H-2} du dv.$$

The operator $K^*_{H,n}$ is an isometry between $\mathcal{E}_{\mathbb{C}}(0,T)^{\otimes n}$ and $L^2((0,T)^n)$, which can be extended to $\mathcal{H}_{\mathbb{C}}(0,T)^{\otimes n}$. In terms of fractional integrals, we have:

$$(K^*_{H,n}\phi)(\mathbf{s}) = (c^*_H)^n \Gamma(H-1/2)^n [\mathbf{s}]^{1/2-H} I^{H-1/2}_{T-,n}([\mathbf{u}]^{H-1/2}\phi(\mathbf{u}))(\mathbf{s}),$$

where $c^*_H = (\frac{\alpha_H}{\beta(H-1/2, 2-2H)})^{1/2}$, $\mathbf{s} = (s_1, \ldots, s_n)$, $[\mathbf{s}] = s_1 \ldots s_n$, and

$$I^{\alpha}_{T-,n} f(\mathbf{s}) := \frac{1}{\Gamma(\alpha)^n} \int_{s_1}^T \ldots \int_{s_n}^T [\mathbf{u}-\mathbf{s}]^{\alpha-1} f(\mathbf{u}) d\mathbf{u}$$

is a multi-dimensional fractional integral of $f \in L^1((0,T)^n)$, of order $\alpha \in (0,1)$.

For any function $\phi \in \mathcal{H}_{\mathbb{C}}(0,T)^{\otimes n}$, we have:

$$\alpha_H^n \int_{(0,T)^{2n}} \phi(\mathbf{u})\overline{\phi(\mathbf{v})}[\mathbf{u}-\mathbf{v}]^{2H-2} d\mathbf{u} d\mathbf{v} = d_H^n \int_{(0,T)^n} |I^{H-1/2}_{T-,n}([\mathbf{u}]^{H-1/2}\phi(\mathbf{u}))(\mathbf{s})|^2 \lambda_{H,n}(d\mathbf{s}), \tag{15}$$

where $d_H = (c^*_H)^2 \Gamma(H-1/2)^2$ and $\lambda_{H,n}(d\mathbf{s}) = [\mathbf{s}]^{1-2H} d\mathbf{s}$.

Let $\mathcal{E}_T$ be the class of elementary functions on $(0,T)\times\mathbb{R}^d$. For any $\varphi \in \mathcal{E}_T^{\otimes n}$,

$$\|\varphi\|^2_{\mathcal{HP}^{\otimes n}} = d_H^n \int_{\mathbb{R}^{nd}} \int_{(0,T)^n} |I^{H-1/2}_{T-,n}([\mathbf{u}]^{H-1/2} \mathcal{F}\varphi(u_1, \cdot, \ldots, u_n, \cdot)(\boldsymbol{\xi}))(\mathbf{s})|^2 \lambda_{H,n}(d\mathbf{s})\mu(d\xi_1)\ldots\mu(d\xi_n), \tag{16}$$

where $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^{nd}$.

The following theorem is the multidimensional analogue of Theorem 2.1 of [2].

Theorem 2.2 *Let $(0,T)^n \ni \mathbf{t} \mapsto \varphi(t_1, \cdot, \ldots, t_n, \cdot) \in \mathcal{S}'(\mathbb{R}^{nd})$ be a deterministic function such that $\mathcal{F}f(t_1, \cdot, \ldots, t_n \cdot)$ is a function on $\mathbb{R}^{nd}$, for all $\mathbf{t} \in (0,T)^n$.*

Suppose that:
(i) the function $\mathbf{t} \mapsto \mathcal{F}\varphi(t_1, \cdot, \ldots, t_n, \cdot)(\boldsymbol{\xi})$ belongs to $\mathcal{H}_{\mathbb{C}}(0,T)^{\otimes n}$ for all $\boldsymbol{\xi} \in \mathbb{R}^{nd}$;

(ii) the function $(\mathbf{t}, \boldsymbol{\xi}) \mapsto \mathcal{F}\varphi(t_1, \cdot, \ldots, t_n, \cdot)(\boldsymbol{\xi})$ *is measurable on* $(0,T)^n \times \mathbb{R}^{nd}$*;*
(iii) $\int_{(0,T)^n} \prod_{i=1}^n 1_{\{u_i \geq s_i\}} [\mathbf{u}]^{H-1/2} [\mathbf{u} - \mathbf{s}]^{H-3/2} |\mathcal{F}\varphi(u_1, \cdot, \ldots, u_n, \cdot)(\boldsymbol{\xi})| d\mathbf{u} < \infty$ *for all* $(\mathbf{s}, \boldsymbol{\xi}) \in (0,T)^n \times \mathbb{R}^{nd}$ *(or* $\mathcal{F}\varphi(s_1, \cdot, \ldots, s_n, \cdot)(\boldsymbol{\xi}) \geq 0$ *for all* $(\mathbf{s}, \boldsymbol{\xi})$*).*

If

$$I_T := \alpha_H^n \int_{\mathbb{R}^{nd}} \int_{(0,T)^{2n}} \mathcal{F}\varphi(u_1, \cdot, \ldots, u_n, \cdot)(\boldsymbol{\xi}) \overline{\mathcal{F}\varphi(v_1, \cdot, \ldots, v_n, \cdot)(\boldsymbol{\xi})}$$

$$\prod_{i=1}^n |u_i - v_i|^{2H-2} d\mathbf{u} d\mathbf{v} \mu(d\xi_1) \ldots \mu(d\xi_n) < \infty, \tag{17}$$

then $\varphi \in \mathcal{HP}^{\otimes n}$ *and* $\|\varphi\|^2_{\mathcal{HP}^{\otimes n}} = I_T$*. (By convention, we set* $\varphi(t_1, \cdot, \ldots, t_n, \cdot) = 0$ *if* $t_i > T$ *for some* $i = 1, \ldots, n$*.)*

Proof: The argument is similar to the one used in the proof of Theorem 2.1 of [2], being based on relations (15) and (16) above. We omit the details. □

Remark 2.3 In our case, we apply Theorem 2.2 to the function

$$(0,t)^n \ni \mathbf{t} \mapsto \varphi(t_1, \cdot, \ldots, t_n, \cdot) = f_n(t_1, \cdot, \ldots, t_n, \cdot, t, x).$$

(We define $f_n(t_1, \cdot, \ldots, t_n, \cdot, t, x)$ to be 0 if the relation $t_1 < \ldots < t_n$ is not satisfied.) To see that this function satisfies hypothesis (i)-(iii) of Theorem 2.2, we use (9) and the fact that:

$$|\mathcal{F}G(t, \cdot)(\xi)| \leq C_T \quad \forall t \in [0,T], \xi \in \mathbb{R}^d, \tag{18}$$

From here, we infer that the map $\mathbf{t} \mapsto \mathcal{F}f_n(t_1, \cdot, \ldots, t_n, \cdot, t, x)$ belongs to $L^2_{\mathbb{C}}(0,T)^{\otimes n}$, which is included in $\mathcal{H}_{\mathbb{C}}(0,T)^{\otimes n}$.

Therefore, to show that $f_n(\cdot, t, x) \in \mathcal{HP}^{\otimes n}$, it suffices to prove that (17) holds. This will be done in Section 3.

2.3 Malliavin Calculus

In this subsection, we introduce the basic elements of the Malliavin calculus with respect to the isonormal Gaussian process W (see [10] for more details).

We first introduce the multiple Wiener integral with respect to W. Let $\mathcal{G}$ be the σ-field generated by $\{W(\varphi); \varphi \in \mathcal{HP}\}$. By Theorem 1.1.1 of [10], $L^2(\Omega, \mathcal{G}, P) = \oplus_{n=0}^\infty \mathcal{HP}_n$, where $\mathcal{HP}_n$ be the n-th Wiener chaos of W. Hence, every $F \in L^2(\Omega, \mathcal{G}, P)$ admits the following Wiener chaos expansion:

$$F = \sum_{n=0}^\infty J_n F, \tag{19}$$

where J_n is the projection on $\mathcal{HP}_n$, for $n \geq 1$. By convention, $J_0 F = E(F)$.

The definition of the multiple Wiener integral I_n is similar to the white noise case (see Subsection 1.1.2 of [10]). More precisely, I_n is a linear and continuous operator from $\mathcal{HP}^{\otimes n}$ onto $\mathcal{HP}_n$. For any $f \in \mathcal{HP}^{\otimes n}$, we write

$$I_n(f) = \int_{(\mathbb{R}_+ \times \mathbb{R}^d)^n} f(t_1, x_1, \dots, t_n, x_n) W(dt_1, dx_1) \dots W(dt_n, dx_n),$$

even if f_n is not a function in $(t_1, x_1), \dots, (t_n, x_n)$. Note that $I_n(f) = I_n(\tilde{f})$ and

$$E(I_n(f) I_n(g)) = n! \langle \tilde{f}, \tilde{g} \rangle_{\mathcal{HP}^{\otimes n}}, \tag{20}$$

where $\tilde{f}$ denotes the symmetrization of f, i.e.

$$\tilde{f}(t_1, x_1, \dots, t_n, x_n) = \frac{1}{n!} \sum_{\rho \in S_n} f(t_{\rho(1)}, x_{\rho(1)}, \dots, t_{\rho(n)}, x_{\rho(n)}).$$

(Here S_n denotes the set of all permutations of $\{1, \dots, n\}$.)

Any random variable $F \in L^2(\Omega, \mathcal{G}, P)$ admits the decomposition:

$$F = \sum_{n \geq 0} I_n(f_n) \tag{21}$$

for some $f_n \in \mathcal{HP}^{\otimes n}$ symmetric, and

$$E|F|^2 = \sum_{n=0}^{\infty} E|I_n(f_n)|^2 = \sum_{n=0}^{\infty} n! \, \|f_n\|^2_{\mathcal{HP}^{\otimes n}}.$$

(By convention, $f_0 = E(F)$ and $I_0(x) = x$ for all $x \in \mathbb{R}$.)

We now introduce the derivative operator. Let $\mathcal{S}$ be the class of smooth random variables of the form

$$F = f(W(\varphi_1), \dots, W(\varphi_n)), \tag{22}$$

where $f \in C_b^{\infty}(\mathbb{R}^n)$, $\varphi_i \in \mathcal{HP}$, $n \geq 1$, and $C_b^{\infty}(\mathbb{R}^n)$ is the class of bounded C^{∞}-functions on $\mathbb{R}^n$, whose partial derivatives are bounded. The Malliavin derivative of F of the form (22) is an $\mathcal{HP}$-valued random variable given by:

$$DF := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(\varphi_1), \dots, W(\varphi_n)) \varphi_i.$$

We endow $\mathcal{S}$ with the norm $\|F\|^2_{\mathbb{D}^{1,2}} := E|F|^2 + E\|DF\|^2_{\mathcal{HP}}$. The operator D can be extended to the space $\mathbb{D}^{1,2}$, the completion of $\mathcal{S}$ with respect to $\|\cdot\|_{\mathbb{D}^{1,2}}$.

The following result is the analogue of Proposition 1.2.7 of [10].

Proposition 2.4 *Let F be a random variable given by (21). If $F \in \mathbb{D}^{1,2}$, then*

$$D_{\bullet} F = \sum_{n \geq 1} n I_{n-1}(f_n(\cdot, \bullet)),$$

where $\bullet$ denotes the missing (t, x) variable.

Proof: It is enough to assume that $F = I_n(f_n)$, for some symmetric elementary function f_n of the form $f_n = \sum_{i_1,\ldots,i_n=1}^m a_{i_1\ldots i_n} 1_{A_{i_1}\times\ldots\times A_{i_n}}$, where $m \geq n$, $A_1,\ldots,A_m$ are pairwise-disjoint bounded Borel sets in $\mathbb{R}_+ \times \mathbb{R}^d$, and $a_{i_1\ldots i_n} = 0$ if any two of the indices $i_1,\ldots,i_n$ are equal. Then

$$\begin{aligned} f_n &= \sum_{1\leq i_1<\ldots<i_n\leq m} a_{i_1\ldots i_n} \sum_{\rho\in S(\{i_1\ldots i_n\})} 1_{A_{\rho(i_1)}\times\ldots\times A_{\rho(i_n)}} &(23)\\ I_n(f_n) &= n! \sum_{1\leq i_1<\ldots<i_n\leq m} a_{i_1\ldots i_n} W(A_{i_1})\ldots W(A_{i_n}), &(24)\end{aligned}$$

where $S(\{i_1,\ldots,i_n\})$ denotes the set of all permutations of $\{i_1,\ldots,i_n\}$ and $W(A) = W(1_A)$. Using (24), the fact that $D(FG) = (DF)G + F(DG)$, and $DW(\varphi) = \varphi$ for any $\varphi \in \mathcal{HP}$, we infer that

$$D_\bullet F = n! \sum_{1\leq i_1<\ldots<i_n\leq m} a_{i_1\ldots i_n} \sum_{j=1}^n 1_{A_{i_j}}(\bullet) \left(\prod_{k\neq j} W(A_{i_k}) \right).$$

Using (23), we obtain:

$$\begin{aligned} I_{n-1}(f_n(\cdot,\bullet)) &= \sum_{1\leq i_1<\ldots<i_n\leq m} a_{i_1\ldots i_n} \sum_{\rho\in S(\{i_1\ldots i_n\})} 1_{A_{\rho(i_n)}}(\bullet) \prod_{j=1}^{n-1} W(A_{\rho(i_j)})\\ &= \sum_{1\leq i_1<\ldots<i_n\leq m} a_{i_1\ldots i_n} \sum_{j=1}^n 1_{A_{i_j}}(\bullet) \sum_{\rho\in S(\{i_1\ldots i_n\}\setminus\{i_j\})} \prod_{k\neq j} W(A_{\rho(i_k)})\\ &= (n-1)! \sum_{1\leq i_1<\ldots<i_n\leq m} a_{i_1\ldots i_n} \sum_{j=1}^n 1_{A_{i_j}}(\bullet) \prod_{k\neq j} W(A_{i_k}).\end{aligned}$$

□

The divergence operator δ is defined as the adjoint of the operator D. The domain of δ, denoted by Dom δ, is the set of $u \in L^2(\Omega;\mathcal{HP})$ such that

$$|E\langle DF, u\rangle_{\mathcal{HP}}| \leq c(E|F|^2)^{1/2}, \quad \forall F \in \mathbb{D}^{1,2},$$

where c is a constant depending on u. If $u \in$ Dom δ, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the following duality relation:

$$E(F\delta(u)) = E\langle DF, u\rangle_{\mathcal{HP}}, \quad \forall F \in \mathbb{D}^{1,2}. \tag{25}$$

If $u \in$ Dom δ, we will use the notation

$$\delta(u) = \int_0^\infty \int_{\mathbb{R}^d} u(t,x) W(\delta t, \delta x),$$

even if u is not a function in (t,x), and we say that $\delta(u)$ is the Skorohod integral of u with respect to W.

