

A CHARACTERIZATION OF REPRESENTABLE INTERVALS

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ABSTRACT. In this note we provide a characterization, in terms of additional algebraic structure, of those strict intervals (certain cocategory objects) in a symmetric monoidal closed category \mathcal{E} that are representable in the sense of inducing on \mathcal{E} the structure of a finitely bicomplete 2-category. Several examples and connections with the homotopy theory of 2-categories are also discussed.

INTRODUCTION

Approached from an abstract perspective, a fundamental feature of the category of spaces which enables the development of homotopy theory is the presence of an object I with which the notions of *path* and *deformation* thereof are defined. When dealing with topological spaces, I is most naturally taken to be the closed unit interval $[0, 1]$, but there are other instances where the homotopy theory of a category is determined in an appropriate way by an interval object I . For example, the simplicial interval $I = \Delta[1]$ determines — in a sense clarified by the recent work of Cisinski [2] — the classical model structure on the category of simplicial sets and the infinite dimensional sphere J is correspondingly related to the quasi-category model structure studied by Joyal [5]. Similarly, the category $\mathbf{2}$ gives rise to the natural model structure — in which the weak equivalences are categorical equivalences, the fibrations are isofibrations and the cofibrations are functors injective on objects — on the category \mathbf{Cat} of small categories [7]. This model structure is, moreover, well-behaved with respect to the usual 2-category structure on \mathbf{Cat} (it is a *model \mathbf{Cat} -category* in the sense of [10]). One special property of the category $\mathbf{2}$, which is in part responsible for these facts, is that it is a cocategory in \mathbf{Cat} .

In this paper we study, with a view towards homotopy theory, one (abstract) notion of *strict interval object* — namely, a cocategory with object of coobjects the tensor unit in a symmetric monoidal closed category — of which $\mathbf{2}$ is a leading example. Every such interval I gives rise to a 2-category structure on its ambient category and it is our principal goal to investigate certain properties of the induced 2-category structure in terms of the interval itself. In particular, our main theorem (Theorem 2.10) gives a characterization of those strict intervals I for which the induced 2-category structure is *finitely bicomplete* in the 2-categorical sense. A strict interval I with this property is said to be *representable* and the content of Theorem 2.10 is that a strict interval I is representable whenever it is a distributive lattice with top and bottom elements which are, in a suitable sense, its generators.

We note here that neither the closed unit interval in the category of spaces nor the simplicial interval in the category of simplicial sets are examples of strict interval

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objects in the sense of the present paper. For example, although the closed unit interval can be equipped with suitable structure maps, it fails to satisfy the defining equations for cocategories, which are only satisfied up to homotopy. Instead it is expected that these are examples of “weak ω -intervals” in the sense that they are weak co- ω -categories. As such, the present paper may be regarded as, in part, laying the groundwork for later investigation of these intervals and the corresponding weak higher-dimensional completeness properties of the model structures to which they give rise.

The plan of this paper is as follows. Section 1 is concerned with introducing the basic definitions and examples. In particular, we give the leading examples of strict intervals and explain the induced 2-category structure. In Section 2 we recall the 2-categorical notion of finite bicompleteness and prove our main results including Theorem 2.10. Lack [10] has shown that every finitely bicomplete 2-category can be equipped with a model structure in which the weak equivalences are categorical equivalences and the fibrations are isofibrations and in Section 3 we briefly explain when, given the presence of a strict interval I which is representable, this model structure can be lifted, using a theorem due to Berger and Moerdijk [1], to the category of reduced operads.

Notation and conventions. Throughout we assume, unless otherwise stated, that the ambient category \mathcal{E} is a (finitely) bicomplete symmetric monoidal closed category (for further details regarding which we refer the reader to [12]). We employ common notation $(A \otimes B)$ and $[B, A]$ for the tensor product and internal hom of objects A and B , respectively. We denote the tensor unit by U (instead of the more common I) and the natural isomorphisms associated to the symmetric monoidal closed structure of \mathcal{E} are denoted by $\lambda : U \otimes A \rightarrow A$, $\rho : A \otimes U \rightarrow A$, $\alpha : A \otimes (B \otimes C) \rightarrow (A \otimes B) \otimes C$, and $\tau : A \otimes B \rightarrow B \otimes A$. Associated to the closed structure we denote the isomorphism $[U, A] \rightarrow A$ by ∂ and write $\varepsilon : [U, A] \otimes U \rightarrow A$ for the evaluation map.

We will frequently deal with pushouts and, if the following is a pushout diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & B \\ g \downarrow & & \downarrow g' \\ A & \xrightarrow{f'} & P \end{array}$$

then, when $h : A \rightarrow X$ and $k : B \rightarrow X$ are maps for which $h \circ g = k \circ f$, we denote the induced map $P \rightarrow X$ by $[h, k]$. Likewise, we employ the notation $\langle h, k \rangle$ for canonical maps into pullbacks.

Finally, we refer the reader to [9] for further details regarding 2-categories.

1. INTERVALS

The definition of *cocategory objects* in \mathcal{E} is exactly dual to that of a category objects in \mathcal{E} . In order to fix notation and provide motivation we will rehearse the definition in full. For us, the principal impetus for the definition of cocategories is that a cocategory in \mathcal{E} provides (more than) sufficient data to define a reasonable notion of homotopy in \mathcal{E} and this induced notion of homotopy is directly related to a 2-category structure on \mathcal{E} . In thinking about cocategory objects it is often

instructive to view them as analogous to the unit interval in the category of topological spaces. However, the unit interval is *not* a cocategory object in the category of topological spaces and continuous functions.

1.1. The definition. A **cocategory** \mathbb{C} in a category \mathcal{E} consists objects C_0 (**object of coobjects**), C_1 (**object of coarrows**) and C_2 (**object of cocomposable coarrows**) together with arrows

$$\begin{array}{ccccc}
 & \perp & & \downarrow & \\
 C_0 & \begin{array}{c} \curvearrowright \\ \longleftarrow \\ \longrightarrow \\ \curvearrowleft \end{array} & C_1 & \begin{array}{c} \curvearrowright \\ \longleftarrow \\ \longrightarrow \\ \curvearrowleft \end{array} & C_2 \\
 & \top & & \uparrow &
 \end{array}$$

satisfying the following list of requirements.

- The following square is a pushout:

$$\begin{array}{ccc}
 C_0 & \xrightarrow{\perp} & C_1 \\
 \top \downarrow & & \downarrow \uparrow \\
 C_1 & \longrightarrow & C_2.
 \end{array}$$

- The following diagram commutes:

$$\begin{array}{ccc}
 C_0 & \xrightarrow{\perp} & C_1 & \xleftarrow{\top} & C_0 \\
 & \searrow & \downarrow i & \swarrow & \\
 & & C_0. & &
 \end{array}$$

- The following diagrams commute:

$$\begin{array}{ccc}
 C_0 & \xrightarrow{\perp} & C_1 \\
 \perp \downarrow & & \downarrow * \\
 C_1 & \longrightarrow & C_2,
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 C_0 & \xrightarrow{\top} & C_1 \\
 \top \downarrow & & \downarrow * \\
 C_1 & \longrightarrow & C_2.
 \end{array}$$

- The following **co-unit laws** hold:

$$\begin{array}{ccc}
 & C_1 & \\
 & \parallel & \\
 C_1 & \xleftarrow{[\perp \circ i, C_1]} & C_2 & \xrightarrow{[C_1, \top \circ i]} & C_1. \\
 & \parallel & \\
 & C_1 &
 \end{array}$$

- Finally, let the object C_3 (**the object of cocomposable triples**) be defined as the the following pushout:

$$\begin{array}{ccc}
 C_1 & \xrightarrow{\uparrow} & C_2 \\
 \downarrow & & \downarrow \Downarrow \\
 C_2 & \longrightarrow & C_3, \\
 & \uparrow &
 \end{array}$$

The **coassociative law** then states that the following diagram commutes:

$$\begin{array}{ccc} C_1 & \xrightarrow{\star} & C_2 \\ \star \downarrow & & \downarrow [\downarrow \circ \downarrow, \uparrow \circ \star] \\ C_2 & \xrightarrow{[\downarrow \circ \star, \uparrow \circ \uparrow]} & C_3 \end{array}$$

Remark. The map \perp is the dual of the domain map, \top is the dual of the codomain map, and \downarrow and \uparrow are dual to the first and second projections, respectively. This notation, and the other notation occurring in the definition, is justified by the interpretation of these arrows in the examples considered below. We refer to i and \star as the *coidentity* and *cocomposition* maps, respectively.

