MAT1332 Spring/Summer 2009 (Instructor: Michael A. Warren)

Handout: The substitution method for definite integrals.

For indefinite integrals, the method of substitution is a technique for finding antiderivatives which corresponds to the chain rule for differentiation. One sometimes sees the result of the method expressed as the (somewhat misleading) equation $\int f(x)dx = \int f(g(x))g'(x)dx$. However, remembering that $\int f(x)dx$ should properly be regarded as the set of antiderivatives of f, substitution should be really be thought of as two procedures: one which takes an antiderivative of $f(x)$ and gives an antiderivative of $f(g(x))g'(x)$, and the other which, assuming g possesses an inverse g^{-1}, takes an antiderivative of $f(g(x))g'(x)$ and gives an antiderivative of $f(x)$. Let us examine each of these procedures in turn.

In each case below I first give the "formal" mathematically precise description of the method and then the "practical" version you should already be familiar with. All functions are assumed to be continuous.

Going from an antiderivative of $f(x)$ to an antiderivative of $f(g(x))g'(x)$

This direction of the procedure is easy. Given an antiderivative $F(x)$ of $f(x)$, we have that $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$ since

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x),$$

where the first equation is by the chain rule and the second is by the fact that $F(x)$ is an antiderivative of $f(x)$. This formal description is the real mathematical content of (this direction of) the substitution method.

In practice one is trying to integrate some (possibly) complicated function $\theta(x)$ which we may not immediately recognize as being of the form $\int f(g(x))g'(x)dx$ for suitable f and g. Instead we follow the steps:

1. Make a "guess" by setting $u := g(x)$ for $g(x)$ some other function which occurs somewhere in the $\theta(x)$ we are trying to integrate.

2. Differentiate $\frac{du}{dx} = g'(x)$ and write $dx = \frac{du}{g'(x)}$.

3. If we have chosen our u correctly, then $\theta(x)$ should now turn out to be of the form $f(u)g'(x)$, for some function f, so that the $g'(x)$ cancels to give $\int \theta(x)dx = \int f(u)du$. If not, then it is back to the drawing board.

4. If it did work in step (3), then we now integrate $f(u)$ and obtain an antiderivative $F(u)$.

5. Finally, substitute $g(x)$ for u to obtain $F(g(x))$.

Step (1) is a bit of a gamble if we do not already recognize our $\theta(x)$ as being in the correct form to apply substitution. Steps (2) and (3) really amount, in terms of the formal description of the method given above, to checking that indeed $\theta(x)$ is in the correct form.

Going from an antiderivative of $f(g(x))g'(x)$ to an antiderivative of $f(x)$ when g has an inverse g^{-1}

Assume $F(x)$ is an antiderivative of $f(g(x))g'(x)$. We claim that $F(g^{-1}(x))$ is an antiderivative of $f(x)$. To see this we reason as follows:

$$\begin{aligned}\frac{d}{dx}F(g^{-1}(x)) &= F'(g^{-1}(x))(g^{-1})'(x)\\ &= f(g(g^{-1}(x)))\cdot g'(g^{-1}(x))\cdot (g^{-1})'(x)\\ &= f(x)\cdot g'(g^{-1}(x))\cdot \frac{1}{g'(g^{-1}(x))}\\ &= f(x),\end{aligned}$$

where the first equation is by the chain rule, the second equation is by the fact that $F(x)$ is an antiderivative of $f(g(x))g'(x)$, the third is by the fact that g^{-1} is the inverse of g together with the fact that the derivative of any inverse function $g^{-1}(x)$ is $1/g'(g^{-1}(x))$.

In practice, we are again given some function $\theta(x)$ which we would like to integrate and so we gamble and hope that it will be easier to integrate $f(g(x))g'(x)$ for g some invertible function. Here is the corresponding approach you should have seen before:

1. Set $u := g^{-1}(x)$ for g and invertible function. So $x = g(u)$.
2. Take the derivative $\frac{dx}{du} = g'(u)$.
3. Write $dx = g'(u)du$ in our original integral to give $\int \theta(u)g'(u)du$.
4. Try to integrate this new function. If it works, we obtain an antiderivative $F(u)$.
5. Replace u with $g^{-1}(x)$ to obtain $F(g^{-1}(x))$, which is an antiderivative of our original function.

All of the business in step (2) just amounts to a fancy way of saying that we are going to try to integrate $f(g(x))g'(x)$ instead of $f(x)$ and then use $F(g^{-1}(x))$ as an antiderivative for $f(x)$ when F is an antiderivative of $f(g(x))g'(x)$.