The next result gives an important calculus rule, which plays a crucial role in the present article. This rule states, in particular, that the Skorohod integral of a multiple Wiener integral of order n coincides with a multiple Wiener integral of order $n+1$, i.e.

$$\int_{\mathbb{R}_+\times\mathbb{R}^d}\left(\int_{(\mathbb{R}_+\times\mathbb{R}^d)^n} f_n(t_1,x_1,\ldots,t_n,x_n,t,x)W(dt_1,dx_1)\ldots W(dt_n,dx_n)\right)W(\delta t,\delta x)$$
$$= \int_{(\mathbb{R}_+\times\mathbb{R}^d)^{n+1}} f_n(t_1,x_1,\ldots,t_n,x_n,t,x)W(dt_1,dx_1)\ldots W(dt_n,dx_n)W(dt,dx). \tag{26}$$

Proposition 2.5 *Assume that $u\in L^2(\Omega;\mathcal{HP})$ has the Wiener chaos expansion:*

$$u(\bullet)=\sum_{n\geq 0} I_n(f_n(\cdot,\bullet)), \tag{27}$$

where $\bullet$ denotes the missing (t,x)-variable, $\cdot$ denotes the missing n variables $(t_1,x_1),\ldots,(t_n,x_n)$, and f_n is symmetric and lies in $\mathcal{HP}^{\otimes n}$ (in the first n variables). Then $u\in$ Dom δ if and only if the series $\sum_{n\geq 0} I_{n+1}(\tilde{f}_n)$ converges in $L^2(\Omega)$, where $\tilde{f}_n$ is the symmetrization of f_n in all $n+1$ variables. In this case,

$$\delta(u)=\sum_{n\geq 0} I_{n+1}(\tilde{f}_n)=\sum_{n\geq 0} I_{n+1}(f_n).$$

Remark 2.6 (a) If $u(t,x)$ is a function in (t,x), relation (27) is interpreted as follows: for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$,

$$u(t,x)=\sum_{n\geq 0} I_n(f_n(\cdot,t,x)) \quad \text{in } L^2(\Omega). \tag{28}$$

(b) If $u(t,x)$ is a distribution in (t,x), relation (27) is interpreted as follows: for any $\phi\in C_0^\infty(\mathbb{R}_+\times\mathbb{R}^d)$,

$$(u(\bullet),\phi)=\sum_{n\geq 0} I_n((f_n(\cdot,\bullet),\phi)) \quad \text{in } L^2(\Omega). \tag{29}$$

(c) If $u(t,x)$ is a function in t and a distribution in x, relation (27) is interpreted as follows: for any $t>0, \phi\in C_0^\infty(\mathbb{R}^d)$,

$$(u(t,*),\phi)=\sum_{n\geq 0} I_n((f_n(\cdot,t,*),\phi)) \quad \text{in } L^2(\Omega),$$

where $*$ denotes the missing x-variable.

Proof: Using the same argument as in the proof of Proposition 1.3.7 of [10], it suffices to prove that for any $G=I_n(g)$ with $g\in\mathcal{HP}^{\otimes n}$ symmetric, we have:

$$E\langle DG,u\rangle_{\mathcal{HP}}=E(I_n(\tilde{f}_{n-1})G). \tag{30}$$

Without loss of generality, we may assume that g is a function in all variables. By Proposition 2.4, DG is a function given by

$$D_{s,y}G = nI_{n-1}(g(\cdot,s,y)) \quad \forall s>0, y \in \mathbb{R}^d. \tag{31}$$

We consider separately the following three cases.

Case 1. $u(t,x)$ is a function in (t,x). Using (28), (31), the orthogonality of the Wiener chaos spaces, and (20), we obtain

$$\begin{aligned}
E\langle DG,u\rangle_{\mathcal{HP}} &= \alpha_H E\int_{(\mathbb{R}_+\times\mathbb{R}^d)^2} u(t,x)(D_{s,y}G)|t-s|^{2H-2}f(x-y)dxdydtds \\
&= n\alpha_H\int_{(\mathbb{R}_+\times\mathbb{R}^d)^2} E(I_{n-1}(f_{n-1}(\cdot,t,x))I_{n-1}(g(\cdot,s,y))) \\
&\qquad |t-s|^{2H-2}f(x-y)dxdydtds \\
&= n(n-1)!\ \alpha_H\int_{(\mathbb{R}_+\times\mathbb{R}^d)^2} \langle f_{n-1}(\cdot,t,x),g(\cdot,s,y)\rangle_{\mathcal{HP}^{\otimes(n-1)}} \\
&\qquad |t-s|^{2H-2}f(x-y)dxdydtds \\
&= n!\ \langle f_{n-1},g\rangle_{\mathcal{HP}^{\otimes n}} = n!\ \langle \tilde{f}_{n-1},g\rangle_{\mathcal{HP}^{\otimes n}} = E(I_n(\tilde{f}_{n-1})I_n(g)),
\end{aligned}$$

where for the second-last equality, we used the symmetry of g. This proves (30).

Case 2. $u(t,x)$ is a distribution in (t,x) (in $\mathcal{S}'(\mathbb{R}^{d+1})$). In this case, we regularize f_n as follows. Let $\psi \in C_0^\infty(\mathbb{R}^{d+1})$ be such that $\psi \geq 0$, the support of ψ is included in $(0,1)\times\{x\in\mathbb{R}^d; |x|\leq 1\}$ and $\int_{\mathbb{R}^{d+1}}\psi(t,x)dtdx = 1$. Let $\psi_\varepsilon(t,x) = \varepsilon^{-d-1}\psi(t/\varepsilon, x/\varepsilon)$ and $f_{n,\varepsilon}(\cdot,\bullet) := \psi_\varepsilon * f_n(\cdot,\bullet)$, where $*$ denotes the convolution with respect to the missing (t,x)-variable, denoted by $\bullet$. Note that $f_{n,\varepsilon}(\cdot,t,x)$ is a function in (t,x) (see p. 245 of [14]). Let

$$u_\varepsilon(t,x) = \sum_{n\geq 0} I_n(f_{n,\varepsilon}(\cdot,t,x)).$$

We claim that $u_\varepsilon = \psi_\varepsilon * u$. To see this, note that for any $\phi \in C_0^\infty(\mathbb{R}_+\times\mathbb{R}^d)$,

$$\begin{aligned}
((\psi_\varepsilon * u)(\bullet),\phi) &= (u(\bullet),\psi_\varepsilon * \tilde{\phi}) = \sum_{n\geq 0} I_n((f_n(\cdot,\bullet),\psi_\varepsilon*\tilde{\phi})) \\
&= \sum_{n\geq 0} I_n((f_{n,\varepsilon}(\cdot,\bullet),\phi)) = \sum_{n\geq 0}(I_n(f_{n,\varepsilon}(\cdot,\bullet)),\phi) = (u_\varepsilon(\bullet),\phi),
\end{aligned}$$

where we used (29) for the second equality, and the stochastic Fubini's theorem for the second-last equality.

Applying the result of Case 1 to u_ε, we get:

$$E\langle DG,u_\varepsilon\rangle_{\mathcal{HP}} = E(I_n(\tilde{f}_{n-1,\varepsilon})G). \tag{32}$$

Relation (30) follows by letting $\varepsilon \to 0$. On the left-hand side of (32), we have:

$$\begin{aligned}
E\|u_\varepsilon - u\|^2_{\mathcal{HP}} &= a_{H,d}\int_{\mathbb{R}} d\tau|\tau|^{1-2H}\int_{\mathbb{R}^d}\mu(d\xi)|\mathcal{F}u_\varepsilon(\tau,\xi)-\mathcal{F}u(\tau,\xi)|^2 \\
&= a_{H,d}\int_{\mathbb{R}} d\tau|\tau|^{-(2H-1)}\int_{\mathbb{R}^d}\mu(d\xi)|\mathcal{F}u(\tau,\xi)|^2|\mathcal{F}\psi_\varepsilon(\tau,\xi)-1|^2 \to 0,
\end{aligned}$$

as $\varepsilon \to 0$, by the Dominated Convergence Theorem, where $\mathcal{F}$ denotes the Fourier transform in the (t,x)-variable and $a_{H,d}$ is a constant depending on H, d and μ.

On the right-hand side of (32), we have:

$$E|I_n(\tilde{f}_{n-1,\varepsilon}) - I_n(\tilde{f}_{n-1})|^2 = n!\|\tilde{f}_{n-1,\varepsilon} - \tilde{f}_{n-1}\|^2_{\mathcal{HP}^{\otimes n}} =$$
$$n! a^n_{H,d} \int_{\mathbb{R}^n} d\tau_1 \dots d\tau_{n-1} d\tau \prod_{i=1}^{n-1} |\tau_i|^{1-2H} |\tau|^{1-2H} \int_{\mathbb{R}^{nd}} \mu(d\xi_1) \dots \mu(d\xi_{n-1}) \mu(d\xi)$$
$$|\mathcal{F}^{(n)} \tilde{f}_{n-1}(\tau_1, \xi_1, \dots, \tau_{n-1}, \xi_{n-1}, \tau, \xi)|^2 |\mathcal{F}\psi_\varepsilon(\tau,\xi) - 1|^2 \to 0, \quad \text{as } \varepsilon \to 0,$$

by the Dominated Convergence Theorem, where $\mathcal{F}^{(n)}$ denotes the Fourier transform in all n variables $(t_1, x_1), \dots, (t_{n-1}, x_{n-1}), (t,x)$.

Case 3. $u(t,x)$ is a function in t and a distribution in x. The argument is similar to Case 2, based on a regularization of u in space. We omit the details. □

We now return to our framework.

Definition 2.7 *We say that $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ is a* **solution of (1)** *if $u(0,x) = 1$ for all $x \in \mathbb{R}^d$, and for any $t > 0, x \in \mathbb{R}^d$:*
(i) $E|u(t,x)|^2 < \infty$;
(ii) $u(t,x)$ is $\mathcal{F}_t$-measurable, where $\mathcal{F}_t = \sigma\{W_s(A); 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R}^d)\}$;
(iii) the process $v^{(t,x)} := G(t-\cdot, x-*)u$ belongs to Dom δ *and*

$$u(t,x) = 1 + \delta(v^{(t,x)}).$$

Here, $\cdot$ denotes the missing s-variable, $*$ denotes the missing y-variable, and $G(t-s, x-*)u(s,*)$ denotes the multiplication of the distribution $G(t-s,x-*)$ with the function $u(s,*)$, for any $s \in (0,t)$.

The following result concludes our preliminary discussion.

Theorem 2.8 *Suppose that for any $t > 0, x \in \mathbb{R}^d, n \geq 1$, $f_n(\cdot, t, x) \in \mathcal{HP}^{\otimes n}$, $f_n(\cdot, t, x)$ being the kernels introduced in Subsection 2.1. Then equation (1) has a solution if and only if for any $t > 0, x \in \mathbb{R}^d$,*

$$\text{the series } \sum_{n\geq 1} I_n(f_n(\cdot,t,x)) \text{ converges in } L^2(\Omega). \tag{33}$$

In this case, the solution is given by: $u(0,x) = 1$ for all $x \in \mathbb{R}^d$, and

$$u(t,x) = 1 + \sum_{n\geq 1} I_n(f_n(\cdot,t,x)), \quad \text{for all } t > 0, x \in \mathbb{R}^d. \tag{34}$$

Proof: Let $v^{(t,x)}$ be given by Definition 2.7. We claim that $v^{(t,x)}$ has the Wiener chaos expansion:

$$v^{(t,x)}(\bullet) = \sum_{n\geq 0} I_n(f_{n+1}(\cdot, \bullet, t, x)), \tag{35}$$

where $\bullet$ denotes the missing (s,y)-variable.