If $\mathbb{C} = (C_0, C_1, C_2)$ is a cocategory object and A is any object of \mathcal{E} , then $A \otimes \mathbb{C} = (A \otimes C_0, A \otimes C_1, A \otimes C_2)$ is also a cocategory in \mathcal{E} . Moreover, if \mathbb{C} is a cocategory in \mathcal{E} and A is any object, then one obtains a category $[\mathbb{C}, A]$ in \mathcal{E} by taking internal hom.

1.2. Cocategories with additional structure. We will be concerned with cocategories which possess additional structure.

Definition 1.1. A cocategory \mathbb{C} in \mathcal{E} is a **cogroupoid** if there exists a **symmetry** or **coinverse** map $\sigma : C_1 \rightarrow C_1$ such that the following diagrams commute:

$$\begin{array}{ccc} C_0 & \xrightarrow{\perp} & C_1 & \xleftarrow{\top} & C_0 \\ & \searrow & \downarrow \sigma & \swarrow & \\ & & C_1 & & \end{array}$$

$$\begin{array}{ccc} C_1 & \xrightarrow{\star} & C_2 \\ i \downarrow & & \downarrow [\sigma, C_1] \\ C_0 & \xrightarrow{\top} & C_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} C_1 & \xrightarrow{\star} & C_2 \\ i \downarrow & & \downarrow [C_1, \sigma] \\ C_0 & \xrightarrow{\perp} & C_1 \end{array}$$

When \mathbb{C} is a cogroupoid in \mathcal{E} and A is an object of \mathcal{E} , $[\mathbb{C}, A]$ is a groupoid in \mathcal{E} .

Definition 1.2. A cocategory object \mathbb{C} in a category \mathcal{E} is said to be a **strict interval** if the object C_0 of coobjects is the tensor unit U . When \mathbb{C} is a strict interval we often write I instead of C_1 and I_2 instead of C_2 . When a strict interval I is a cogroupoid it is said to be **invertible**.

Remark. Because we will be dealing throughout exclusively with strict intervals the adjective “strict” will henceforth be omitted.

Cocategories in \mathcal{E} together with their obvious morphisms form a category $\mathbf{Cocat}(\mathcal{E})$. There is also a category $\mathbf{Int}(\mathcal{E})$ of strict intervals in \mathcal{E} .

Example 1.3. The following are examples of cocategories and intervals:

- (1) Every object A of a category \mathcal{E} determines a cocategory object given by setting $A_i := A$ for $i = 0, 1, 2$ and defining all of the structure maps to be the identity 1_A . This is said to be the **discrete cocategory on A** . The discrete cocategory on the tensor unit U is the terminal object in $\mathbf{Int}(\mathcal{E})$.

- (2) There is an (invertible) interval in \mathcal{E} obtained by taking the object of coarrows to be $U + U$ with \perp and \top the coproduct injections. This is the initial object in $\mathbf{Int}(\mathcal{E})$. Indeed, a topos \mathcal{E} is Boolean if and only if its subobject classifier Ω is an invertible interval with \perp and \top the usual “truth-values”.
- (3) In \mathbf{Cat} the category $\mathbf{2}$ which is the free category on the graph consisting of two vertices and one edge between them is a cocategory object. Similarly, the free groupoid \mathbf{I} on $\mathbf{2}$ is an invertible interval in \mathbf{Cat} and in \mathbf{Gpd} with the following structure:

$$\begin{array}{ccc} & d & \\ \perp & \curvearrowright & \top \\ & u & \end{array}$$

such that u and d are inverse and where $\perp, \top : \mathbf{1} \rightrightarrows \mathbf{I}$ are the obvious functors. \mathbf{I}_2 is then the result of gluing \mathbf{I} to itself along the top and bottom:

$$\perp \quad \bullet \quad \top$$

Cocomposition $\star : \mathbf{I} \rightarrow \mathbf{I}_2$ is the functor given by $\star(\perp) := \perp$ and $\star(\top) := \top$, and the initial and final segment functors are defined in the evident way. Finally, $\sigma : \mathbf{I} \rightarrow \mathbf{I}$ is defined by $\sigma(\perp) := \top$ and $\sigma(\top) := \perp$. We note that these examples also generalize to the case of internal categories in a suitably complete and cocomplete category \mathcal{E} .

- (4) Assume R is a commutative ring (with 1) and let $\mathbf{Ch}_{0\leq}(R)$ be the category of (non-negatively graded) chain complexes of R -modules, then there exists an (invertible) interval \mathbb{I} in $\mathbf{Ch}_{0\leq}(R)$ which we now describe. \mathbb{I}^0 is the chain complex which consists of R in degree 0 and is 0 in all other degrees. \mathbb{I}^1 is given by

$$\begin{array}{ccccccc} \dots & \xrightarrow{d} & 0 & \xrightarrow{d} & R & \longrightarrow & R \oplus R \\ & & & & & & x \longmapsto (x, -x), \end{array}$$

where x is an arbitrary integer. \mathbb{I}^2 consists of

$$\begin{array}{ccccccc} \dots & \xrightarrow{d} & 0 & \xrightarrow{d} & R \oplus R & \xrightarrow{d} & R \oplus R \oplus R \\ & & & & (x, y) \longmapsto & & (x, y - x, -y), \end{array}$$

for x and y in R . \downarrow and \uparrow are the left and right inclusions (in both non-trivial degrees), respectively. Similarly, \perp and \top are the left and right inclusions, respectively. $i : \mathbb{I}^1 \rightarrow \mathbb{I}^0$ is given by addition in degree 0 and the zero map in all other degrees. Finally, cocomposition $\star : \mathbb{I}^1 \rightarrow \mathbb{I}^2$ is given defined as follows:

$$\begin{aligned} \star_1(x) &:= (x, x) \\ \star_0(x, y) &:= (x, 0, y), \end{aligned}$$

for x and y in R . The symmetry $\sigma : \mathbb{I} \rightarrow \mathbb{I}$ is given by taking additive inverse in degree 1 and by sending (x, y) to (y, x) in degree 0, for x, y in R .

As we have already noted, the topological unit interval $I = [0, 1]$ in \mathbf{Top} fails to satisfy the co-associativity and co-unit laws on the nose and is therefore not an interval in the present sense.

Remark. The question of what kinds of cocategories can exist in a topos has been addressed by Lumsdaine [11] who shows that in a coherent category the only cocategories are “coequivalence relations”. I.e. any such cocategory must have \perp and \top jointly epimorphic.

Remark. The composites

$$[I, A] \xrightarrow{t^*} [U, A] \xrightarrow{\partial} A,$$

with $t = \perp, \top$, are denoted by ∂_0 and ∂_1 , respectively.

Remark (Comonoid structure). Every interval I in \mathcal{E} has an associated comonoid structure (cf. Appendix A). For the comonoid comultiplication $\Delta : I \rightarrow I \otimes I$ we first form, using the fact that I_2 is the pushout of \perp along \top , the canonical map $[(\perp \otimes I) \circ \lambda^{-1}, (I \otimes \top) \circ \rho^{-1}] : I_2 \rightarrow I \otimes I$. We then define

$$\Delta := [(\perp \otimes I) \circ \lambda^{-1}, (I \otimes \top) \circ \rho^{-1}] \circ \star$$

The comonoid counit $\epsilon : I \rightarrow U$ is the coidentity map $i : I \rightarrow U$. With these definitions the comonoid axioms follow from the counit and coassociativity laws for cocategories. In the case where the symmetric monoidal closed structure on \mathcal{E} is cartesian, Δ is precisely the usual diagonal map.

1.3. Homotopy. The first way in which we make use of the existence of an interval object in \mathcal{E} is to define homotopy.