Substitution for definite integrals

Corresponding fact for definite integrals is that if g is a function with continuous derivative, then

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g'(x)dx. \tag{1}$$

For this we do not require that g be invertible, for given any antiderivative $F(x)$ of $f(x)$ we have, by the first part of the substitution method for indefinite integrals, that $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$. By the Fundamental Theorem, $\int_{g(a)}^{g(b)} f(x)dx = F(g(b)) - F(g(a))$, and similarly

$$\int_a^b f(g(x))g'(x)dx = F(g(b)) - F(g(a)).$$

Thus we have proved (1). When g has an inverse g^{-1} we also obtain

$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(x))g'(x)dx \tag{2}$$

since, given an antiderivative $F(x)$ of $f(x)$, we have that $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$ and so the right-hand side of (2) is equal to $F(g(g^{-1}(b))) - F(g(g^{-1}(a)))$ which is just $F(b) - F(a)$.

In practice there are three ways to apply substitution to definite integrals. On the one hand, in order to find the definite integral $\int_a^b \theta(x)dx$ we may first apply substitution for *indefinite* integrals to obtain an antiderivative $G(x)$ and then calculate $\int_a^b \theta(x)dx = G(b) - G(a)$ using the Fundamental Theorem.

On the other hand, we can apply substitution directly to find the definite integral $\int_a^b \theta(x)dx$ in "one go" by transforming the limits of integration as follows:

1. Make a "guess" and set $u := g(x)$ for $g(x)$ some function occurring in $\theta(x)$.
2. Differentiate $\frac{du}{dx} = g'(x)$ and write $dx = \frac{du}{g'(x)}$.
3. Again, if we have chosen correctly then $\theta(x)$ should now turn out to be of the form $f(u)g'(x)$. If not, then it is back to the drawing board.
4. If we obtained the correct form in the third step, then we change the limits of integration by applying $g(x)$ to obtain
$$\int_{g(a)}^{g(b)} f(u)du.$$
5. Find the definite integral of $f(u)$ in these bounds. This is equal to $\int_a^b \theta(x)dx$.

You should compare this method with the equation (1).

For the final method we assume that g is an invertible function and that we wish to integrate a function of the form $\int_a^b f(x)dx$.

1. Make a "guess" and set $u := g^{-1}(x)$ so that $x = g(u)$.
2. Differentiate $\frac{dx}{du} = g'(u)$.
3. Rewrite our original integral $\int_a^b f(x)dx$ with new bounds and with $dx = g'(u)du$:
$$\int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)du.$$
4. Compute this definite integral. It is equal to $\int_a^b f(x)dx$.

This method corresponds to (2).

Examples

The following examples should help illustrate the method of substitution.

1. Let $f(x) := \sin(2x)$ and suppose we wish to find $\int_0^{\pi/4} \sin(2x)\,dx$. Set $u := 2x$ (i.e., $g^{-1}(x) = 2x$) so that $x = \frac{u}{2}$ (i.e., $g(u) := \frac{u}{2}$) and $\frac{dx}{du} = \frac{1}{2}$. Then:

$$\int_0^{\frac{\pi}{4}} \sin(2x)\,dx \;=\; \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(u)\,du \;=\; \frac{1}{2}\Big(-\cos(u)\Big)\Big|_0^{\frac{\pi}{2}} \;=\; \frac{1}{2}.$$

   We could also have simply computed the indefinite integral $\int \sin(2x)\,dx = -\frac{1}{2}\cos(2x) + C$ using substitution for indefinite integrals and then applied the Fundamental Theorem to obtain

$$\int_0^{\frac{\pi}{4}} \sin(2x)\,dx = -\frac{1}{2}\cos(\frac{\pi}{2}) + \frac{1}{2} = \frac{1}{2}.$$

2. Let $f(x) := \sqrt{1-x^2}$. Then setting $u := \arcsin(x)$ so that $x = \sin(u)$ we have

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx &= \int_{\arcsin(-1)}^{\arcsin(1)} \sqrt{1-\sin^2(u)}\cos(u)\,du \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(u)\,du \\ &= \frac{1}{2}\Big(\sin(u)\cos(u) + u\Big)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{\pi}{2}\end{aligned}$$

3. Consider $f(x) := \frac{x}{(1+x^2)^2}$. Set $u := 1 + x^2$ (i.e., $g(x) = (1+x^2)$). So $\frac{du}{dx} = 2x$. Then

$$\int_0^2 f(x)\,dx \;=\; \frac{1}{2}\int_1^5 \frac{1}{u^2}\,du \;=\; \frac{1}{2}\Big(-\frac{1}{u}\Big)\Big|_1^5 \;=\; \frac{2}{5}$$