From (35), by Proposition 2.5, it will follow that $v^{(t,x)} \in \text{Dom}\,\delta$ if and only if the series $\sum_{n\geq 0} I_{n+1}(f_{n+1}(\cdot,t,x))$ converges in $L^2(\Omega)$, and in this case,

$$\delta(v^{(t,x)}) = \sum_{n\geq 0} I_{n+1}(f_{n+1}(\cdot,t,x)) = u(t,x) - 1.$$

It remains to prove (35). If $d \leq 2$, then $G(t,x)$ is a function in x, and (35) is clear, since for any $s > 0$ and $y \in \mathbb{R}^d$,

$$\begin{aligned} v^{(t,x)}(s,y) &= G(t-s,x-y)\sum_{n\geq 0} I_n(f_n(\cdot,s,y)) = \sum_{n\geq 0} I_n(G(t-s,x-y)f_n(\cdot,s,y)) \\ &= \sum_{n\geq 0} I_n(f_{n+1}(\cdot,s,y,t,x)). \end{aligned}$$

Suppose now that $d \geq 3$. Then $G(t,x)$ is a distribution in x. Recalling the interpretation given to (35) in Remark 2.6.(c), we show that for any $s \in (0,t)$ and $\phi \in C_0^\infty(\mathbb{R}^d)$ fixed,

$$(v^{(t,x)}(s,*),\phi) = \sum_{n\geq 0} I_n((f_{n+1}(\cdot,s,*,t,x),\phi)) \quad \text{in } L^2(\Omega). \tag{36}$$

Since $v^{(t,x)}(s,*)$ is the product between the distribution $G(t-s,x-*)$ and the function $u(s,*)$, the action of $v^{(t,x)}(s,*)$ on ϕ is given by:

$$\begin{aligned} (v^{(t,x)}(s,*),\phi) &= (G(t-s,x-*),\phi u(s,*)) = [\phi u(s,*) * G(t-s,*)](x) \\ &= \sum_{n\geq 0}[\phi J_n(s,*) * G(t-s,*)](x), \end{aligned}$$

where $J_n(s,y) = I_n(f_n(\cdot,s,y))$, and we used (34).

To prove (36), it suffices to show that for any $n \geq 0$,

$$[\phi J_n(s,*) * G(t-s,*)](x) = I_n((f_{n+1}(\cdot,s,*,t,x),\phi)). \tag{37}$$

Let G_ε be a regularization of G in space, i.e. $G_\varepsilon(s,*) = \psi_\varepsilon * G(s,*)$, where $\psi_\varepsilon = \varepsilon^{-d}\psi(x/\varepsilon)$, $\psi \in C_0^\infty(\mathbb{R}^d)$, $\psi \geq 0$, supp $\psi \subset \{x \in \mathbb{R}^d; |x| \leq 1\}$ and $\int_{\mathbb{R}^d}\psi(x)dx = 1$. Then $\mathcal{F}G_\varepsilon(s,*)(\xi) = \mathcal{F}G(s,*)(\xi)\mathcal{F}\psi_\varepsilon(\xi)$.

Since $G_\varepsilon(s,y)$ is a function in y, for any $n \geq 0$,

$$[\phi J_{n,\varepsilon}(s,*) * G_\varepsilon(t-s,*)](x) = I_n((f_{n+1,\varepsilon}(\cdot,s,*,t,x),\phi)), \tag{38}$$

where $J_{n,\varepsilon}(s,y) = I_n(f_{n,\varepsilon}(\cdot,s,y))$,

$$f_{n,\varepsilon}(t_1,x_1,\ldots,t_n,x_n,s,y) = G_\varepsilon(s-t_n,y-x_n)\ldots G_\varepsilon(t_2-t_1,x_2-x_1)1_{\{t_1<\ldots<t_n<s\}},$$

$$f_{n+1,\varepsilon}(t_1,x_1,\ldots,t_n,x_n,s,y,t,x) = G_\varepsilon(t-s,x-y)f_{n,\varepsilon}(t_1,x_1,\ldots,t_n,x_n,s,y)1_{\{s<t\}}.$$

Relation (37) follows by taking the limit as $\varepsilon \to 0$ in (38). This is justified below.

On the left hand side of (38), we use the fact that for any $y \in \mathbb{R}^d$,

$$E|J_{n,\varepsilon}(s,y) - J_n(s,y)|^2 \to 0, \quad \text{as } \varepsilon \to 0. \tag{39}$$

To see this, note that:

$$E|J_{n,\varepsilon}(s,y) - J_n(s,y)|^2 = E|I_n(f_{n,\varepsilon}(\cdot,s,y)) - I_n(f_n(\cdot,s,y))|^2 = n!\|(\tilde{f}_{n,\varepsilon} - \tilde{f}_n)(\cdot,s,y)\|^2_{\mathcal{HP}^{\otimes n}}$$
$$= n!\alpha_H^n \int_{(0,s)^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} \int_{\mathbb{R}^{nd}} \mathcal{F}(\tilde{f}_{n,\varepsilon} - \tilde{f}_n)(t_1,\cdot,\ldots,t_n,\cdot,s,y)(\boldsymbol{\xi})$$
$$\overline{\mathcal{F}(\tilde{f}_{n,\varepsilon} - \tilde{f}_n)(s_1,\cdot,\ldots,s_n,\cdot,s,y)(\boldsymbol{\xi})}\mu(d\xi_1)\ldots\mu(d\xi_n)d\mathbf{t}d\mathbf{s},$$

where $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_n)$ and $\mathbf{t} = (t_1,\ldots,t_n)$. Here, $\tilde{f}_n(\cdot,s,y)$ is the symmetrization of $f_n(\cdot,s,y)$ in the first n variables $(t_1,x_1),\ldots,(t_n,x_n)$, and $\mathcal{F}$ denotes the Fourier transform with respect to the missing variables $x_1,\ldots,x_n$.

Using (9), one can prove that:

$$\mathcal{F}(\tilde{f}_{n,\varepsilon} - \tilde{f}_n)(t_1,\cdot,\ldots,t_n,\cdot,s,y)(\boldsymbol{\xi}) = \mathcal{F}\tilde{f}_n(t_1,\cdot,\ldots,t_n,\cdot,s,y)(\boldsymbol{\xi})[\overline{\mathcal{F}\psi_\varepsilon(\xi_{\rho(1)})}$$
$$\overline{\psi_\varepsilon(\xi_{\rho(1)} + \xi_{\rho(2)})}\ldots\overline{\psi_\varepsilon(\xi_{\rho(1)} + \ldots + \xi_{\rho(n)})} - 1],$$

where ρ is a permutation such that $t_{\rho(1)} < \ldots < t_{\rho(n)}$. Relation (39) follows by the Dominated Convergence theorem, since $\mathcal{F}\psi_\varepsilon(\xi) \to 0$. The application of this theorem is justified since $|\mathcal{F}\psi_\varepsilon(\xi)| \le 1$ and $\|\tilde{f}_n(\cdot,s,y)\|^2_{\mathcal{HP}^{\otimes n}} < \infty$.

On the right hand side of (38), we use the fact that:

$$E|I_n((f_{n+1,\varepsilon}(\cdot,s,*,t,x),\phi)) - I_n((f_{n+1}(\cdot,s,*,t,x),\phi))|^2 \to 0, \quad \text{as } \varepsilon \to 0. \tag{40}$$

To see this, note that

$$E|I_n((f_{n+1,\varepsilon}(\cdot,s,*,t,x),\phi)) - I_n((f_{n+1}(\cdot,s,*,t,x),\phi))|^2 = n!\|g_\varepsilon(\cdot,s,t,x)\|^2_{\mathcal{HP}^{\otimes n}}$$
$$= n!\alpha_H^n \int_{(0,s)^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} \int_{\mathbb{R}^{nd}} \mathcal{F}g_\varepsilon(t_1,\cdot,\ldots,t_n,\cdot,s,t,x)(\boldsymbol{\xi})$$
$$\overline{\mathcal{F}g_\varepsilon(s_1,\cdot,\ldots,s_n,\cdot,s,t,x)(\boldsymbol{\xi})}\mu(d\xi_1)\ldots\mu(d\xi_n)d\mathbf{t}d\mathbf{s}, \tag{41}$$

where $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_n)$, $\mathbf{t} = (t_1,\ldots,t_n)$ and

$$g_\varepsilon(\cdot,s,t,x) := ((\tilde{f}_{n+1,\varepsilon} - \tilde{f}_{n+1})(\cdot,s,*,t,x),\phi).$$

Here, the action of ϕ is on the missing y-variable (denoted by $*$), $\tilde{f}_{n+1}(\cdot,s,*,t,x)$ is the symmetrization of $f_{n+1}(\cdot,s,*,t,x)$ in the first n variables $(t_1,x_1),\ldots,(t_n,x_n)$, and the Fourier transform is taken with respect to the missing variables $x_1,\ldots,x_n$.

Note that

$$\mathcal{F}g_\varepsilon(t_1,\cdot,\ldots,t_n,\cdot,s,t,x)(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} \mathcal{F}(\tilde{f}_{n+1,\varepsilon} - \tilde{f}_{n+1})(t_1,\cdot,\ldots,t_n,\cdot,s,*,t,x)(\boldsymbol{\xi},\xi)\overline{\mathcal{F}\phi(\xi)}d\xi,$$

where the first Fourier transform under the integral is taken with respect to the $n+1$ missing variables $x_1, \ldots, x_n, y$. Using (9), one can prove that:

$$\mathcal{F}(\tilde{f}_{n+1,\varepsilon} - \tilde{f}_{n+1})(t_1, \cdot, \ldots, t_n, \cdot, s, *, t, x)(\boldsymbol{\xi}, \xi) = \mathcal{F}\tilde{f}_n(t_1, \cdot, \ldots, t_n, \cdot, s, x)(\boldsymbol{\xi}) k_{\varepsilon,\rho}(\boldsymbol{\xi}, \xi),$$

where

$$k_{\varepsilon,\rho}(\boldsymbol{\xi}, \xi) = e^{-i\xi\cdot x}\overline{\mathcal{F}G(t-s,\cdot)(\xi_1 + \ldots + \xi_n + \xi)}\,[\overline{\psi_\varepsilon(\xi_{\rho(1)})}\ldots\overline{\psi_\varepsilon(\xi_{\rho(1)} + \ldots + \xi_{\rho(n)} + \xi)} - 1],$$

and ρ is the permutation for which $t_{\rho(1)} < \ldots < t_{\rho(n)}$. Hence

$$\begin{aligned} \mathcal{F}g_\varepsilon(t_1, \cdot, \ldots, t_n, \cdot, s, t, x)(\boldsymbol{\xi}) &= \mathcal{F}\tilde{f}_n(t_1, \cdot, \ldots, t_n, \cdot, s, x)(\boldsymbol{\xi}) \int_{\mathbb{R}^d} k_{\varepsilon,\rho}(\boldsymbol{\xi}, \xi)\overline{\mathcal{F}\phi(\xi)}d\xi \\ &=: \mathcal{F}\tilde{f}_n(t_1, \cdot, \ldots, t_n, \cdot, s, x)(\boldsymbol{\xi}) K_{\varepsilon,\rho}(\boldsymbol{\xi}). \qquad (42) \end{aligned}$$

Since $\mathcal{F}\psi_\varepsilon(\xi) \to 0$, by the Dominated Convergence Theorem, it follows that $K_{\varepsilon,\rho}(\boldsymbol{\xi}) \to 0$ as $\varepsilon \to 0$. (To justify this, we use (18) and $|\mathcal{F}\psi_\varepsilon(\xi)| \leq 1$.)

Relation (40) follows from (41) and (42), again by the Dominated Convergence Theorem, whose application is justified by the fact that $|K_{\varepsilon,\rho}(\boldsymbol{\xi})| \leq 2C_t \int_{\mathbb{R}^d} |\mathcal{F}\phi(\xi)|d\xi =: C_{t,\phi}$, and $\|\tilde{f}_n(\cdot, s, x)\|^2_{\mathcal{HP}^{\otimes n}} < \infty$. $\Box$

Remark 2.9 Let $u_0(t,x) = 1$ and $u_n(t,x) = 1 + \sum_{k=1}^n I_k(f_k(\cdot, t, x))$ for $n \geq 1$. Let $v_n^{(t,x)} = G(t-\cdot, x-*)u_n$ for any $n \geq 0$. Using the same argument as above, one can show that $\delta(v_n^{(t,x)}) = u_{n+1}(t,x) - 1$, i.e

$$u_{n+1}(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)u_n(s,y)W(\delta s, \delta y), \quad \forall n \geq 0.$$

In other words, $\{u_n\}_{n\geq 0}$ plays the role of the Picard's iteration sequence used in the case $H = 1/2$.

3 Existence of the Solution

In this section, we examine condition (33), in the particular case when f is the Riesz kernel of order $\alpha \in (0,d)$, i.e. $f(x) = c_{\alpha,d}|x|^{-(d-\alpha)}$, $\mu(d\xi) = |\xi|^{-\alpha}d\xi$.

Our main result shows that $\alpha > d-2$ is a sufficient for (33), and hence a sufficient condition for the existence of a solution to (1) (by Theorem 2.8).

Note that $\alpha > d-2$ is also the sufficient condition for the existence of a solution to (1), in the case when $H = 1/2$ and the solution is interpreted using a martingale measure stochastic integral (see Theorem 5.1 of [3]).

Due to the orthogonality of the Wiener chaos spaces and (20), condition (33) is equivalent to:

$$S(t) := 1 + \sum_{n\geq 1} n! \|\tilde{f}_n(\cdot, t, x)\|^2_{\mathcal{HP}^{\otimes n}} < \infty, \qquad (43)$$

where $\tilde{f}_n(\cdot, t, x)$ is the symmetrization of $f_n(\cdot, t, x)$ in the first n variables (t_1, x_1), $\ldots, (t_n, x_n)$. In this case, $E|u(t,x)|^2 = S(t)$.

We begin with the calculation of $\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{HP}^{\otimes n}}$. At the same time, this calculation will show that $f_n(\cdot,t,x) \in \mathcal{HP}^{\otimes n}$ (see Remark 2.3). (By abuse of notation, we use $\|\cdot\|_{\mathcal{HP}^{\otimes n}}$, even if we do not know yet that $\tilde{f}_n(\cdot,t,x) \in \mathcal{HP}^{\otimes n}$.)