Definition 1.4. Let I be an interval object in \mathcal{E} . A **homotopy (with respect to I)** $\eta : f \Rightarrow g$ **from f to g** , for $f, g \in \mathcal{E}(A, B)$, is a map $\eta : A \otimes I \rightarrow B$ such that the following diagram commutes:

$$\begin{array}{ccccc} A \otimes U & \xrightarrow{A \otimes \perp} & A \otimes I & \xleftarrow{A \otimes \top} & A \otimes U \\ \rho \downarrow & & \downarrow \eta & & \downarrow \rho \\ A & \xrightarrow{f} & B & \xleftarrow{g} & A \end{array}$$

Example 1.5. The intervals from Example 1.3 give rise to the following notions of homotopy:

- (1) The discrete interval on U generates the finest notion of homotopy (in terms of the number of homotopy classes of maps). I.e., there exists a homotopy between f and g with respect to this cocategory if and only if f and g are identical.
- (2) The initial object of $\mathbf{Int}(\mathcal{E})$ generates the coarsest homotopy relation: all maps are homotopic. Indeed, given maps f and g there exists, with respect to this cocategory, a unique homotopy $f \Rightarrow g$.
- (3) In \mathbf{Cat} , homotopies $f \Rightarrow g$ with respect to $\mathbf{2}$ correspond to natural transformations $f \Rightarrow g$ and, similarly, homotopies with respect to \mathbf{I} correspond to natural isomorphisms.
- (4) In $\mathbf{Ch}_{0 \leq}(R)$, \mathbb{I} induces the usual notion of chain homotopy.

1.4. Induced 2-category structure. When it possesses an interval I , \mathcal{E} can be equipped with the structure of a 2-category by taking 2-cells to be homotopies with respect to I . I.e., we define

$$\mathcal{E}(A, B)_1 := \mathcal{E}(A \otimes I, B),$$

which endows $\mathcal{E}(A, B)$ with the structure of a category since $[I, B]$ is an internal category in \mathcal{E} . Explicitly, given φ in $\mathcal{E}(A, B)_1$, the domain of φ is defined to be the arrow $\varphi \circ (A \otimes \perp) \circ \rho^{-1} : A \rightarrow B$ and the codomain is $\varphi \circ (A \otimes \top) \circ \rho^{-1} : A \rightarrow B$. Given arrows $\eta : f \Rightarrow g$ and $\gamma : g \Rightarrow h$ in $\mathcal{E}(A, B)$, the vertical composite $f \Rightarrow h$ is defined as follows. Since \mathcal{E} is monoidal closed the following square is a pushout:

$$\begin{array}{ccc} A \otimes U & \xrightarrow{A \otimes \perp} & A \otimes I \\ A \otimes \top \downarrow & & \downarrow A \otimes \top \\ A \otimes I & \xrightarrow{A \otimes \downarrow} & A \otimes I_2 \end{array}$$

and there exists an induced map $[\eta, \gamma] : A \otimes I_2 \rightarrow B$. Recalling the third clause from the definition of cocategory object, it is easily verified that $[\eta, \gamma] \circ (A \otimes \star)$ is the required vertical composite $(\gamma \cdot \eta) : f \Rightarrow h$. The composition functor

$$\mathcal{E}(A, B) \times \mathcal{E}(B, C) \longrightarrow \mathcal{E}(A, C)$$

is then obtained by defining the horizontal composite $(\gamma * \eta)$ of a pair of 2-cells

$$\begin{array}{ccccc} & f & & h & \\ & \curvearrowright & & \curvearrowright & \\ A & \eta \Downarrow & B & \gamma \Downarrow & C \\ & \curvearrowleft & & \curvearrowleft & \\ & g & & k & \end{array}$$

to be the composite

$$A \otimes I \xrightarrow{A \otimes \Delta} A \otimes (I \otimes I) \xrightarrow{\alpha} (A \otimes I) \otimes I \xrightarrow{\eta \otimes I} B \otimes I \xrightarrow{\gamma} C.$$

The proof that the structure defined above constitutes a 2-category is routine and is therefore left to the reader. Thus we have the following proposition (a discussion of the generalization of this result, in the cartesian setting, to co- n -categories can be found in [15]):

Proposition 1.6. *Suppose I is an interval object in \mathcal{E} . Then \mathcal{E} is a 2-category with the same objects and arrows, and with 2-cells the homotopies.*

Proposition 1.6 has the following evident corollary:

Corollary 1.7. *An interval I in \mathcal{E} is invertible if and only if, for all objects A and B of \mathcal{E} , the category $\mathcal{E}(A, B)$ is a groupoid.*

Remark. Assuming \mathcal{E} is a 2-category which is finitely cocomplete in the 2-categorical sense (as discussed, e.g., in Section 2 below) there exists for every object A of \mathcal{E} a cocategory $(A \cdot \mathbf{2})$ obtained by taking the tensor product of A with the category $\mathbf{2}$ (this fact can be found in its dual form in [14]). When \mathcal{E} is simultaneously equipped with a **Cat**-enriched symmetric monoidal closed structure it follows that the 2-category structure on \mathcal{E} is induced, in the sense of Proposition 1.6, by the interval $(U \cdot \mathbf{2})$. Note that the assumption of **Cat**-enrichedness is necessary.

2. REPRESENTABILITY

We now turn to the proof of our main Theorem 2.10 which gives necessary and sufficient conditions under which the 2-category structure on \mathcal{E} induced by an interval I is *finitely bicomplete* in the 2-categorical sense [14, 3]. We will also

see that, when \mathcal{E} is finitely bicomplete, I can be shown to possess additional useful structure. For example, we will see that such an interval is necessarily both a lattice and a *Hopf object* in the sense of Berger and Moerdijk [1].

First we recall the 2-categorical notion of finite (co)completeness due to Gray [3] and Street [14]. Namely, a 2-category \mathcal{K} **finitely complete** whenever it has all finite conical limits in the 2-categorical sense and, for each object A , the cotensor $(\mathbf{2} \pitchfork A)$ with the category $\mathbf{2}$ exists. Similarly, \mathcal{K} is **finitely cocomplete** if and only if it possesses all finite conical colimits and tensors $(A \cdot \mathbf{2})$ with $\mathbf{2}$ exist. It is straightforward to verify that, when \mathcal{E} possesses an interval I , the resulting 2-category possesses whatever conical limits and colimits \mathcal{E} has in the ordinary 1-dimensional sense:

Lemma 2.1. *Assume that \mathcal{E} possesses an interval I and regard \mathcal{E} as a 2-category with respect to the 2-category structure induced by I . Then the conical (co)limit of a functor $F : \mathcal{C} \rightarrow \mathcal{E}$ from a (small) category \mathcal{C} exists if and only if the ordinarily 1-dimensional (co)limit of F exists.*

2.1. Lattice structure of representable intervals. In order to show that the 2-category structure on \mathcal{E} induced by an interval I is finitely bicomplete it suffices, by Lemma 2.1, to prove that tensor and cotensor products with the category $\mathbf{2}$ exist. Indeed, if $(\mathbf{2} \pitchfork A)$ exists, it is necessarily isomorphic to the internal hom $[I, A]$ since the 2-natural isomorphism

$$(1) \quad \mathcal{E}(B, \mathbf{2} \pitchfork A) \cong \mathcal{E}(B, A)^{\mathbf{2}}$$

of categories restricts to a natural isomorphism of their respective collections of objects:

$$\mathcal{E}(B, \mathbf{2} \pitchfork A) \cong \mathcal{E}(B \otimes I, A).$$

Similar reasoning implies that when the tensor product $(A \cdot \mathbf{2})$ exists it is necessarily $(A \otimes I)$. Note though that it does not *a priori* follow that $[I, A]$ is $(\mathbf{2} \pitchfork A)$ in the sense of possessing the full 2-categorical universal property of $(\mathbf{2} \pitchfork A)$, and similarly for $(A \otimes I)$ and $(A \cdot \mathbf{2})$. This remark should be compared with the familiar fact that a 2-category with all 1-dimensional conical limits need not possess all 2-dimensional conical limits (cf. [8]).

As the reader may easily verify, if I is an interval in \mathcal{E} , then there exist isomorphisms of categories

$$\mathcal{E}(B \otimes I, A) \cong \mathcal{E}(B, [I, A])$$

natural in A and B . Thus, it follows that \mathcal{E} possesses tensors with $\mathbf{2}$ if and only if it possesses cotensors with $\mathbf{2}$.

Definition 2.2. An interval I in \mathcal{E} is **representable** if cotensors with $\mathbf{2}$ exist with respect to the 2-category structure on \mathcal{E} induced by I .

Thus, an interval I is representable if and only if \mathcal{E} is a finitely bicomplete 2-category with respect to the induced 2-category structure of Section 1.4. The following useful lemma is a straightforward consequence of the definition:

Lemma 2.3. *If I is representable, then, for all objects A and B of \mathcal{E} , the following diagram in \mathbf{Cat} commutes:*

$$\begin{array}{ccc} \mathcal{E}(B, [I, A]) & \xrightarrow{\cong} & \mathcal{E}(B, A)^2 \\ & \searrow \mathcal{E}(B, \partial_i) \quad \swarrow \partial_i & \\ & & \mathcal{E}(B, A) \end{array}$$

when $i = 0, 1$.

Remark. Given objects A and B of \mathcal{E} there exists, regardless of whether I is representable, a functor $\Phi : \mathcal{E}(B, [I, A]) \rightarrow \mathcal{E}(B, A)^2$ which acts by transpose under the tensor-hom adjunction. I.e., given an object f of $\mathcal{E}(B, [I, A])$, the arrow $\Phi(f)$ in $\mathcal{E}(B, A)$ is defined to be the transpose $\tilde{f} : B \otimes I \rightarrow A$ of f . Similarly, an arrow $\varphi : f \Rightarrow g$ in $\mathcal{E}(B, [I, A])$ is obtained by projecting the transpose $\tilde{\varphi}$ to the following commutative square:

$$(2) \quad \begin{array}{ccc} \partial_0 f & \xrightarrow{\partial_0 \circ \varphi} & \partial_0 g \\ \tilde{f} \downarrow & & \downarrow \tilde{g} \\ \partial_1 f & \xrightarrow{\partial_1 \circ \varphi} & \partial_1 g \end{array}$$

Moreover, by Lemma 2.3, if I is representable, then Φ is necessarily the natural isomorphism witnessing this fact.

When A and B are objects of \mathcal{E} , we say that a map $\varphi : B \otimes (I \otimes I) \rightarrow A$ is a **square in A parameterized by B** . The **boundary** of such a square, written $\partial(\varphi)$, is the tuple consisting of the composites

$$\begin{array}{l} B \otimes I \xrightarrow{B \otimes \lambda^{-1}} B \otimes (U \otimes I) \xrightarrow{B \otimes (t \otimes I)} B \otimes (I \otimes I) \xrightarrow{\varphi} A, \text{ and} \\ B \otimes I \xrightarrow{B \otimes \rho^{-1}} B \otimes (I \otimes U) \xrightarrow{B \otimes (I \otimes t)} B \otimes (I \otimes I) \xrightarrow{\varphi} A \end{array}$$

for $t = \perp, \top$, respectively.

Definition 2.4. An interval I has **injective boundaries** if parameterized squares are completely determined by their boundaries. I.e., when φ and ψ are parameterized squares, $\partial(\varphi) = \partial(\psi)$ implies that $\varphi = \psi$.

Lemma 2.5. *All representable intervals I have injective boundaries.*

Proof. Let squares φ and ψ in A parameterized by B be given which agree on their boundaries. Both squares determine arrows $\tilde{\varphi}$ and $\tilde{\psi}$ in the category $\mathcal{E}(B, [I, A])$. Moreover, because they agree on their boundaries, $\Phi(\tilde{\varphi}) = \Phi(\tilde{\psi})$. Thus, because Φ is a natural isomorphism it follows that $\varphi = \psi$, as required. \square

We will now prove that if I is representable, then it is necessarily a unital distributive lattice in the sense of Appendix A.

Proposition 2.6. *If I is representable, then it possesses the structure of a unital distributive lattice such that \perp is the unit for join $\vee : I \otimes I \rightarrow I$ and \top is the unit for meet $\wedge : I \otimes I \rightarrow I$. Moreover, this structure is unique in the strong sense that meet and join are the canonical maps $I \otimes I \rightarrow I$ such that both \wedge and $\wedge \circ \tau$ are 2-cells $\perp \circ i \Rightarrow 1_I$, and both \vee and $\vee \circ \tau$ are homotopies $1_I \Rightarrow \top \circ i$.*

Proof. Because I is representable it follows that there exists a 2-natural isomorphism

$$(3) \quad \mathcal{E}(U, I)^2 \xrightarrow{\cong} \mathcal{E}(U, [I, I])$$

of categories which is given at the level of objects by exponential transpose. In $\mathcal{E}(U, I)$ the following diagram commutes

$$\begin{array}{ccc} \perp & \xrightarrow{\perp \circ i \circ \lambda} & \perp \\ \perp \circ i \circ \lambda \downarrow & & \downarrow \lambda \\ \perp & \xrightarrow{\lambda} & \top \end{array}$$

Thus, applying (3) to this arrow of $\mathcal{E}(U, I)^2$ yields a map $\square : U \otimes I \rightarrow [I, I]$. Denote by $\wedge : I \otimes I \rightarrow I$ the transpose of the composite $\square \circ \lambda^{-1}$ and observe, by definition and Lemma 2.3, that \top is a unit for this operation and following diagram commutes:

$$(4) \quad \begin{array}{ccccc} U \otimes I & \xrightarrow{\perp \otimes I} & I \otimes I & \xleftarrow{I \otimes \perp} & I \otimes U \\ \lambda \downarrow & & \downarrow \wedge & & \downarrow \lambda \\ I & \xrightarrow{\perp \circ i} & I & \xleftarrow{\perp \circ i} & I \end{array}$$

also commutes. In the same way, applying the isomorphism (3) to the arrow

$$\begin{array}{ccc} \perp & \xrightarrow{\lambda} & \top \\ \lambda \downarrow & & \downarrow \top \circ i \circ \lambda \\ \top & \xrightarrow{\top \circ i \circ \lambda} & \top \end{array}$$

of $\mathcal{E}(U, I)^2$ yields a map $\boxplus : U \otimes I \rightarrow [I, I]$ for which the transpose $\vee : I \times I \rightarrow I$ of $\boxplus \circ \lambda^{-1}$ is an operation which has as a unit \perp and satisfies the dual of (4). Moreover, by Lemma 2.5, it follows that \vee and \wedge are the canonical maps $I \otimes I \rightarrow I$ with these properties. For example, the idempotent law which states that $\vee \circ \Delta = 1_I$ holds since

$$\begin{aligned} \vee \circ \Delta &= [\vee \circ (\perp \otimes I) \circ \lambda^{-1}, \vee \circ (I \otimes \top) \circ \rho^{-1}] \circ \star \\ &= [1_I, \top \circ i] \circ \star \\ &= 1_I \end{aligned}$$

where the final equation is by the cocategory counit law. The other idempotent law is similar. Commutativity of the additional diagrams for distributive lattices also follow from Lemma 2.5 by a routine (but lengthy) series of diagram chases. \square

Using join $\vee : I \otimes I \rightarrow I$ we see that I is a commutative Hopf object in the sense of [1] (see Appendix A for the definition).

Corollary 2.7. *If I is representable, then it is a commutative Hopf object.*

Proof. As we have already seen (I, Δ, i) is a comonoid and both (I, \vee, \perp) and (I, \wedge, \top) are commutative monoids. In fact, I can be made into a commutative Hopf object using either of these monoid structures. To see this it remains to

verify that \vee and \perp , as well as \wedge and \top , are comonoid homomorphisms. Since $\star \circ \perp = \downarrow \circ \perp$ it follows that \perp is a homomorphism. \vee is seen to be a homomorphism by testing on boundaries. A dual proof shows that \wedge and \top are also comonoid homomorphisms. \square

Remark. We note that if $\varphi : I \rightarrow H$ is an arrow in $\mathbf{Int}(\mathcal{E})$ between representable intervals, then it is necessarily also a morphism of Hopf objects provided that H and I are both equipped with “meet” (respectively, “join”) Hopf object structures from the proof of Corollary 2.7.