Note that

$$\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{HP}^{\otimes n}} = \alpha_H^n \int_{(0,t)^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} \int_{\mathbb{R}^d} \mathcal{F}\tilde{f}_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)(\boldsymbol{\xi})$$

$$\overline{\mathcal{F}\tilde{f}_n(s_1,\cdot,\ldots,s_n,\cdot,t,x)(\boldsymbol{\xi})}\mu(d\xi_1)\ldots\mu(d\xi_n)d\mathbf{t}d\mathbf{s},$$

where $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_n)$, $\mathbf{t} = (t_1,\ldots,t_n)$ and $\mathbf{s} = (s_1,\ldots,s_n)$. Let

$$g_{\mathbf{t}}^{(n)}(\cdot,t,x) := n!\tilde{f}_n(t_1,\cdot,\ldots,t_n,\cdot,t,x).$$

Hence,

$$\mathcal{F}\tilde{f}_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)(\boldsymbol{\xi}) = \frac{1}{n!}\mathcal{F}g_{\mathbf{t}}^{(n)}(\cdot,t,x)(\boldsymbol{\xi}),$$

and

$$\|\tilde{f}_n(\cdot\ ,t,x)\|^2_{\mathcal{HP}^{\otimes n}} = \frac{1}{(n!)^2}\alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^n |s_j - t_j|^{2H-2}\tilde{\psi}^{(n)}(\mathbf{t},\mathbf{s})d\mathbf{t}d\mathbf{s},$$

where

$$\tilde{\psi}^{(n)}(\mathbf{t},\mathbf{s}) := \int_{\mathbb{R}^{nd}} \mathcal{F}g_{\mathbf{t}}^{(n)}(\cdot,t,x)(\boldsymbol{\xi})\overline{\mathcal{F}g_{\mathbf{s}}^{(n)}(\cdot,t,x)(\boldsymbol{\xi})}\mu(d\xi_1)\ldots\mu(d\xi_n). \tag{44}$$

We let

$$\tilde{\alpha}_n(t) := (n!)^2\|\tilde{f}_n(\cdot\ ,t,x)\|^2_{\mathcal{HP}^{\otimes n}}.$$

With this notation, relation (43) becomes:

$$S(t) = \sum_{n\geq 0}\frac{1}{n!}\tilde{\alpha}_n(t) < \infty. \tag{45}$$

We proceed to the evaluation of $\tilde{\alpha}_n(t)$, which relies on the evaluation of $\tilde{\psi}^{(n)}(\mathbf{t},\mathbf{s})$. Using relation (9), one can prove that:

$$\mathcal{F}g_{\mathbf{t}}^{(n)}(\cdot,t,x)(\boldsymbol{\xi}) = e^{-i(\xi_1+\ldots+\xi_n)\cdot x}\sum_{\rho\in S_n}\overline{\mathcal{F}G(u_1,\cdot\ )(\xi_{\rho(1)})}\ \overline{\mathcal{F}G(u_2,\cdot\ )(\xi_{\rho(1)}+\xi_{\rho(2)})}$$

$$\ldots\overline{\mathcal{F}G(u_n,\cdot\ )(\xi_{\rho(1)}+\ldots+\xi_{\rho(n)})}1_{\{t_{\rho(1)}<\ldots<t_{\rho(n)}\}}, \tag{46}$$

where S_n is the set of all permutations of $\{1,\ldots,n\}$ and $u_j = t_{\rho(j+1)} - t_{\rho(j)}$ for $1 \leq j \leq n$, with $t_{\rho(n+1)} = t$.

In the argument below, since there is no risk of confusion, we omit writing the variable (t,x) of $g_{\mathbf{t}}^{(n)}(\cdot,t,x)$.

It is known that, for any $t>0$, $G(t,\cdot)$ is a distribution with rapid decrease in $\mathcal{S}'(\mathbb{R}^d)$, whose Fourier transform is given by: (see e.g. [15])

$$\mathcal{F}G(t,\cdot)(\xi)=\frac{\sin(t|\xi|)}{|\xi|},\quad \forall \xi\in\mathbb{R}^d.$$

The following central result was announced in the introduction, and will allow us to estimate $\tilde{\psi}^{(n)}(\mathbf{t},\mathbf{s})$.

Lemma 3.1 *Assume that $d-2<\alpha<d$. Then,*

$$I:=\int_{\mathbb{R}^d}\frac{\sin^2(t|\xi|)}{|\xi|^2}|\xi-\eta|^{-\alpha}d\xi\le C_{\alpha,d}t^{\alpha-d+2},\quad \textit{for any } t>0,\eta\in\mathbb{R}^d.$$

Proof: Using the change of variable $\xi'=t\xi$, we obtain,

$$I=t^{\alpha-d+2}\int_{\mathbb{R}^d}\frac{\sin^2(|\xi'|)}{|\xi'|^2}|\xi'-t\eta|^{-\alpha}d\xi'.$$

We claim that:

$$I(a):=\int_{\mathbb{R}^d}\frac{\sin^2(|\xi|)}{|\xi|^2}|a-\xi|^{-\alpha}d\xi\le C_{\alpha,d},\quad \forall a\in\mathbb{R}^d.$$

To see this, we change the variable $a-\xi$ into ξ, and we write

$$I(a)=\int_{|\xi|\le 1}\frac{\sin^2(|\xi-a|)}{|\xi-a|^2}|\xi|^{-\alpha}d\xi+\int_{|\xi|>1}\frac{\sin^2(|\xi-a|)}{|\xi-a|^2}|\xi|^{-\alpha}d\xi=:I_1(a)+I_2(a).$$

For $I_1(a)$, we use the fact that $|\frac{\sin x}{x}|\le 1$ for any $x>0$. Hence

$$I_1(a)\le\int_{|\xi|\le 1}|\xi|^{-\alpha}d\xi=c_d\int_0^1\lambda^{-\alpha+d-1}d\lambda=c_d\frac{1}{d-\alpha}.$$

For $I_2(a)$, we use the fact that

$$\frac{\sin^2(t|\xi|)}{|\xi|^2}\le 2(t^2+1)\frac{1}{1+|\xi|^2},\quad \forall t>0,\forall\xi\in\mathbb{R}^d.$$

(see p. 81 of [13]). In our case, $t=1$. Hence

$$I_2(a)\le 4\int_{|\xi|>1}\frac{1}{1+|\xi-a|^2}|\xi|^{-\alpha}d\xi\le 4\sup_{a\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{1+|\xi-a|^2}|\xi|^{-\alpha}d\xi.$$

Finally, we observe that $\alpha>d-2$ is equivalent to $\int_{\mathbb{R}^d}\frac{1}{1+|\xi|^2}|\xi|^{-\alpha}d\xi<\infty$, which in turn is equivalent to

$$\sup_{a\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{1+|\xi-a|^2}|\xi|^{-\alpha}d\xi<\infty$$

(see (5.5) of [3]). $\Box$

Based on the previous lemma, we estimate $\tilde{\psi}^{(n)}(\mathbf{t},\mathbf{s})$.

Lemma 3.2 *If f is the Riesz kernel of order $\alpha > d-2$, then for any $\mathbf{t}, \mathbf{s} \in [0,t]^n$,*

$$\tilde{\psi}^{(n)}(\mathbf{t},\mathbf{s}) \leq C_{\alpha,d}^n [\beta(\mathbf{t})\beta(\mathbf{s})]^{(\alpha-d+2)/2},$$

where $\beta(\mathbf{t}) = \prod_{j=1}^n (t_{\rho(j+1)} - t_{\rho(j)})$, $\beta(\mathbf{s}) = \prod_{j=1}^n (s_{\sigma(j+1)} - s_{\sigma(j)})$, and the permutations ρ and σ of $\{1,\ldots,n\}$ are chosen such that

$$t_{\rho(1)} < t_{\rho(2)} < \ldots < t_{\rho(n)} \quad \text{and} \quad s_{\sigma(1)} < s_{\sigma(2)} < \ldots < s_{\sigma(n)}, \tag{47}$$

with $t_{\rho(n+1)} = s_{\sigma(n+1)} = t$.

Proof: By the Cauchy-Schwartz inequality,

$$\tilde{\psi}^{(n)}(\mathbf{t},\mathbf{s}) \leq \tilde{\psi}^{(n)}(\mathbf{t},\mathbf{t})^{1/2} \tilde{\psi}^{(n)}(\mathbf{s},\mathbf{s})^{1/2}.$$

Let $u_j = t_{\rho(j+1)} - t_{\rho(j)}$ for $j = 1,\ldots,n$. Using (44) and (46), we obtain:

$$\begin{aligned}
\tilde{\psi}^{(n)}(\mathbf{t},\mathbf{t}) &= \int_{\mathbb{R}^{nd}} |\mathcal{F}g_{\mathbf{t}}^{(n)}(\boldsymbol{\xi})|^2 \mu(d\xi_1)\ldots\mu(d\xi_n) \\
&= \int_{\mathbb{R}^{nd}} |\mathcal{F}G(u_1,\cdot\,)(\xi_{\rho(1)})|^2 \ldots |\mathcal{F}G(u_n,\cdot\,)(\xi_{\rho(1)}+\ldots+\xi_{\rho(n)})|^2 \mu(d\xi_1)\ldots\mu(d\xi_n) \\
&= \int_{\mathbb{R}^{nd}} |\mathcal{F}G(u_1,\cdot\,)(\xi_1')|^2 \ldots |\mathcal{F}G(u_n,\cdot\,)(\xi_1'+\ldots+\xi_n')|^2 \mu(d\xi_1')\ldots\mu(d\xi_n') \\
&= \int_{\mathbb{R}^{nd}} \frac{\sin^2(u_1|\xi_1'|)}{|\xi_1'|^2} \cdot \frac{\sin^2(u_2|\xi_1'+\xi_2'|)}{|\xi_1'+\xi_2'|^2} \ldots \frac{\sin^2(u_n|\xi_1'+\ldots+\xi_n'|)}{|\xi_1'+\ldots+\xi_n'|^2} |\xi_1'|^{-\alpha} \ldots |\xi_n'|^{-\alpha} d\xi_1' \ldots d\xi_n',
\end{aligned}$$

where we used the change of variable $\xi_j' = \xi_{\rho(j)}, j = 1,\ldots,n$.

We now use the change of variable

$$\eta_j = \xi_1' + \ldots + \xi_j', \quad j = 1,\ldots,n.$$

The inverse transformation is: $\xi_1' = \eta_1, \xi_j' = \eta_j - \eta_{j-1}, j = 2,\ldots,n$. We get

$$\begin{aligned}
\psi^{*(n)}(\mathbf{t},\mathbf{t}) &= \int_{\mathbb{R}^d} d\eta_1 \frac{\sin^2(u_1|\eta_1|)}{|\eta_1|^2} |\eta_1|^{-\alpha} \int_{\mathbb{R}^d} d\eta_2 \frac{\sin^2(u_2|\eta_2|)}{|\eta_2|^2} |\eta_2 - \eta_1|^{-\alpha} \ldots \\
&\quad \int_{\mathbb{R}^d} d\eta_n \frac{\sin^2(u_n|\eta_n|)}{|\eta_n|^2} |\eta_n - \eta_{n-1}|^{-\alpha}.
\end{aligned}$$

Using Lemma 3.1 iteratively, we obtain $\tilde{\psi}^{(n)}(\mathbf{t},\mathbf{t}) \leq C_{\alpha,d}^n (u_1 \ldots u_n)^{\alpha-d+2}$. The result follows. □

Proposition 3.3 *If f is the Riesz kernel of order $\alpha > d-2$, then for any $t > 0$ and $n \geq 1$,*

$$\tilde{\alpha}_n(t) \leq C(t)^n \frac{1}{(n!)^{\alpha-d+2}}, \tag{48}$$

where $C(t) = C_{\alpha,d,H} t^{2H+\alpha-d+2}$.

Proof: Let $h = (\alpha - d + 2)/(2H)$. As in the proof of Proposition 3.6 of [1], using Lemma 3.2 and inequality (16) of [1], we obtain that:

$$\begin{aligned}
\tilde{\alpha}_n(t) &\leq C^n_{\alpha,d,H}(n!)^{2H}\left(\int_{0<s_1<s_2<\ldots<s_n<t}[(t-s_n)\ldots(s_2-s_1)]^h d\mathbf{s}\right)^{2H} \\
&\leq C^n_{\alpha,d,H}t^{n(1+h)2H}\left(\frac{n!}{\Gamma(n(1+h)+1)}\right)^{2H} \\
&\leq C^n_{\alpha,d,H}t^{n(1+h)2H}\frac{1}{(n!)^{\alpha-d+2}}.
\end{aligned}$$

For the last inequality above, we used the fact that for $a > 0$, $\Gamma(an+1) = C_n(n!)^a$, where C_n is a constant such that $\lambda^{-n} \leq C_n \leq \lambda^n$ for some $\lambda > 1$. □

The existence of the solution is immediate.

Proposition 3.4 *If f is the Riesz kernel of order $\alpha > d - 2$, then condition (33) holds, and consequently, equation (1) has a solution.*

Proof: As we mentioned earlier, condition (33) is equivalent to (45), which in turn is satisfied, since by Proposition 3.3,

$$S(t) = \sum_{n\geq 0}\frac{1}{n!}\tilde{\alpha}_n(t) \leq \sum_{n\geq 0}\frac{C(t)^n}{(n!)^{\alpha-d+3}} < \infty.$$

The second statement follows by Theorem 2.8. □

Since $C(t)$ is an increasing function in t, the previous argument shows that:

$$S(t) = \sum_{n\geq 0}\frac{1}{n!}\tilde{\alpha}_n(t) \leq C_T < \infty, \quad \forall t \in [0,T], \tag{49}$$

for any $T > 0$, i.e. $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E|u(t,x)|^2 < \infty$ for all $T > 0$.

Remark 3.5 In the case of the heat equation, it was shown in [1] that, if f is the Riesz kernel of order $\alpha > d - 4H$, then for any $t > 0$ and $n \geq 1$,

$$\tilde{\alpha}_n(t) \leq C(t)^n\frac{1}{(n!)^{-(d-\alpha)/2}},$$

where $C(t) = C_{\alpha,d,H}t^{2H-(d-\alpha)/2}$. In this case, (45) holds if $\alpha > d - 2$.

Remark 3.6 Using the same method as above, one can prove that $\sum_{n\geq 0}\frac{1}{n!}\alpha_n(t) \leq C_T < \infty$ for all $t \in [0,T]$, where $\alpha_n(t) = (n!)^2\|f_n(\cdot,t,x)\|^2_{\mathcal{HP}^{\otimes n}}$.

4 Moments of the Solution

In this section, we show that the solution is $L^2(\Omega)$-continuous and has uniformly bounded moments of order $p \geq 1$. With the obvious modifications, the results presented in this section remain valid for the heat equation (see Remark 3.5).

Let $u(t,x) = \sum_{n \geq 0} J_n(t,x)$, where $J_n(t,x)$ is the projection of $u(t,x)$ on the Wiener chaos $\mathcal{HP}_n$. By the orthogonality of the $J_n(t,x)$'s, we have:

$$E|u(t,x)|^2 = \sum_{n \geq 0} E|J_n(t,x)|^2. \tag{50}$$

Note that

$$E|J_n(t,x)|^2 = E|I_n(\tilde{f}_n(\cdot\ ,t,x))|^2 = \frac{1}{n!}\tilde{\alpha}_n(t). \tag{51}$$

It is known that, for any $1 < p < q < \infty$, the norms $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent on any Wiener chaos $\mathcal{HP}_n$, where $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$. This is a consequence of the hypercontractivity property of the Ornstein-Uhlenbeck semigroup $(T_t)_{t \geq 0}$, defined by:

$$T_t F = \sum_{n \geq 0} e^{-nt} J_n F, \quad F \in L^2(\Omega),$$

where we denote by $J_n F$ the projection of F on the n-th Wiener chaos $\mathcal{HP}_n$. The property says that for any $p > 1$ and $t > 0$,

$$\|T_t F\|_{q(t)} \leq \|F\|_p,$$

where $q(t) = e^{2t}(p-1)+1$ (see Theorem 1.4.1 of [10]). Hence, for any $1 < p < q < \infty$ and for any $F \in \mathcal{HP}_n$,

$$e^{-nt}\|F\|_q = \|T_t F\|_q \leq \|F\|_p,$$

where $t > 0$ is chosen such that $q = e^{2t}(p-1)+1$. In particular, for any $p > 2$ and for any $F \in \mathcal{HP}_n$,

$$\|F\|_p \leq e^{nt}\|F\|_2 = (p-1)^{n/2}\|F\|_2, \tag{52}$$

where $t > 0$ is chosen such that $p = e^{2t}+1$.

Applying these results in our case, we obtain the following result.

Theorem 4.1 *Let f be the Riesz kernel of order $\alpha > d-2$ and u be the solution of (1). Then u is $L^2(\Omega)$-continuous, and for any $p \geq 1$, $T > 0$*

$$\sup_{t \leq T} \sup_{x \in \mathbb{R}^d} E|u(t,x|^p < \infty. \tag{53}$$

Proof: We apply (52) for $F = J_n(t,x) \in \mathcal{HP}_n$. Using (51), we obtain that:

$$\|J_n(t,x)\|_p \leq (p-1)^{n/2}\|J_n(t,x)\|_2 = (p-1)^{n/2}\left(\frac{1}{n!}\tilde{\alpha}_n(t)\right)^{1/2}.$$

Using (48), we obtain that:

$$\sum_{n\geq 0} \|J_n(t,x)\|_p \leq \sum_{n\geq 0}(p-1)^{n/2}\left\{C(t)^n \frac{1}{(n!)^{\alpha-d+3}}\right\}^{1/2} < \infty.$$

Since $\alpha(t)$ does not depend on x and $C(t)$ is an increasing function of t, we have: for any $T>0$,

$$\sum_{n\geq 0} \sup_{t\leq T} \sup_{x\in\mathbb{R}^d} \|J_n(t,x)\|_p \leq C_{T,p} < \infty. \tag{54}$$

From here, we conclude that for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$, the sequence $\{u_n(t,x)=\sum_{k=0}^{n} J_k(t,x), n\geq 0\}$ is Cauchy in $L^p(\Omega)$, since

$$\|u_n(t,x)-u_m(t,x)\|_p \leq \sum_{k=m+1}^{n} \|J_k(t,x)\|_p \to 0, \quad \text{as } n,m\to\infty, n>m.$$

Therefore, there exists a random variable $v(t,x)\in L^p(\Omega)$ such that $u_n(t,x)\to v(t,x)$ in $L^p(\Omega)$. But $u_n(t,x)\to u(t,x)$ in $L^2(\Omega)$, and hence $u(t,x)=v(t,x)$ a.s. Using (54), one can show that

$$u_n(t,x)\to u(t,x) \text{ in } L^p(\Omega), \text{ uniformly in } (t,x)\in[0,T]\times\mathbb{R}^d \tag{55}$$

and $\|u_n(t,x)\|_p \leq \sum_{k=0}^{n}\|J_k(t,x)\|_p \leq C_{T,p}$ for all $(t,x)\in[0,T]\times\mathbb{R}^d, n\geq 0$. Taking $n\to\infty$, we obtain (53).

By Lemma 4.2 below, J_n is $L^2(\Omega)$-continuous. Hence u_n is $L^2(\Omega)$-continuous. Due to (55), it follows that u is $L^2(\Omega)$-continuous. □

Lemma 4.2 *a) For any $n\geq 1$ and $t>0$,*

$$E|J_n(t+h,x)-J_n(t,x)|^2 \to 0 \text{ as } h\to 0, \text{ uniformly in } x\in\mathbb{R}^d.$$

b) For any $n\geq 1, t>0, x\in\mathbb{R}^d$,

$$E|J_n(t,x+z)-J_n(t,x)|^2 \to 0 \text{ as } |z|\to 0, z\in\mathbb{R}^d.$$

Proof: a) Suppose that $h\in[0,1]$. (The case $h<0$ is similar.) Then,

$$\begin{aligned} E|J_n(t+h,x)-J_n(t,x)|^2 &= E|I_n(\tilde{f}_n(\cdot,t+h,x)-\tilde{f}_n(\cdot,t,x))|^2 \\ &= n!\ \|\tilde{f}_n(\cdot,t+h,x)-\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{HP}^{\otimes n}} \\ &\leq \frac{2}{n!}(E_1(t,h)+E_2(t,h)), \end{aligned}$$

where

$$E_1(t,h) := (n!)^2\|\tilde{f}_n(\cdot,t+h,x)1_{[0,t]^n}-\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{HP}^{\otimes n}} \tag{56}$$

$$E_2(t,h) := (n!)^2\|\tilde{f}_n(\cdot,t+h,x)1_{[0,t+h]^n\setminus[0,t]^n}\|^2_{\mathcal{HP}^{\otimes n}}. \tag{57}$$

We treat $E_1(t,h)$ first. Note that

$$E_1(t,h) = \alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^{n} |t_j - s_j|^{2H-2} \psi_h^{(n)}(\mathbf{t},\mathbf{s}) d\mathbf{t} d\mathbf{s}, \tag{58}$$

where

$$\psi_h^{(n)}(\mathbf{t},\mathbf{s}) = \int_{\mathbb{R}^{nd}} \mathcal{F}(g_{\mathbf{t}}^{(n)}(\cdot,t,x+h) - g_{\mathbf{t}}^{(n)}(\cdot,t,x))(\boldsymbol{\xi}) \overline{\mathcal{F}(g_{\mathbf{s}}^{(n)}(\cdot,t,x+h) - g_{\mathbf{s}}^{(n)}(\cdot,t,x))(\boldsymbol{\xi})} \mu(d\xi_1)\ldots\mu(d\xi_n)$$

By the Cauchy-Schwartz inequality, $\psi_h^{(n)}(\mathbf{t},\mathbf{s}) \leq \psi_h^{(n)}(\mathbf{t},\mathbf{t})^{1/2} \cdot \psi_h^{(n)}(\mathbf{s},\mathbf{s})^{1/2}$.

To evaluate $\psi_h^{(n)}(\mathbf{t},\mathbf{t})$, we use (46), denoting $u_j = t_{\rho(j+1)} - t_{\rho(j)}$, when $0 < t_{\rho(1)} < \ldots < t_{\rho(n)} < t_{\rho(n+1)} = t$:

$$\begin{aligned} \psi_h^{(n)}(\mathbf{t},\mathbf{t}) &= \int_{\mathbb{R}^{nd}} |\mathcal{F}(g_{\mathbf{t}}^{(n)}(\cdot\,,t+h,x) - g_{\mathbf{t}}^{(n)}(\cdot\,,t,x))(\boldsymbol{\xi})|^2 \mu(d\xi_1)\ldots\mu(d\xi_n) \\ &= \int_{\mathbb{R}^{nd}} |\mathcal{F}G(u_1,\cdot\,)(\xi_{\rho(1)})|^2 \ldots |\mathcal{F}G(u_{n-1},\cdot\,)(\xi_{\rho(1)} + \ldots + \xi_{\rho(n-1)})|^2 \\ &\quad |\mathcal{F}[G(u_n+h,\cdot\,) - G(u_n,\cdot\,)](\xi_{\rho(1)} + \ldots + \xi_{\rho(n)})|^2 \mu(d\xi_1)\ldots\mu(d\xi_n). \end{aligned}$$

Proceeding as in the evaluation of $\psi^{*(n)}(\mathbf{t},\mathbf{t})$, we obtain,

$$\begin{aligned} \psi_h^{(n)}(\mathbf{t},\mathbf{t}) &= \int_{\mathbb{R}^d} d\eta_1 \frac{\sin^2(u_1|\eta_1|)}{|\eta_1|^2} |\eta_1|^{-\alpha} \int_{\mathbb{R}^d} d\eta_2 \frac{\sin^2(u_2|\eta_2|)}{|\eta_2|^2} |\eta_2 - \eta_1|^{-\alpha} \ldots \\ &\quad \int_{\mathbb{R}^d} d\eta_n \frac{|\sin((u_n+h)|\eta_n|) - \sin(u_n|\eta_n|)|^2}{|\eta_n|^2} |\eta_n - \eta_{n-1}|^{-\alpha}. \end{aligned} \tag{59}$$

By the Dominated Convergence Theorem,

$$\psi_h^{(n)}(\mathbf{t},\mathbf{t}) \to 0 \quad \text{as } h \to 0.$$

The application of this theorem is justified, since

$$\frac{|\sin((t+h)|\xi|) - \sin(t|\xi|)|}{|\xi|} \leq \left(\frac{4}{1+|\xi|^2}\right)^{1/2},$$

for all $\xi \in \mathbb{R}^d, t > 0, h \in [0,1]$ (see p.4 of Erratum of [4]). The fact that $E_1(t,h) \to 0$ follows by applying the Dominated Convergence Theorem in (58).

We now treat $E_2(t,h)$. Let $A = [0,t+h]^n \backslash [0,t]^n$. We have

$$E_2(t,h) = \alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^{n} |t_j - s_j|^{2H-2} 1_A(\mathbf{t}) 1_A(\mathbf{s}) \gamma_h^{(n)}(\mathbf{t},\mathbf{s}) d\mathbf{t} d\mathbf{s}, \tag{60}$$

where

$$\gamma_h^{(n)}(\mathbf{t},\mathbf{s}) = \int_{\mathbb{R}^{nd}} \mathcal{F}g_{\mathbf{t}}^{(n)}(\cdot\,,t+h,x)(\boldsymbol{\xi}) \overline{\mathcal{F}g_{\mathbf{s}}^{(n)}(\cdot\,,t+h,x)(\boldsymbol{\xi})} \mu(d\xi_1)\ldots\mu(d\xi_n).$$

By the Cauchy-Schwartz inequality, $\gamma_h^{(n)}(\mathbf{t},\mathbf{s}) \leq \gamma_h^{(n)}(\mathbf{t},\mathbf{t})^{1/2}\gamma_h^{(n)}(\mathbf{s},\mathbf{s})^{1/2}$. To evaluate $\gamma_h^{(n)}(\mathbf{t},\mathbf{t})$, we use again (46):

$$\begin{aligned}
\gamma_h^{(n)}(\mathbf{t},\mathbf{t}) &= \int_{\mathbb{R}^{nd}} |\mathcal{F}g_{\mathbf{t}}^{(n)}(\cdot\,,t+h,x)(\boldsymbol{\xi})|^2 \mu(d\xi_1)\dots\mu(d\xi_n) \\
&= \int_{\mathbb{R}^d} d\eta_1 \frac{\sin^2(u_1|\eta_1|)}{|\eta_1|^2}|\eta_1|^{-\alpha}\int_{\mathbb{R}^d} d\eta_2 \frac{\sin^2(u_2|\eta_2|)}{|\eta_2|^2}|\eta_2-\eta_1|^{-\alpha}\dots \\
&\quad \int_{\mathbb{R}^d} d\eta_n \frac{\sin^2((u_n+h)|\eta_n|)}{|\eta_n|^2}|\eta_n-\eta_{n-1}|^{-\alpha}, \qquad (61)
\end{aligned}$$

where $u_j = t_{\rho(j+1)} - t_{\rho(j)}$. By Lemma 3.1,

$$\gamma_h^{(n)}(\mathbf{t},\mathbf{t}) \leq C_{\alpha,d}^n [u_1 \dots u_{n-1}(u_n+h)]^{\alpha-d+2},$$

and hence

$$\gamma_h^{(n)}(\mathbf{t},\mathbf{s}) \leq C_{\alpha,d}^n \left[(u_n+h)(v_n+h)\prod_{j=1}^{n-1} u_j v_j\right]^{(\alpha-d+2)/2},$$

where $v_j = s_{\sigma(j+1)} - s_{\sigma(j)}$ and $s_{\sigma(1)} < \dots < s_{\sigma(n)} < s_{\sigma(n+1)} = t$.