2.2. The characterization of representable intervals. We would now like to investigate the extent to which Proposition 2.6 characterizes representable intervals. For the remainder of this section, unless otherwise stated we do *not* assume that I is representable. We do however assume that there exist meet $\wedge : I \otimes I \rightarrow I$ and join $\vee : I \otimes I \rightarrow I$ operations which have \perp and \top as respective units and satisfy condition (10) from Appendix A. In this case, given $\varphi : B \otimes I \rightarrow A$, there are maps $\varphi^b, \varphi^\sharp$ and φ^\natural defined as follows. First, φ^b and φ^\sharp are given by

$$(B \otimes I) \otimes I \xrightarrow{\alpha^{-1}} B \otimes (I \otimes I) \xrightarrow{B \otimes \diamond} B \otimes I \xrightarrow{\varphi} A$$

for $\diamond = \wedge, \vee$, respectively. On the other hand, φ^\natural is the composite

$$(B \otimes I) \otimes I \xrightarrow{(B \otimes i) \otimes I} (B \otimes U) \otimes I \xrightarrow{\rho \otimes I} B \otimes I \xrightarrow{\varphi} A.$$

Given a composable pair of arrows

$$\begin{array}{ccc} & f & \\ & \downarrow \varphi & \\ B & \xrightarrow{g} & A \\ & \downarrow \psi & \\ & h & \end{array}$$

in $\mathcal{E}(B, A)$, it follows that, *qua* arrows in $\mathcal{E}(B \otimes I, A)$, both of the composites $(\psi^\natural \cdot \varphi^\sharp)$ and $(\psi^b \cdot \varphi^\natural)$ are defined. Moreover, the transpose $B \otimes I \rightarrow [I, A]$ of $(\psi^b \cdot \varphi^\natural)$, which we denote by $\theta_{\varphi, \psi}$, is itself an arrow from the transpose $\tilde{\varphi} : B \rightarrow [I, A]$ of φ to the transpose $\widetilde{(\psi \cdot \varphi)}$ of $(\psi \cdot \varphi)$ in $\mathcal{E}(B, [I, A])$. Similarly, it follows that the transpose $v_{\varphi, \psi} : B \otimes I \rightarrow [I, A]$ of $(\psi^\natural \cdot \varphi^\sharp)$ is a homotopy $\widetilde{(\psi \cdot \varphi)} \Rightarrow \tilde{\psi}$.

Lemma 2.8. *Assume I has injective boundaries and let arrows $\varphi : f \Rightarrow g$, $\psi : g \Rightarrow h$ and $\chi : h \Rightarrow k$ in $\mathcal{E}(B, A)$ be given. Then $\theta_{-, -}$ and $v_{-, -}$ satisfy the following “cocycle conditions”:*

$$\begin{aligned} \theta_{\varphi, 1_g} &= 1_{\tilde{\varphi}} \\ &= v_{1_f, \varphi} \\ (5) \quad \theta_{(\psi \cdot \varphi), \chi} \cdot \theta_{\varphi, \psi} &= \theta_{\varphi, (\chi \cdot \psi)} \\ v_{\psi, \chi} \cdot v_{\varphi, (\chi \cdot \psi)} &= v_{(\psi \cdot \varphi), \chi} \end{aligned}$$

Proof. It suffices to test (the transposes of) these maps on their boundaries. To see that they agree on the boundaries is a straightforward calculation. For example, the result of evaluating the transpose of the left-hand side of (5) at $(B \otimes I) \otimes \perp$ is

$$[\chi^b \cdot (\psi \cdot \varphi)^\natural, \psi^b \cdot \varphi^\natural] \circ ((B \otimes \star) \otimes I) \circ ((B \otimes I) \otimes \perp)$$

which itself is equal to the identity homotopy $1_f \circ \rho$ for $\rho : (B \otimes I) \otimes U \longrightarrow B \otimes I$. On the other hand,

$$\begin{aligned} ((\chi \cdot \psi)^\flat \cdot \varphi^\sharp) \circ ((B \otimes I) \otimes \perp) &= \varphi^\sharp \circ ((B \otimes I) \otimes \perp) \\ &= 1_f \circ \rho. \end{aligned}$$

The additional boundaries and cases are by similar calculations. \square

Lemma 2.9. *Assume I satisfies the conclusion of Lemma 2.5 and let a commutative diagram*

$$\begin{array}{ccccc} f & \xrightarrow{\gamma} & f' & \xrightarrow{\gamma'} & f'' \\ \varphi \downarrow & & \downarrow \psi & & \downarrow \chi \\ g & \xrightarrow{\delta} & g' & \xrightarrow{\delta'} & g'' \end{array}$$

be given in $\mathcal{E}(B, A)$, then

$$(6) \quad v_{\gamma, (\chi \cdot \gamma')} \cdot \theta_{(\delta \cdot \varphi), \delta'} = \theta_{\psi, \delta'} \cdot v_{\gamma, \psi}.$$

Proof. As with the proof of Lemma 2.8 it suffices to test the boundaries of these two maps, and these are straightforward calculations. \square

Theorem 2.10. *An interval I in \mathcal{E} is representable if and only if it has injective boundaries and possesses binary meet and join operations such that both \wedge and $\wedge \circ \tau$ are 2-cells $\perp \circ i \Longrightarrow 1_I$, and both \vee and $\vee \circ \tau$ are 2-cells $1_I \Longrightarrow \top \circ i$.*

Proof. It follows from Proposition 2.6 and Lemma 2.5 that a representable interval possesses the required properties.

For the other direction of the equivalence, by Lemmata 2.1 and the earlier observation that if it exists $\mathbf{2} \pitchfork A$ is necessarily $[I, A]$, it suffices to prove that there exist 2-natural isomorphisms

$$\mathcal{E}(B, [I, A]) \cong \mathcal{E}(B, A)^{\mathbf{2}}$$

of categories. Moreover, we have already seen that the functor

$$\mathcal{E}(B, [I, A]) \xrightarrow{\Phi} \mathcal{E}(B, A)^{\mathbf{2}}$$

must send an object $f : B \longrightarrow [I, A]$ to its transpose

$$\begin{array}{ccc} & \xrightarrow{\partial_0 f} & \\ B & \begin{array}{c} \Downarrow \tilde{f} \\ \Downarrow \tilde{f} \\ \Downarrow \tilde{f} \end{array} & A, \\ & \xrightarrow{\partial_1 f} & \end{array}$$

and an arrow $\varphi : f \Longrightarrow g$ in $\mathcal{E}(B, [I, A])$ to the associated “boundary diagram” (2). Functoriality of Φ follows, and the 2-naturality of this construction, is a trivial consequence of the definitions.

The inverse $\Psi : \mathcal{E}(B, A)^{\mathbf{2}} \longrightarrow \mathcal{E}(B, [I, A])$ of Φ is defined as follows. An arrow $\varphi : f \Longrightarrow f'$ in $\mathcal{E}(B, A)$ is sent to its transpose $\tilde{\varphi} : B \longrightarrow [I, A]$. Next, let a

commutative diagram

$$(7) \quad \begin{array}{ccc} f & \xrightarrow{\gamma} & f' \\ \varphi \downarrow & & \downarrow \psi \\ g & \xrightarrow{\delta} & g' \end{array}$$

be given in $\mathcal{E}(B, A)$ and denote by ζ the composite $(\delta \cdot \varphi) = (\psi \cdot \gamma)$. We also write $\tilde{\zeta}$ for the transpose of ζ . By the discussion above, we have that $\theta_{\varphi, \delta} : B \otimes I \rightarrow [I, A]$ is an arrow $\tilde{\varphi} \Rightarrow \tilde{\zeta}$ in $\mathcal{E}(B, [I, A])$. Similarly, $v_{\gamma, \psi} : B \otimes I \rightarrow [I, A]$ is an arrow $\tilde{\zeta} \Rightarrow \tilde{\psi}$ in $\mathcal{E}(B, [I, A])$. Let Ψ send the arrow (7) to composite $(v_{\gamma, \psi} \cdot \theta_{\varphi, \delta})$. Functoriality of Ψ follows from Lemmata 2.8 and 2.9.

Φ and Ψ are easily seen to be inverse on objects. For arrows, let an arrow (7) be given. We must show that, where γ and δ are as above, $\partial_0 \circ (v_{\gamma, \psi} \cdot \theta_{\varphi, \delta}) = \gamma$ and $\partial_1 \circ (v_{\gamma, \psi} \cdot \theta_{\varphi, \delta}) = \delta$. For the first equation, observe that, since

$$\begin{aligned} (\delta^\flat \cdot \varphi^\sharp) \circ (B \otimes I) \otimes \perp &= \varphi^\sharp \circ ((B \otimes I) \otimes \perp) \\ &= f \circ \rho \circ \rho \circ ((B \otimes i) \otimes U) \end{aligned}$$

and $\gamma^\sharp \circ ((B \otimes I) \otimes \perp) = \gamma \circ \rho$, it follows that

$$[1_f, \gamma] = [\partial_0 \circ \theta_{\varphi, \delta}, \partial_0 \circ v_{\gamma, \psi}].$$

Thus, $\partial_0 \circ (\delta \cdot \gamma) = \gamma$ and, by a similar calculation, $\partial_1 \circ (\delta \cdot \gamma) = \delta$.