By inequality (16) in [1],

$$E_2(t,h) \leq b_H^{2n} C_{\alpha,d}^n \left(\int_{[0,t+h]^n} 1_A(\mathbf{t}) \left[\prod_{j=1}^{n-1}(t_{\rho(j+1)} - t_{\rho(j)})(t+h-t_{\rho(n)})\right]^{\delta} d\mathbf{t}\right)^{2H},$$

where $\delta = (\alpha - d + 2)/(2H)$. Using the fact that

$$A = \bigcup_{\rho \in S_{n-1}} \{(t_1,\dots,t_n); 0 < t_{\rho(1)} < \dots < t_{\rho(n-1)} < t_n \text{ and } t_n \in [t,t+h]\},$$

we obtain that:

$$E_2(t,h) \leq b_H^{2n} C_{\alpha,d}^n \left[(n-1)!\int_t^{t+h}(t+h-t_n)^{\delta} I_{n-1}(t_n,\delta)dt_n\right]^{2H},$$

where

$$\begin{aligned}
I_{n-1}(t_n,\delta) &:= \int_{0<t_1<\dots<t_{n-1}<t_n} \prod_{j=1}^{n-1}(t_{j+1}-t_j)^{\delta} dt_1\dots dt_{n-1} \\
&= \frac{\Gamma(1+\delta)^n}{\Gamma((n-1)(1+\delta)+1)} t_n^{(n-1)(1+\delta)},
\end{aligned}$$

(see Lemma 3.5 of [1]). We obtain:

$$
\begin{aligned}
E_2(t,h) &\le b_H^{2n} C_{\alpha,d}^n \Gamma(1+\delta)^{2Hn} \left[\frac{(n-1)!}{\Gamma((n-1)(1+\delta)+1)} \int_t^{t+h} (t+h-t_n)^{\delta} t_n^{(n-1)(1+\delta)} dt_n \right]^{2H} \\
&\le C_{\alpha,d,H}^n (t+1)^{2H(n-1)(1+\delta)} \frac{1}{[(n-1)!]^{2H\delta}} \left(\int_t^{t+h} (t+h-t_n)^{\delta} dt_n \right)^{2H} \\
&= C_{\alpha,d,H}^n (t+1)^{2H(n-1)(1+\delta)} \frac{1}{[(n-1)!]^{2H\delta}} h^{2H(\delta+1)} \to 0, \quad \text{as } h \to 0.
\end{aligned}
$$

b) We have:

$$
E|J_n(t,x) - J_n(t,y)|^2 = E|I_n(f_n(\cdot,t,x)) - I_n(f_n(\cdot,t,y))|^2 = \frac{1}{n!} E_3(t,x,y),
$$

where

$$
\begin{aligned}
E_3(t,x,y) &:= (n!)^2 \|\tilde{f}_n(\cdot,t,x) - \tilde{f}_n(\cdot,t,y)\|^2_{\mathcal{HP}^{\otimes n}} && (62) \\
&= \alpha_H^n \int_{[0,t]^n} \prod_{j=1}^n |t_j - s_j|^{2H-2} \psi_{x,y}^{(n)}(\mathbf{t},\mathbf{s}) d\mathbf{t} d\mathbf{s} && (63)
\end{aligned}
$$

and

$$
\psi_{x,y}^{(n)}(\mathbf{t},\mathbf{s}) = \int_{\mathbb{R}^{nd}} \mathcal{F}(g_{\mathbf{t}}^{(n)}(\cdot\, t,x) - g_{\mathbf{t}}^{(n)}(\cdot\,,t,y))(\boldsymbol{\xi}) \overline{\mathcal{F}(g_{\mathbf{s}}^{(n)}(\cdot\, t,x) - g_{\mathbf{s}}^{(n)}(\cdot\,,t,y))(\boldsymbol{\xi})} \mu(d\xi_1)\ldots\mu(d\xi_n).
$$

By the Cauchy-Schwartz inequality, $\psi_{x,y}^{(n)}(\mathbf{t},\mathbf{s}) \le \psi_{x,y}^{(n)}(\mathbf{t},\mathbf{t})^{1/2} \psi_{x,y}^{(n)}(\mathbf{s},\mathbf{s})^{1/2}$. To evaluate $\psi_{x,y}^{(n)}(\mathbf{t},\mathbf{t})$, we use (46):

$$
\begin{aligned}
\psi_{x,y}^{(n)}(\mathbf{t},\mathbf{t}) &= \int_{\mathbb{R}^{nd}} |\mathcal{F}(g_{\mathbf{t}}^{(n)}(\cdot\,,t,x) - g_{\mathbf{t}}^{(n)}(\cdot\,,t,y))(\boldsymbol{\xi})|^2 \mu(d\xi_1)\ldots\mu(d\xi_n) \\
&= \int_{\mathbb{R}^d} d\eta_1 \frac{\sin^2(u_1|\eta_1|)}{|\eta_1|^2} |\eta_1|^{-\alpha} \int_{\mathbb{R}^d} d\eta_2 \frac{\sin^2(u_2|\eta_2|)}{|\eta_2|^2} |\eta_2 - \eta_1|^{-\alpha} \ldots \\
&\quad \int_{\mathbb{R}^d} d\eta_n \frac{\sin^2(u_n|\eta_n|)}{|\eta_n|^2} |\eta_n - \eta_{n-1}|^{-\alpha} |1 - e^{-i\eta_n\cdot(y-x)}|^2. \qquad (64)
\end{aligned}
$$

By the Dominated Convergence Theorem, $E_3(t,x,y) \to 0$ as $|x-y| \to 0$. $\Box$

5 Hölder Continuity

In this section, we obtain some bounds for the p-th moments of the solution, from which we infer that the solution has a γ-Hölder continuous modification, with $0 < \gamma < \frac{\alpha-d+2}{2}$.

If f is the Riesz kernel of order $\alpha > d-2$, then for any $1 > \beta > (d-\alpha)/2$, $\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^{\beta}} < \infty$, which is equivalent to

$$
\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi+\eta|^2)^{\beta}} < \infty. \qquad (65)
$$

(see (7.26) of [3]).

By Proposition 7.4 of [3], the fundamental solution G of the wave equation satisfies hypothesis (H3)-(H5) of [3], for any $0 < \gamma_i \leq 1-\beta < \frac{\alpha-d+2}{2}$, $i = 1,2,3$. This fact is used in the proof of the next result.

Theorem 5.1 *Let f be the Riesz kernel of order $\alpha > d-2$ and u be the solution of (1). Then for any $p \geq 2, T > 0$ and $K \subset \mathbb{R}^d$ compact,*

$$\begin{aligned} E|u(t+h,x) - u(t,x)|^p &\leq C|h|^{p[\gamma_1 \wedge (\gamma_2+H)]}, \quad \forall t \in [0,T], \forall x \in \mathbb{R}^d, \forall h \in \mathbb{R}, t+h \in [0,T], \\ E|u(t,x+z) - u(t,x)|^p &\leq C|z|^{p\gamma_3}, \quad \forall t \geq 0, \forall x \in K, \forall z \in \mathbb{R}^d, x+z \in K \end{aligned}$$

for any $0 < \gamma_i < \frac{\alpha-d+2}{2}$, where C is a constant which depends on α, d, H, p, T.

In particular, $\{u(t,x); (t,x) \in [0,T] \times K\}$ has a modification which is a.s. jointly γ-Hölder continuous in time and space, for any $\gamma \in (0, \frac{\alpha-d+2}{2})$.

Proof: We first treat the time increments. By Minkowski's inequality and (52),

$$\begin{aligned} &\|u(t+h,x) - u(t,x)\|_p = \|\sum_{n\geq 0}(J_n(t+h,x) - J_n(t,x))\|_p \\ &\leq \sum_{n\geq 0}\|J_n(t+h,x) - J_n(t,x)\|_p \leq \sum_{n\geq 0}(p-1)^{n/2}\|J_n(t+h,x) - J_n(t,x)\|_2 \\ &= \sum_{n\geq 0}(p-1)^{n/2}\left\{\frac{2}{n!}(E_1(t,h) + E_2(t,h))\right\}^{1/2}, \end{aligned} \tag{66}$$

where $E_1(t,h)$ and $E_2(t,h)$ are given by (56), respectively (57).

To estimate $E_1(t,h)$, we use (58) and (59). The inner integral in (59) is

$$\int_{\mathbb{R}^d} |\mathcal{F}G(u_n+h,\cdot)(\xi+\eta_{n-1}) - \mathcal{F}G(u_n,\cdot)(\xi+\eta_{n-1})|^2\mu(d\xi).$$

This integral is bounded by $Ch^{2\gamma_1}$ for some $0 < \gamma_1 < \frac{\alpha-d+2}{2}$, due to (H3). The remaining $(n-1)$-fold integral in (59) is bounded above by $C_{\alpha,d}^{n-1}(u_1 \ldots u_{n-1})^{\alpha-d+2}$, by Lemma 3.1. Hence,

$$\psi_h^{(n)}(\mathbf{t},\mathbf{s}) \leq Ch^{2\gamma_1}C_{\alpha,d}^{n-1}(u_1 \ldots u_{n-1})^{(\alpha-d+2)/2}(v_1 \ldots v_{n-1})^{(\alpha-d+2)/2},$$

where $u_j = t_{\rho(j+1)} - t_{\rho(j)}$ and $v_j = s_{\sigma(j+1)} - s_{\sigma(j)}$. By inequality (16) of [1],

$$\begin{aligned} E_1(t,h) &\leq Ch^{2\gamma_1}C_{\alpha,d}^{n-1}b_H^{2n}\left(n!\int_0^t \int_{0<t_1<\ldots<t_{n-1}<t_n}\prod_{j=1}^{n-1}(t_j-t_{j-1})^{\delta}dt_1\ldots dt_{n-1}dt_n\right)^{2H} \\ &= Ch^{2\gamma_1}C_{\alpha,d}^{n-1}b_H^{2n}\Gamma(1+\delta)^{2Hn}\left(\frac{n!}{\Gamma((n-1)(1+\delta)+1)}\int_0^t t_n^{(n-1)(1+\delta)}dt_n\right)^{2H}, \\ &\leq Ch^{2\gamma_1}C_{\alpha,d,H}^{n}\frac{T^{n(2H+\alpha-d+2)}}{(n!)^{\alpha-d+2}}, \end{aligned} \tag{67}$$

where $\delta = (\alpha - d + 2)/(2H)$.

To estimate $E_2(t,h)$, we use (60) and (61). The inner integral in (61) is

$$\int_{\mathbb{R}^d} |\mathcal{F}G(u_n + h, \cdot\ )(\xi + \eta_{n-1})|^2 \mu(d\xi) \leq C(u_n + h)^{2\gamma_2},$$

for some $0 < \gamma_2 < \frac{\alpha-d+2}{2}$, by (H4). Using the same ideas as above, we get:

$$\begin{aligned} E_2(t,h) &\leq CC_{\alpha,d,H}^n \left(\frac{(n-1)!}{\Gamma((n-1)(1+\delta)+1)} \int_t^{t+h} (t+h-t_n) t_n^{(n-1)(1+\delta)} dt_n \right)^{2H} \\ &\leq Ch^{2(\gamma_2+H)} C_{\alpha,d,H}^n \frac{T^{n(2H+\alpha-d+2)}}{(n!)^{\alpha-d+2}}. \end{aligned} \quad (68)$$

From (66), (67) and (68), we get:

$$\begin{aligned} \|u(t+h,x) - u(t,x)\|_p &\leq Ch^{\gamma_1 \wedge (\gamma_2+H)} \sum_{n\geq 0} \frac{(p-1)^{n/2} C_{\alpha,d,H}^{n/2} T^{n(2H+\alpha-d+2)/2}}{(n!)^{(\alpha-d+3)/2}} \\ &=: h^{\gamma_1 \wedge (\gamma_2+H)} C(\alpha,d,H,p,T). \end{aligned}$$

We now treat the spatial increments. As above, we obtain:

$$\|u(t,x+z) - u(t,x)\|_p \leq \sum_{n\geq 0} (p-1)^{n/2} \left(\frac{1}{n!} E_3(t,x,x+z) \right)^{1/2},$$

where $E_3(t,x,y)$ is given by (62). To estimate $E_3(x,x+z)$ we use (63) and (64). Using the fact that $\mathcal{F}G(u, \cdot - z)(\xi) = e^{-i\xi\cdot z}\mathcal{F}G(u,\cdot)(\xi)$, we see that the inner integral in (64) is:

$$\int_{\mathbb{R}^d} |\mathcal{F}G(u_n, \cdot - z)(\xi + \eta_{n-1}) - \mathcal{F}G(u_n, \cdot\ )(\xi + \eta_{n-1})|^2 \mu(d\xi) \leq C|z|^{2\gamma_3},$$

for some $0 < \gamma_3 < \frac{\alpha-d+2}{2}$, by (H5). The rest of the proof is the same as above.

The final statement follows by a version of Kolmogorov's criterion for multi-parameter processes (see e.g. Problem 2.9 of [8]). $\square$

6 Malliavin differentiability of the solution

In this section, we show that $u(t,x)$ is Malliavin differentiable of any order. When $d \leq 2$, we show that the Malliavin derivative of the solution satisfies a certain integral equation. These results are valid for the heat equation in any dimension d.

Recall that if F is a smooth random variable of the form (22), the iterated Malliavin derivative $D^k F$ is an $\mathcal{HP}^{\otimes k}$-valued random variable, defined by:

$$D^k F = \sum_{i_1,\ldots,i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1} \ldots x_{i_k}} (W(\varphi_1), \ldots, W(\varphi_n)) \varphi_{i_1} \otimes \ldots \otimes \varphi_{i_k}.$$

The space $\mathbb{D}^{k,p}$ is the completion of the space $\mathcal{S}$ of smooth random variables, with respect to the the norm $\|\cdot\|_{\mathbb{D}^{k,p}}$ defined by:

$$\|F\|_{\mathbb{D}^{k,p}}^p = E|F|^p + \sum_{j=1}^k \|D^j F\|_{\mathcal{HP}^{\otimes j}}^p. \tag{69}$$

Using a known criterion (see e.g. p.28 of [10]), it follows that $u(t,x) \in \mathbb{D}^{k,2}$, since by (51) and (48), we have:

$$\sum_{n\geq 1} n^k E|J_n(t,x)|^2 = \sum_{n\geq 1} n^k \frac{1}{n!}\tilde{\alpha}_n(t) \leq \sum_{n\geq 1} \frac{[2^k C(t)]^n}{(n!)^{\alpha-d+3}} < \infty.$$

Next, we show that $u(t,x) \in \mathbb{D}^{k,p}$ for all $k \geq 1, p > 1$.

Let $F \in L^p(\Omega)$ be such that $D^k F$ exists. By Meyer's inequalities (Theorem 1.5.1 of [10]), for any $p > 1$,

$$E\|D^k F\|_{\mathcal{HP}^{\otimes k}}^p < \infty \text{ if and only if } E|C^k F|^p < \infty, \tag{70}$$

where $C^k : \text{Dom } C^k \subset L^2(\Omega) \to L^2(\Omega)$ is the operator defined by

$$C^k F = \sum_{n\geq 1} (-\sqrt{n})^k J_n F,$$

and $\text{Dom } C^k = \{F \in L^2(\Omega); \sum_{n\geq 1} n^k E|J_n F|^2 < \infty\} = \mathbb{D}^{k,2}$ for any $k \geq 1$.

By Minkowski's inequality and (52), we have:

$$\|C^k F\|_p \leq \sum_{n\geq 1} n^{k/2} \|J_n F\|_p \leq \sum_{n\geq 1} n^{k/2}(p-1)^{n/2}\|J_n F\|_2. \tag{71}$$

Combining (69), (70) and (71), we infer that $\|F\|_{\mathbb{D}^{k,p}} < \infty$ (i.e. $F \in \mathbb{D}^{k,p}$), if

$$\sum_{n\geq 1} n^{k/2}(p-1)^{n/2}\|J_n F\|_2 < \infty. \tag{72}$$

Applying this in our case, we obtain the following result.

Proposition 6.1 *Let f be the Riesz kernel of order $\alpha > d-2$ and u be the solution of (1). Then $u(t,x) \in \mathbb{D}^{k,p}$ for all $k \geq 1$ and $p > 1$.*

Proof: We verify (72). By (51) and (48), we have:

$$\begin{aligned}\sum_{n\geq 1} n^{k/2}(p-1)^{n/2}\|J_n(t,x)\|_2 &= \sum_{n\geq 1} n^{k/2}(p-1)^{n/2}\left(\frac{1}{n!}\tilde{\alpha}_n(t)\right)^{1/2} \\ &\leq \sum_{n\geq 1} \frac{[2^k(p-1)C(t)]^{n/2}}{(n!)^{(\alpha-d+3)/2}} < \infty.\end{aligned}$$

□

In the final part of this section, we show that the Malliavin derivative $Du(t,x)$ satisfies a certain integral equation. For this, we assume that $G(t,x)$ is a function in x (i.e. $d \leq 2$).

Recall that u satisfies the integral equation (3). Intuitively, using the commutativity between the operators D and δ, the derivative Du should satisfy:

$$Du(t,x) = G(t-\cdot, x-*)u + \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)Du(s,y)W(\delta s, \delta y), \quad (73)$$

where $\cdot$ denotes the missing r variable and $*$ denotes the missing z variable.

The integrand of the stochastic integral above is an $\mathcal{HP}\otimes\mathcal{HP}$-valued random variable, and the integral needs to be defined as an $\mathcal{HP}$-valued random variable.

For this reason, we introduce a Hilbert-space-valued Skorohod integral.

If $\mathcal{A}$ is an arbitrary Hilbert space, we let $\mathcal{S}(\mathcal{A})$ be the class of smooth $\mathcal{A}$-valued random variables $F = \sum_{j=1}^m F_j v_j$, with $F_j \in \mathcal{S}, v_j \in \mathcal{A}, m \geq 1$. The Malliavin derivative of such F is defined as $DF = \sum_{j=1}^m DF \otimes v_j$. We denote by $\mathbb{D}^{1,2}(\mathcal{A})$ the completion of $\mathcal{S}(\mathcal{A})$ with respect to the norm $\|\cdot\|_{\mathbb{D}^{1,2}(\mathcal{A})}$, where

$$\|F\|^2_{\mathbb{D}^{1,2}(\mathcal{A})} := E\|F\|^2_{\mathcal{A}} + E\|DF\|^2_{\mathcal{HP}\otimes\mathcal{A}}.$$

Similarly to the case $\mathcal{A} = \mathbb{R}$ (considered in Subsection 2.3), we let δ^* be the adjoint of the operator D. The domain of δ^*, denoted by Dom δ^*, is the set of $U \in L^2(\Omega; \mathcal{HP}\otimes\mathcal{A})$ such that:

$$|E\langle DF, U\rangle_{\mathcal{HP}\otimes\mathcal{A}}| \leq c(E\|F\|^2_{\mathcal{A}})^{1/2}, \quad \forall F \in \mathbb{D}^{1,2}(\mathcal{A}),$$

where c is a constant depending on U. If $U \in$ Dom δ^*, then $\delta^*(U)$ is the element of $L^2(\Omega;\mathcal{A})$ characterized by the following duality relation:

$$E\langle F, \delta^*(U)\rangle_{\mathcal{A}} = E\langle DF, U\rangle_{\mathcal{HP}\otimes\mathcal{A}}, \quad \forall F \in \mathbb{D}^{1,2}(\mathcal{A}). \quad (74)$$

If $U \in$ Dom δ^*, we use the notation

$$\delta^*(U) = \int_0^\infty \int_{\mathbb{R}^d} U(t,x) W(\delta^* t, \delta^* x),$$

even if $U(t,x)$ is not a function in (t,x), and we say that $\delta^*(U)$ is the $\mathcal{A}$-valued Skorohod integral of U with respect to W.

Similar to the case $\mathcal{A} = \mathbb{R}$, we have the following result.

Proposition 6.2 *Let $U, V \in \mathbb{D}^{1,2}(\mathcal{HP}\otimes\mathcal{A})$. Then*

$$E\langle\delta^*(U),\delta^*(V)\rangle_{\mathcal{A}} = E\langle U,V\rangle_{\mathcal{HP}\otimes\mathcal{A}} + E\left(\sum_{i,j,k=1}^\infty D^{e_i}\langle U, e_j\otimes a_k\rangle_{\mathcal{HP}\otimes\mathcal{A}} D^{e_j}\langle V, e_i\otimes a_k\rangle_{\mathcal{HP}\otimes\mathcal{A}}\right),$$

where $(e_i)_{i\geq 1}$, $(a_k)_{k\geq 1}$ are complete orthonormal systems in $\mathcal{HP}$, respectively $\mathcal{A}$. Consequently, if $U \in \mathbb{D}^{1,2}(\mathcal{HP}\otimes\mathcal{A})$, then $U \in$ Dom δ^* and

$$E\|\delta^*(U)\|^2_{\mathcal{A}} \leq E\|U\|^2_{\mathcal{HP}\otimes\mathcal{A}} + E\|DU\|^2_{\mathcal{HP}\otimes\mathcal{HP}\otimes\mathcal{A}} = \|U\|^2_{\mathbb{D}^{1,2}(\mathcal{HP}\otimes\mathcal{A})}. \quad (75)$$

Proof: The proof is similar to Proposition 1.3.1 of [10]. For this, one needs to revisit the basic rules of the Malliavin calculus. We omit the details, but we list below these rules, which apply to any isonormal Gaussian process $\{W(h)\}_{h\in\mathcal{H}}$:

1) For any $F \in \mathcal{S}(\mathcal{A}), h \in \mathcal{H}, v \in \mathcal{A}$,

$$E\langle DF, h\otimes v\rangle_{\mathcal{H}\otimes\mathcal{A}} = E(\langle F, v\rangle_{\mathcal{A}} W(h)).$$

2) For any $F \in \mathcal{S}(\mathcal{A})$, $G \in \mathcal{S}$, $h \in \mathcal{H}$, $v \in \mathcal{A}$,

$$E(G\langle DF, h\otimes v\rangle_{\mathcal{H}\otimes\mathcal{A}}) = -E(\langle F, v\rangle_{\mathcal{A}}\langle DG, h\rangle_{\mathcal{H}}) + E(\langle F, v\rangle_{\mathcal{A}} G W(h)).$$

3) If $U = \sum_{j=1}^m F_j(h_j \otimes v_j) \in \mathcal{S}(\mathcal{H}\otimes\mathcal{A})$ for some $F_j \in \mathcal{S}$, $h_j \in \mathcal{H}$, $v_j \in \mathcal{A}$, then $U \in \text{Dom }\delta^*$ and

$$\delta^*(U) = \sum_{j=1}^m F_j W(h_j) - \sum_{j=1}^m \langle DF_j, h_j\rangle_{\mathcal{H}} v_j.$$

4) For any $U = \sum_{j=1}^m F_j(h_j \otimes v_j) \in \mathcal{S}(\mathcal{H}\otimes\mathcal{A})$ and $h \in \mathcal{H}$, $v \in \mathcal{A}$,

$$D^{h\otimes v}(\delta^*(U)) = \langle U, h\otimes v\rangle_{\mathcal{H}\otimes\mathcal{A}} + \langle \delta^*(D^h u), v\rangle_{\mathcal{A}},$$

where $D^{h\otimes v}(\delta^*(U)) = \langle D(\delta^*(U)), h\otimes v\rangle_{\mathcal{H}\otimes\mathcal{A}}$ and $D^h U = \sum_{j=1}^m (D^h F_j)(h_j \otimes v_j)$. Here $D^h F = \langle DF, h\rangle_{\mathcal{H}}$. $\Box$

In what follows, we let δ^* be the operator corresponding to the case $\mathcal{A} = \mathcal{HP}$. We begin with some preliminary results.

Lemma 6.3 *For any $t > 0, x \in \mathbb{R}^d$, $Du_n(t,x) \to Du(t,x)$ in $L^2(\Omega; \mathcal{HP})$, and*

$$C_T^{(1)} := \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\|Du(t,x)\|_{\mathcal{HP}}^2 < \infty, \quad \textit{for all } T > 0. \tag{76}$$

Proof: Using Lemma 1.2.3 of [10], it suffices to prove that:

$$\sup_{n\geq 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\|Du_n(t,x)\|_{\mathcal{HP}}^2 < \infty.$$

By Proposition 2.4, $D_{r,z}u_n(t,x) = \sum_{k=1}^n kI_{k-1}(f_k(\cdot\ , r, z, t, x))$. Using the orthogonality of the Wiener chaos spaces, (20) and (49), we get: for any $t \in [0,T]$

$$\begin{aligned}
E\|Du_n(t,x)\|^2_{\mathcal{HP}} &= \alpha_H \int_{(0,t)^2}\int_{\mathbb{R}^{2d}} |r-r'|^{2H-2} f(z-z') \left(\sum_{k=1}^n k I_{k-1}(f_k(\cdot\ ,r,z,t,x))\right) \\
&\quad \left(\sum_{l=1}^n l I_{l-1}(f_l(\cdot\ ,r',z',t,x))\right) dz dz' dr dr' \\
&= \alpha_H \sum_{k=1}^n k^2(k-1)! \int_{(0,t)^2}\int_{\mathbb{R}^{2d}} |r-r'|^{2H-2} f(z-z') \\
&\quad \langle \bar{f}_k(\cdot\ ,r,z,t,x), \bar{f}_k(\cdot\ ,r',z',t,x)\rangle_{\mathcal{HP}^{\otimes(k-1)}} dz dz' dr dr' \\
&= \sum_{k=1}^n k k!\ \|\bar{f}_k(\cdot\ ,t,x)\|^2_{\mathcal{HP}^{\otimes k}} \le \sum_{k=1}^n k k!\ \|f_k(\cdot\ ,t,x)\|^2_{\mathcal{HP}^{\otimes k}} \\
&= \sum_{k=1}^n k k! \frac{1}{(k!)^2}\alpha_k(t) = \sum_{k=1}^n \frac{1}{(k-1)!}\alpha_k(t) \le C_T < \infty,
\end{aligned}$$

where $\bar{f}_k(\cdot,r,z,t,x)$ denotes the symmetrization of $f_k(\cdot,r,z,t,x)$ with respect to the first $k-1$ variables. For the first inequality above, we used the fact that the $\|\tilde{f}\|_{\mathcal{HP}^{\otimes n}} \le \|f\|_{\mathcal{HP}^{\otimes n}}$ for any $f \in \mathcal{HP}^{\otimes n}$. For the last inequality, we used Remark 3.6. $\Box$

Remark 6.4 Using Proposition 2.4 iteratively, we obtain that:

$$D^2_{(\tau,w),(r,z)} u(t,x) = \sum_{n\ge 2} n(n-1) I_{n-2}(f_n(\cdot,\tau,w,r,z,t,x)),$$

for any $(\tau,w) \in (0,t)\times\mathbb{R}^d$ and $(r,z) \in (0,t)\times\mathbb{R}^d$. Hence, similarly to (76), one can show that:

$$C_T^{(2)} := \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\|D^2 u(t,x)\|^2_{\mathcal{HP}\otimes\mathcal{HP}} < \infty, \quad \text{for all } T>0. \tag{77}$$

Lemma 6.5 *For any $t>0, x\in\mathbb{R}^d$, the process $U^{(t,x)}$ defined by*

$$U^{(t,x)} = \{U^{(t,x)}(s,y) := G(t-s,x-y)Du(s,y); s\ge 0, y\in\mathbb{R}^d\} \tag{78}$$

belongs to Dom δ^*.