Going the other direction, let an arrow $\varphi : f \Rightarrow g$ in $\mathcal{E}(B, [I, A])$ be given. It suffices, by the hypotheses of the theorem, to prove that $\Psi \circ \Phi(\varphi)$ has the same boundary as φ . But this follows from the fact that, by what we have just proved,

$$\Phi \circ \Psi \circ \Phi(\varphi) = \Phi(\varphi).$$

□

Although most of the examples of intervals studied earlier are already known to give rise to finitely bicomplete 2-category structures, it is nonetheless instructive to consider these cases in light of the theorem.

Example 2.11. Consider the following intervals discussed in Example 1.3:

- (1) The interval I obtained by taking the discrete cocategory on the tensor unit U is representable, with meet and join both the structure map $\lambda = \rho : U \otimes U \rightarrow U$.
- (2) Using the isomorphism $(U + U) \otimes (U + U) \cong (U + U) + (U + U)$ it is easily seen that the interval $(U + U)$ satisfies the necessary and sufficient conditions from Theorem 2.10 for being representable and therefore gives rise to a finitely bicomplete 2-category.
- (3) In **Cat** both **2** and **I** are representable. Of course, this can be easily verified directly, but one can also check that the hypotheses of the theorem are satisfied. For instance, in both cases the meet map \wedge is the functor which sends an object (s, t) of $\mathbf{2} \times \mathbf{2}$ to \top if $s = t = \top$ and to \perp otherwise.

The case of chain complexes of R -modules is worth considering separately.

Example 2.12. First, observe that the interval \mathbb{I} in $\mathbf{Ch}_{0\leq}(R)$ does possess meet and join maps. Namely, in degree 0, the meet \wedge_0 and join \vee_0 operations $\mathbb{I} \otimes \mathbb{I} \rightarrow \mathbb{I}$ are given by

$$\begin{aligned}\wedge_0(x, y, v, w) &:= (x + y + v, w), \text{ and} \\ \vee_0(x, y, v, w) &:= (x, y + v + w),\end{aligned}$$

for x, y, v, w in R . Similarly, in degree 1 we have

$$\begin{aligned}\wedge_1(x, y, v, w) &:= (y + w), \text{ and} \\ \vee_1(x, y, v, w) &:= (x + v),\end{aligned}$$

for x, y, v, w in R . These are easily seen to be chain maps which satisfy the conditions for meet and join from Theorem 2.10 in order for \mathbb{I} to be representable. Nonetheless, \mathbb{I} fails to be representable since it does not possess the injective boundary property. To see this, let us define a new complex C by setting $C_2 := R \oplus R$, $C_1, C_2 := R \oplus R \oplus R \oplus R$ and 0 in all other dimensions. The differentials are given by

$$\begin{array}{ccc} C_2 & \xrightarrow{d} & C_1 \\ (x, y) & \longmapsto & (x + y, -x - y, -x - y, x + y) \end{array}$$

and

$$\begin{array}{ccc} C_1 & \xrightarrow{d} & C_0 \\ (x, y, v, w) & \longmapsto & (x + v, y - v, -x + w, -y - w) \end{array}$$

for x, y, v, w in R . I.e., C is the chain complex associated to the space consisting of two distinct 2-cells bounded by the same square of 1-cells. There are distinct chain maps φ, ψ from $(\mathbb{I} \otimes \mathbb{I}) \rightarrow C$ defined by letting both φ_i and ψ_i be the identity in all degrees $i \neq 2$, and taking φ_2 and ψ_2 to be the left and right inclusions $R \rightarrow R \oplus R$, respectively. It is easily verified that φ and ψ are distinct squares in C which agree on their boundaries. Thus, by Theorem 2.10, $\mathbf{Ch}_{0\leq}(R)$ is not a finitely bicomplete 2-category (with respect to the 2-category structure induced by \mathbb{I}).

Remark. Although we will not consider those intervals I which fail to be representable in our discussion of homotopy theory below, we would like to mention that some effort has been made to investigate the homotopy theory of intervals arising in the homological setting such as the \mathbb{I} just mentioned. In particular, Stanculescu [13] has employed intervals in his work on the homotopy theory of categories enriched in simplicial modules.

3. HOMOTOPY THEORETIC CONSEQUENCES

The purpose of this section is to relate the considerations on intervals from the foregoing sections to several known results from homotopy theory. In particular, we show that, under suitable hypotheses on \mathcal{E} , if I is representable interval in \mathcal{E} , then the ‘‘isofibration’’ model structure on \mathcal{E} due to Lack [10] can be lifted to the category of (reduced) operads using a theorem of Berger and Moerdijk [1]. In order to apply the machinery of *ibid* it is first necessary to construct a *Hopf interval*, which is essentially a cylinder object equipped with the structure of a Hopf object. As such, the principal observation in this section is that, when \mathcal{E} is cocomplete in the 1-dimensional sense, it is possible to construct the free Hopf object generated

by the interval I . We refer the reader to [4] for more information regarding model categories.

3.1. The isofibration model structure. Now, assuming (as we will throughout the remainder of this section) that \mathcal{E} is a finitely bicomplete symmetric monoidal closed category with a representable interval I , it follows from a theorem due to Lack [10] that \mathcal{E} can be equipped with a model structure in which the weak equivalences are the categorical equivalences and the fibrations are isofibrations. Recall that an arrow $f : A \rightarrow B$ in a 2-category is said to be a **categorical equivalence** if there exists a map $f' : B \rightarrow A$ together with isomorphisms $f \circ f' \cong 1_B$ and $f' \circ f \cong 1_A$. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ in **Cat** is said to be an isofibration when isomorphisms in \mathcal{D} whose codomains lie in the image of F can be lifted to isomorphisms in \mathcal{C} . This notion also makes sense in arbitrary 2-categories \mathcal{E} . In particular, we define a map $f : A \rightarrow B$ in \mathcal{E} to be an **isofibration** if, for any object E of \mathcal{E} , the induced map

$$\mathcal{E}(E, A) \xrightarrow{f_*} \mathcal{E}(E, B)$$

is an isofibration in **Cat**.

Definition 3.1. Assume \mathcal{E} is a finitely bicomplete 2-category with a model structure. Then \mathcal{E} is a **model Cat-category** provided that if $p : E \rightarrow B$ is a fibration and $i : X \rightarrow Y$ is a cofibration, then the induced functor

$$\mathcal{E}(Y, E) \xrightarrow{\langle p_*, i^* \rangle} \mathcal{E}(Y, B) \times_{\mathcal{E}(X, B)} \mathcal{E}(X, E)$$

is an isofibration which is simultaneously an equivalence if either p or i is a weak equivalence.

With these definitions, Lack [10] proved the following theorem:

Theorem 3.2 (Lack). *If \mathcal{E} is a finitely bicomplete 2-category, then it is a model **Cat-category** in which the weak equivalences are the equivalences, the fibrations are the isofibrations and the cofibrations are those maps having the left-lifting property with respect to maps which are simultaneously fibrations and weak equivalences.*

We will refer to such a model structure on a 2-category \mathcal{E} as the **isofibration model structure** on \mathcal{E} . Every object is both fibrant and cofibrant in this model structure. It is an immediate consequence of Theorem 3.2 that when \mathcal{E} is a finitely bicomplete symmetric monoidal closed category with a representable interval I it is also a model **Cat-category** with the isofibration model structure.

3.2. The free Hopf interval generated by I . In [1], a (commutative) **Hopf interval** in a symmetric monoidal model category is defined to be a cylinder object

$$\begin{array}{ccc} U + U & \longrightarrow & H \\ & \searrow & \swarrow \\ & U & \end{array}$$

on U such that H is a (commutative) Hopf object, and both maps $U + U \rightarrow H$ and $H \rightarrow U$ are homomorphisms of Hopf objects. Here U has the trivial Hopf object structure given by by structure map $\lambda : U \otimes U \rightarrow U$ and its inverse. On the other hand, $(U + U)$ is given the structure of a commutative Hopf object using the “meet” Hopf object structure described in Corollary 2.7 (which coincides with the Hopf object structure on U described in [1]). We emphasize that a Hopf interval

need not be an interval in the sense of Definition 1.2. The following lemma shows that we cannot in general expect I itself to be a Hopf interval.

Lemma 3.3. *The following are equivalent:*

- (1) I is invertible.
- (2) The structure map $\lambda : U \otimes I \longrightarrow I$, regarded as a 2-cell $\perp \Longrightarrow \top$, possesses an inverse $\neg : \top \Longrightarrow \perp$.
- (3) The meet operation, regarded as a 2-cell $\perp \circ i \Longrightarrow 1_I$, has an inverse. (Or, dually, the join operation has an inverse.)
- (4) The map $i : I \longrightarrow U$ is an equivalence.

Proof. (1) and (2) are equivalent since \neg is defined using the existence of a coinverse map $\sigma : I \longrightarrow I$ to be $\sigma \circ \lambda$, and, going the other way, σ is defined in terms of \neg as $\neg \circ \lambda^{-1}$. To see that (2) implies (3), define $\wedge' : I \otimes I \longrightarrow I$ to be the composite

$$I \otimes I \xrightarrow{I \otimes \lambda^{-1}} I \otimes (U \otimes I) \xrightarrow{I \otimes \neg} I \otimes I \xrightarrow{\wedge} I.$$

It is then easily seen that \wedge' is the inverse of \wedge as arrows in the category $\mathcal{E}(I, I)$.

That (3) implies (4) follows from the fact that \wedge is a 2-cell $\perp \circ i \Longrightarrow 1_I$ and therefore the existence of an inverse for this homotopy implies that i is an equivalence. Going the other way, to see that (4) implies (2), assume given $k : U \longrightarrow I$ together with isomorphisms $\varphi : k \circ i \Longrightarrow 1_I$ and $\psi : 1_U \Longrightarrow i \circ k$. Then we define the inverse $\neg : \top \Longrightarrow \perp$ of λ in $\mathcal{E}(U, I)$ as follows. We first form the vertical composite $\xi := (\varphi * \perp)$ which is, by definition, a 2-cell $k \Longrightarrow \perp$. Similarly, we define $\zeta : \top \Longrightarrow k$ to be the vertical composite $(\varphi^{-1} * \top)$. We then set $\neg := (\xi \cdot \zeta)$. That \neg is the inverse of λ is seen to hold by straightforward calculations. For example, that $\lambda \cdot \neg$ is 1_{\top} follows from the fact that, by the interchange law,

$$\lambda \cdot \neg = (\varphi * \lambda) \cdot \zeta.$$

Moreover, $(\varphi * \lambda) = (\varphi * \top)$ since $(i * \lambda) = 1_U$. Thus, $(\lambda \cdot \neg) = \top$ by another application of the interchange law and the fact that $(\varphi \cdot \varphi^{-1}) = 1_I$. \square

Remark. We mention a further equivalence, which we will not require and which is easily verified using representability of I . Namely, I is invertible if and only if it is a Boolean algebra.

By Lemma 3.3 it follows that, for example, **2** is not a Hopf interval in **Cat**. Nonetheless, when \mathcal{E} is cocomplete (in the 1-dimensional sense) it is possible to construct in the expected manner the free Hopf interval J generated by I . Specifically, we first construct K as the pushout of $[\perp, \top] : U + U \longrightarrow I$ along $[\top, \perp] : U + U \longrightarrow I$ and observe that this is itself an interval. Moreover, K possesses a map $\sigma_K : K \longrightarrow K$ which reverses \perp_K and \top_K . Then J is obtained as the quotient of K by the relations

$$\begin{aligned} [\sigma_K, K] \circ \star_K &= \top_K \circ i_K, \text{ and} \\ [K, \sigma_K] \circ \star_K &= \perp_K \circ i_K. \end{aligned}$$

It is straightforward to verify using these definitions that J is indeed an invertible interval with symmetry map $\sigma_J : J \longrightarrow J$. There exists by construction a morphism of intervals $\iota : I \longrightarrow J$ with which it is possible to state the universal property of J . Namely, J is the free invertible interval generated by I in the sense that, for any morphism of intervals $\xi : I \longrightarrow H$ with H invertible, there exists a canonical map of intervals $\tilde{\xi} : J \longrightarrow H$ extending ξ along the inclusion ι . Additionally, J classifies

the invertible 2-cells in the 2-category structure induced by I as described in the following lemma.

Lemma 3.4. *Given morphisms f and g in $\mathcal{E}(B, A)$ together with a 2-cell $\alpha : f \Rightarrow g$ in the 2-category structure induced by I , α is invertible if and only if there exists a canonical extension $\bar{\alpha} : B \otimes J \rightarrow A$ such that the following diagram commutes:*

$$\begin{array}{ccc} B \otimes J & \xrightarrow{\bar{\alpha}} & A \\ & \swarrow B \otimes \iota & \nearrow \alpha \\ & B \otimes I & \end{array}$$

Proof. Routine. □

Using Lemma 3.4 it is possible to construct meet and join operations on J as such canonical extensions. For example, using representability of I , a straightforward calculation shows that, regarded as a 2-cell $\perp_J \circ i \Rightarrow \iota$, the map $\iota \circ \wedge : I \otimes I \rightarrow J$ has as its inverse the vertical composite $f \cdot (\iota \circ \vee)$ where f is defined to be

$$I \otimes I \xrightarrow{i \otimes I} U \otimes I \xrightarrow{\lambda} I \xrightarrow{\iota} J \xrightarrow{\sigma_J} J.$$

Thus, by Lemma 3.4 there exists a canonical extension $\bar{\wedge} : I \otimes J \rightarrow J$. Applying the same trick one more time, using the symmetry map $\tau : I \otimes J \rightarrow J \otimes I$, yields the required meet operation $\wedge_J : J \otimes J \rightarrow J$. The construction of join is dual. This construction gives us the following lemma.

Lemma 3.5. *J is a representable interval if I is.*

Proof. We have seen that J possesses meet and join operations which satisfy the required equations by construction. Thus, by Theorem 2.10 it suffices to show that if $\varphi, \psi : (B \otimes J) \otimes J \rightrightarrows A$ are parameterized squares which agree on their boundaries, then they are in fact equal. Since ι is a morphism of intervals and I is representable it follows that φ and ψ are equal upon precomposition with $(B \otimes \iota) \otimes \iota$. By construction of J it then follows that they are likewise equal upon precomposition with $(B \otimes J) \otimes \iota$. Finally, by Lemma 3.4, it follows that φ and ψ are equal. □

It follows from Lemma 3.5 that there also exists an isofibration model structure on \mathcal{E} defined with respect to J . However, since only invertible 2-cells feature in the specification of the isofibration model structure, it follows, by Lemma 3.4, that this “ J -model structure” coincides with the original “ I -model structure”. As such, we continue to simply refer to the isofibration model structure on \mathcal{E} without reference to either interval I or J . Note though that the 2-category structures induced by I and J do in general differ. Namely, the 2-cells for J correspond exactly to the invertible 2-cells for I .

Proposition 3.6. *$J = (J, \Delta_J, i_J, \wedge_J, \top_J)$ is the free commutative Hopf interval generated by I in the isofibration model structure.*

Proof. Since J is an invertible interval it follows from Lemma 3.3 that $i_J : J \rightarrow U$ is a weak equivalence. To see that $[\perp_J, \top_J] : U + U \rightarrow J$ is a cofibration we first observe that, rephrasing the usual argument in terms of the present setting, any weak equivalence $p : E \rightarrow B$ is “full” in the sense of possessing the right-lifting property with respect to the map $[\perp, \top] : U + U \rightarrow I$. Moreover, combined with

the fact that any $h : I \rightarrow E$ is invertible (as a 2-cell) whenever $p \circ h : I \rightarrow B$ is, gives by Lemma 3.4 that any weak equivalence p has the right-lifting property with respect to $[\perp_J, \top_J]$ as well.

Observe that $(U + U)$ is the initial object in $\mathbf{Int}(\mathcal{E})$ and $[\perp_J, \top_J]$ is the coobject part of the induced canonical map $(U + U) \rightarrow J$ of intervals. It therefore follows, by the remark following Corollary 2.7, that this is a morphism of Hopf objects. Similarly, i_J is a morphism of Hopf objects since it is the coobject part of the induced map into the terminal object U in $\mathbf{Int}(\mathcal{E})$.