Proof: By Proposition 6.2, it suffices to show that $U^{(t,x)} \in \mathbb{D}^{1,2}(\mathcal{HP}\otimes\mathcal{HP})$, i.e. $\|U^{(t,x)}\|_{\mathbb{D}^{1,2}(\mathcal{HP}\otimes\mathcal{HP})} < \infty$. Note that

$$\|U^{(t,x)}\|^2_{\mathbb{D}^{1,2}(\mathcal{HP}\otimes\mathcal{HP})} = E\|U^{(t,x)}\|^2_{\mathcal{HP}\otimes\mathcal{HP}} + E\|DU^{(t,x)}\|^2_{\mathcal{HP}\otimes\mathcal{HP}\otimes\mathcal{HP}}.$$

By the Cauchy-Schwartz inequality, (76) and (77),

$$\begin{aligned} E\|U\|^2_{\mathcal{HP}^{\otimes 2}} &= \alpha_H \int_0^t \int_0^t G(t-s,x-y)G(t-s',x-y)E\langle Du(s,y), Du(s',y')\rangle_{\mathcal{HP}} \\ & \quad |s-s'|^{2H-2} f(y-y')dydy'dsds' \\ &\leq C_t^{(1)} \|G(t-\cdot,x-*)\|^2_{\mathcal{HP}} < \infty, \\ E\|DU\|^2_{\mathcal{HP}^{\otimes 3}} &= \alpha_H \int_0^t \int_0^t G(t-s,x-y)G(t-s',x-y)E\langle D^2u(s,y), D^2u(s',y')\rangle_{\mathcal{HP}^{\otimes 2}} \\ & \quad |s-s'|^{2H-2} f(y-y')dydy'dsds' \\ &\leq C_t^{(2)} \|G(t-\cdot,x-*)\|^2_{\mathcal{HP}} < \infty. \end{aligned}$$

□

Using the same argument as above, one can show that the process $U_n^{(t,x)}$ defined by $U_n^{(t,x)}(s,y) := G(t-s,x-y)Du_n(s,y)$, belongs to Dom δ^*.

The next result shows that the sequence $\{Du_n(t,x)\}_{n\geq 0}$ satisfies a recurrence relation.

Proposition 6.6 *For any $t>0, x\in\mathbb{R}^d$ and $n\geq 1$,*

$$Du_n(t,x) = G(t-\cdot,x-*)u_{n-1} + \int_0^t \int_{\mathbb{R}^d} U_{n-1}^{(t,x)}(s,y)W(\delta^* s,\delta^* y)$$

in $L^2(\Omega;\mathcal{HP})$.

Proof: *Step 1.* By induction on n, we show that for any $r\in(0,t)$, $z\in\mathbb{R}^d$, $D_{r,z}u_n(t,x) = \sum_{k=1}^n A_k$, where $A_k = \sum_{i=1}^k A_k^{(i)}$,

$$\begin{aligned} A_k^{(1)} &= \int_{r<s_1} G(t-s_{k-1},x-y_{k-1})\ldots G(s_1-r,y_1-z)W(ds_1,dy_1)\ldots W(ds_{k-1},dy_{k-1}) \\ A_k^{(i)} &= \int_{s_{i-1}<r<s_i} G(t-s_{k-1},x-y_{k-1})\ldots G(s_i-r,y_i-z)G(r-s_{i-1},z-y_{i-1}) \\ & \quad \ldots G(s_2-s_1,y_2-y_1)W(ds_1,dy_1)\ldots W(ds_{k-1},dy_{k-1}), \quad i=2,\ldots,k-1 \\ A_k^{(k)} &= \int_{s_{k-1}<r} G(t-r,x-z)G(r-s_{k-1},z-y_{k-1})\ldots G(s_2-s_1,y_2-y_1) \\ & \quad W(ds_1,dy_1)\ldots W(ds_{k-1},dy_{k-1}), \end{aligned}$$

and the integrals above are taken over the set $\{0<s_1<\ldots<s_{k-1}<t\}\times\mathbb{R}^{nd}$.

By definition, $u_n(t,x) = u_{n-1}(t,x) + I_n(\tilde{f}_n(\cdot,t,x))$. Using Proposition 2.4 and the induction hypothesis, we obtain that:

$$D_{r,z}u_n(t,x) = \sum_{k=1}^{n-1} A_k + nI_{n-1}(\tilde{f}_n(\cdot,r,z,t,x)).$$

Note that $nI_{n-1}(\tilde{f}_n(\cdot,r,z,t,x)) = A_n$, the n integrals $A_1^{(n)},\ldots,A_n^{(n)}$ corresponding to the n possible locations of r, compared with the variables $s_1<\ldots<s_{n-1}$.

Step 2. We prove that for every $r \in (0,t)$, $z \in \mathbb{R}^d$

$$D_{r,z}u_n(t,x) = G(t-r,x-z)u_{n-1}(r,z)+\int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)D_{r,z}u_{n-1}(s,y)W(\delta s,\delta y).$$

We use the expression of $D_{r,z}u_n(t,x)$ obtained in Step 1. Note that the terms $A_1^{(1)}, A_2^{(2)}, \ldots, A_n^{(n)}$ have the common factor $G(t-r,x-z)$. A quick calculation shows that the sum of these terms is $G(t-r,x-z)u_{n-1}(r,z)$.

For the remaining terms, we change the names of the variables of integration, so that $G(t-s,x-y)$ becomes a common factor. More precisely, we call (s,y) the variable (s_{k-1},y_{k-1}) in $A_k^{(i)}$, for any $i=1,\ldots,k-1$ and $k=2,\ldots,n$. The sum of these terms turns out to be

$$\int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)D_{r,z}u_{n-1}(s,y)W(ds,dy),$$

using the expression of $D_{r,z}u_{n-1}(s,y)$ obtained in Step 1. Finally, by (26), we can replace the integral $W(ds,dy)$ by an integral $W(\delta s,\delta y)$.

Step 3. We show that the process $\delta^*(U_{n-1}^{(t,x)})$ coincides (in $L^2(\Omega,\mathcal{HP})$) with the process $V_{n-1}^{(t,x)}$, defined by

$$\begin{aligned} V_{n-1}^{(t,x)}(r,z) &:= \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)D_{r,z}u_{n-1}(s,y)W(\delta s,\delta y) \\ &= \delta(G(t-\cdot,x-*)D_{r,z}u_{n-1}). \end{aligned}$$

By the duality relation (74), it suffices to prove that

$$E\langle F, V_{n-1}^{(t,x)}\rangle_{\mathcal{HP}} = E\langle DF, U_{n-1}^{(t,x)}\rangle_{\mathcal{HP}\otimes\mathcal{HP}}, \quad \forall F \in \mathbb{D}^{1,2}(\mathcal{HP}). \tag{79}$$

Without loss of generality, we may assume that F is smooth, i.e. $F=F_0\varphi$ with $F_0 \in \mathcal{S}$ and $\varphi \in \mathcal{HP}$. Then $DF = DF_0 \otimes \varphi$ and

$$\begin{aligned} E\langle DF, U_{n-1}^{(t,x)}\rangle_{\mathcal{HP}\otimes\mathcal{HP}} &= \alpha_H \int_{(\mathbb{R}_+\times\mathbb{R}^d)^2} E\langle DF_0, G(t-*,x-\cdot)D_{r,z}u_n\rangle_{\mathcal{HP}} \\ &\quad \varphi(r',z')|r-r'|^{2H-2}f(z-z')dzdz'drdr' \\ &= \alpha_H \int_{(\mathbb{R}_+\times\mathbb{R}^d)^2} E(F_0V_{n-1}^{(t,x)}(r,z))\varphi(r',z')|r-r'|^{2H-2}f(z-z')dzdz'drdr' \\ &= E\langle F, V^{(t,x)}\rangle_{\mathcal{HP}}. \end{aligned}$$

Note that for the second-last equality above, we used the duality relation (25) (for the operator δ), whereas for the first and last equality we used Fubini's theorem. This shows (79), and concludes the proof. □

The following result gives the precise meaning of relation (73).

Theorem 6.7 *Let f be the Riesz kernel of order $\alpha > d-2$ and u be a solution of (1). For any $t > 0, x \in \mathbb{R}^d$, let $U^{(t,x)}$ be defined by (78). Then,*

$$Du(t,x) = G(t-\cdot, x-*)u + \int_0^t \int_{\mathbb{R}^d} U^{(t,x)}(s,y) W(\delta^* s, \delta^* y)$$

in $L^2(\Omega; \mathcal{HP})$.

Proof: By the duality relation (74), it suffices to prove that,

$$E\langle Du(t,x) - G(t-\cdot, x-*)u, F\rangle_{\mathcal{HP}} = E\langle DF, U^{(t,x)}\rangle_{\mathcal{HP}\otimes\mathcal{HP}}, \qquad (80)$$

for any $F \in \mathbb{D}^{1,2}(\mathcal{HP})$. By Proposition 6.6, for any $F \in \mathbb{D}^{1,2}(\mathcal{HP})$,

$$E\langle Du_n(t,x) - G(t-\cdot, x-*)u_{n-1}, F\rangle_{\mathcal{HP}} = E\langle DF, U_{n-1}^{(t,x)}\rangle_{\mathcal{HP}\otimes\mathcal{HP}}. \qquad (81)$$

Relation (80) is obtained by taking $n \to \infty$ in (81). We justify this below.

On the right-hand side of (81), we use the duality relation (74), the Cauchy-Schwartz inequality, and (75):

$$\begin{aligned} E\langle DF, U_{n-1}^{(t,x)} - U^{(t,x)}\rangle_{\mathcal{HP}\otimes\mathcal{HP}} &= E\langle \delta^*(U_{n-1}^{(t,x)} - U^{(t,x)}), F\rangle_{\mathcal{HP}} \\ &\leq (E\|\delta^*(U_{n-1}^{(t,x)} - U^{(t,x)})\|_{\mathcal{HP}}^2)^{1/2}(E\|F\|_{\mathcal{HP}}^2)^{1/2} \\ &\leq \|U_{n-1}^{(t,x)} - U^{(t,x)}\|_{\mathbb{D}^{1,2}(\mathcal{HP}\otimes\mathcal{HP})}^2 (E\|F\|_{\mathcal{HP}}^2)^{1/2}. \end{aligned}$$

To show that $\|U_{n-1}^{(t,x)} - U^{(t,x)}\|_{\mathbb{D}^{1,2}(\mathcal{HP}\otimes\mathcal{HP})}^2 \to 0$ as $n \to \infty$, we use the same argument as in Lemma 6.5.

For the first term on the left-hand side of (81), by the Cauchy-Schwartz inequality and Lemma 6.3,

$$E\langle Du_n(t,x) - Du(t,x), F\rangle_{\mathcal{HP}} \leq (E\|Du_n(t,x) - Du(t,x)\|_{\mathcal{HP}}^2)^{1/2}(E\|F\|_{\mathcal{HP}}^2)^{1/2} \to 0,$$

as $n \to \infty$. For the second term on the left-hand side of (81), we assume that F is smooth, i.e. $F = F_0\varphi$ with $F_0 \in \mathcal{S}$ and $\varphi \in \mathcal{HP}$. By the Cauchy-Schwartz inequality and the Dominated Convergence theorem,

$$\begin{aligned} &E\langle G(t-\cdot, x-*)(u_{n-1} - u), F\rangle_{\mathcal{HP}} = \alpha_H \int_{(\mathbb{R}_+\times\mathbb{R}^d)^2} E[F_0(u_{n-1}(s,y) - u(s,y))] \\ &\quad G(t-s, x-y)\varphi(s',y')|s-s'|^{2H-2} f(y-y') dy dy ds ds' \\ &\quad \leq \alpha_H E(F_0^2)^{1/2} \int_{(\mathbb{R}_+\times\mathbb{R}^d)^2} (E|u_{n-1}(s,y) - u(s,y))|^2)^{1/2} G(t-s, x-y)\varphi(s',y') \\ &\qquad |s-s'|^{2H-2} f(y-y') dy dy ds ds', \quad \text{as } n \to \infty. \end{aligned}$$

This concludes the proof of (80). □

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