Finally, for freeness, suppose given another Hopf interval H . Then freeness of J means that if

$$U + U \xrightarrow{[a,b]} H \xrightarrow{g} U$$

is any other Hopf interval and $\xi : I \rightarrow H$ is a morphism of commutative Hopf objects, then, by definition the map $g : H \rightarrow U$ factoring the codiagonal $\nabla : U + U \rightarrow U$ is a weak equivalence. As such, there exists an arrow $g' : U \rightarrow H$ together with an invertible 2-cell $\varphi : g' \circ g \Rightarrow 1_H$. Then the vertical composite $(\varphi * a) \cdot (\varphi^{-1} * b)$ in $\mathcal{E}(U, H)$ is a 2-cell $b \Rightarrow a$ which is, by the fact that $g\xi = i$, the inverse of ξ . Thus, by Lemma 3.4 there exists a canonical extension $\tilde{\xi} : J \rightarrow H$ of ξ which is easily seen to commute with $[a, b]$ and g . Additionally, it follows that $\tilde{\xi}$ is a morphism of Hopf objects since both ι and ξ are. Finally, uniqueness of $\tilde{\xi}$ with these properties is trivial. \square

Berger and Moerdijk [1] have shown that the existence of a commutative Hopf interval is one of several conditions which allow one to lift a model structure from a symmetric monoidal closed category \mathcal{E} to the category of reduced operads over \mathcal{E} .

Theorem 3.7 (Berger and Moerdijk). *If a symmetric monoidal closed category \mathcal{E} is a monoidal model category such that \mathcal{E} is cofibrantly generated with cofibrant tensor unit U , \mathcal{E}/U has a symmetric monoidal fibrant replacement functor and \mathcal{E} possesses a commutative Hopf interval, then there exists a cofibrantly generated model structure on the category of reduced operads in which the weak equivalences and fibrations are “pointwise”.*

Here that the model category is *monoidal* means that the appropriate internal form of the condition from Definition 3.1 is satisfied (cf. [4]). As such, in light of Theorem 3.7, we obtain the following corollary to Proposition 3.6:

Corollary 3.8. *Assume \mathcal{E} has all small colimits and possesses a representable interval I , then when the isofibration model structure is cofibrantly generated there is a model structure on the category of reduced operads over \mathcal{E} in which the fibrations and weak equivalences are pointwise.*

Proof. By the fact that all objects in the isofibration model structure are fibrant and Proposition 3.6 it remains, in order to be able to apply Theorem 3.7, only to verify that the model structure is monoidal. For this, it suffices, by the definition of fibrations in the isofibration model structure, to note that if $f : E \rightarrow B$ is a (trivial) fibration, then so is $f_* : [X, E] \rightarrow [X, B]$ for any object X . First, that f_* is a fibration when f is by the tensor-hom adjunction. Next, that f_* is an equivalence when f follows from the fact that the map

$$\mathcal{E}(E, B) \longrightarrow \mathcal{E}([X, E], [X, B])$$

which sends an arrow $f : E \rightarrow B$ to f_* is a functor and therefore preserves isomorphic 2-cells. \square

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APPENDIX A. HOPF OBJECTS AND LATTICES IN A SYMMETRIC MONOIDAL CATEGORY

In this appendix we provide the full definitions of comonoids, Hopf objects and distributive lattices in a symmetric monoidal category.

A.1. Monoids, comonoids and Hopf objects. A **monoid** (M, η, m) in a symmetric monoidal category \mathcal{E} is given by an object M of \mathcal{E} together with arrows $\eta : U \rightarrow M$ and $m : M \otimes M \rightarrow M$ satisfying the following diagrams commute:

$$\begin{array}{ccccc} M \otimes U & \xleftarrow{\rho^{-1}} & M & \xrightarrow{\lambda^{-1}} & U \otimes M \\ M \otimes \eta \downarrow & & \downarrow M & & \downarrow \eta \otimes M \\ M \otimes M & \xrightarrow{m} & M & \xleftarrow{m} & M \otimes M \end{array}$$

and

$$\begin{array}{ccc} M \otimes (M \otimes M) & \xrightarrow{\alpha} & (M \otimes M) \otimes M \\ M \otimes m \downarrow & & \downarrow m \otimes M \\ M \otimes M & & M \otimes M \\ & \searrow m & \swarrow m \\ & & M \end{array}$$

A **comonoid** (G, ϵ, Δ) in a symmetric monoidal category \mathcal{E} is given by an object G of \mathcal{E} together with arrows $\epsilon : G \rightarrow U$ and $\Delta : G \rightarrow G \otimes G$ such that the following diagrams commute:

$$\begin{array}{ccccc} M \otimes M & \xleftarrow{\Delta} & M & \xrightarrow{\Delta} & M \otimes M \\ \epsilon \otimes M \downarrow & & \downarrow M & & \downarrow M \otimes \epsilon \\ U \otimes M & \xrightarrow{\lambda} & M & \xleftarrow{\rho} & M \otimes U \end{array}$$

and

$$\begin{array}{ccc}
 & M & \\
 \Delta \swarrow & & \searrow \Delta \\
 M \otimes M & & M \otimes M \\
 \Delta \otimes M \downarrow & & \downarrow M \otimes \Delta \\
 (M \otimes M) \otimes M & \xrightarrow{\alpha^{-1}} & M \otimes (M \otimes M)
 \end{array}$$

A **(commutative) Hopf object** in \mathcal{E} is a structure $(H, \eta, m, \epsilon, \Delta)$ such that (H, ϵ, Δ) is a comonoid, (H, η, m) is a (commutative) monoid, and the maps m and η are comonoid homomorphisms (cf. [1]). Here note that $H \otimes H$ is given the structure of a comonoid via the map, constructed using the symmetry τ , which (schematically) sends $x \otimes y$ to $((x \otimes y) \otimes (x \otimes y))$.

A.2. Lattices. Assume that (L, ϵ, Δ) is a comonoid in \mathcal{E} . Then L is a **lattice** if there are maps $\vee : L \otimes L \rightarrow L$ and $\wedge : L \otimes L \rightarrow L$ such that both \vee and \wedge are associative, commutative, and following diagrams commute:

$$(8) \quad \begin{array}{ccc}
 L & \xrightarrow{\Delta} & L \otimes L \\
 \Delta \downarrow & \searrow L & \downarrow \vee \\
 L \otimes L & \xrightarrow{\wedge} & L
 \end{array}$$

and

$$(9) \quad \begin{array}{ccccc}
 L \otimes L & \xrightarrow{\Delta \otimes L} & (L \otimes L) \otimes L & \xrightarrow{\alpha^{-1}} & L \otimes (L \otimes L) \\
 L \otimes \epsilon \downarrow & & & & \downarrow L \otimes \clubsuit \\
 L \otimes U & & & & L \otimes L \\
 & \searrow \rho & & \swarrow \diamond & \\
 & & L & &
 \end{array}$$

for $\clubsuit = \vee$ and $\diamond = \wedge$, or $\clubsuit = \wedge$ and $\diamond = \vee$.

A lattice L is **unital** if there exist maps $\top, \perp : U \rightrightarrows L$ such that \top is a unit for \wedge , \perp is a unit for \vee , and the following diagram commutes:

$$(10) \quad \begin{array}{ccccc}
 U \otimes L & \xrightarrow{t \otimes L} & L \otimes L & \xleftarrow{L \otimes t} & L \otimes U \\
 \lambda \downarrow & & \downarrow \diamond & & \downarrow \rho \\
 L & \xrightarrow{t \otimes \epsilon} & L & \xleftarrow{t \otimes \epsilon} & L
 \end{array}$$

for $\diamond = \wedge$ and $t = \perp$, or $\diamond = \vee$ and $t = \top$.

A lattice L is distributive if the further diagram commutes:

$$\begin{array}{ccccc}
 & & L \otimes (L \otimes L) & \xrightarrow{L \otimes \clubsuit} & L \otimes L & & \\
 & & \swarrow \alpha \circ (\Delta \otimes (L \otimes L)) & & \searrow \diamond & & \\
 & & ((L \otimes L) \otimes L) \otimes L & & L & & \\
 & & \downarrow \alpha^{-1} \otimes L & & \uparrow \clubsuit & & \\
 & & (L \otimes (L \otimes L)) \otimes L & & L \otimes L & & \\
 & & \searrow \tau \otimes L & & \nearrow \diamond \otimes \diamond & & \\
 (11) & & ((L \otimes L) \otimes L) \otimes L & \xrightarrow[\alpha^{-1}]{} & (L \otimes L) \otimes (L \otimes L) & &
 \end{array}$$

for either $\clubsuit = \vee$ and $\diamond = \wedge$, or $\clubsuit = \wedge$ and $\diamond = \vee$.

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