MAT1332 Spring/Summer 2009 (Instructor: Michael A. Warren)

Handout: Separation of variables.

You can look in the textbook for a description of how separation of variables can be used to solve autonomous differential equations.

Example: The harvest equation

Consider the differential equation

$$\frac{dv}{dt} = (2-v)v - 1.$$

To solve this equation using separation of variables we observe that the right-hand side of the equation is equal to $-(v-1)^2$ and then multiply both sides by dt:

$$dv = -(v-1)^2 dt.$$

Now dividing gives:

$$-\frac{dv}{(v-1)^2} = dt,$$

and integrating yields the equation $-\int \frac{dv}{(v-1)^2} = \int dt$. We have

$$-\int \frac{dv}{(v-1)^2} = \frac{1}{v-1} - C_1$$

where C_1 is a constant of integration. As such our earlier equation between the two integrals yields

$$\frac{1}{v-1} - C_1 = t + C_2,$$

where C_2 is another constant of integration. Solving for v we obtain

$$v = \frac{1+t+C_0}{t+C_0}$$

where we have defined $C_0 := C_2 + C_1$.

To check that this is gives a solution we see that, by the quotient rule for differentiation, $\frac{dv}{dt} = -\frac{1}{(t+C_0)^2}$. On the other hand, substituting our solution for the v in $-(v-1)^2$ also gives the same result as you can check.

The fact about definite integrals corresponding to separation of variables

Remember that for pure time differential equations if you wanted to determine the net amount of change in a quantity $v(t)$ between t_0 and t_1, you can integrate $\int_{t_0}^{t_1} F(t)dt$ when you are given a pure-time differential equation $\frac{dv}{dt} = F(t)$. For autonomous differential equations we use a related fact. Namely, given an autonomous differential equation

$$\frac{dv}{dt} = G(v)$$

and an initial time t_0, if $G(v(t_0)) \neq 0$, then, for any t,

$$t - t_0 = \int_{v(t_0)}^{v(t)} \frac{1}{G(v)} dv. \tag{1}$$

When $G(v(t_0)) = 0$, then $v(t) = v(t_0)$ and we are in an equilibrium state (we will talk more about such states in the next lecture).

Example: Bacteria growth

Suppose we are studying the growth of a population $b(t)$ of bacteria over time and that at time $t = 0$ there are 1 million bacteria (i.e., $b(0) = 1$). Assuming that we know our bacteria grow according to the growth equation

$$\frac{db}{dt} = kb$$

for $k > 0$ a fixed constant. Let us also assume that we know by observation that after one hour there are 2 million bacteria (i.e., $b(1) = 2$). *How many bacteria will there be after* 2.5 *hours?*

There are two ways to solve this problem: (i) by first solving the differential equation or (ii) by using the equation (1). Let us consider these in turn:

(i) By separation of variables we find that a solution b is of the form

$$b(t) = Ce^{kt}$$

for C a constant of integration. In particular, taking $b(0)$ shows that C is equal to the initial state. In this case, $C = 1$. So we may use this information to solve for k. Since $b(1) = 2$ we obtain $2 = e^k$ and, taking the natural logarithm of both sides, $\ln(2) = k$.

Now that we have found the value of k and we have a solution $b(t)$ for the differential equation we know that

$$b(2.5) = e^{2.5\ln(2)} = 2^{2.5} \approx 5.6569.$$

Thus, after 2.5 hours there are 5.6569 million bacteria.[1]

[1] Note that in the lecture I calculated instead the net change in bacteria population (4.6569) between $t = 0$ and $t = 2.5$ instead of the total number at time $t = 2.5$.

(ii) Again we first solve for the constant k, but this time we do so *without* explicitly solving the differential equation. Namely, we observe that, taking t to be 1 the equation (1) in this case yields

$$1 = \int_1^2 \frac{1}{kb} db = \frac{1}{k}\big(\ln(2) - \ln(1)\big) = \frac{1}{k}\ln(2),$$

so that we have found $k = \ln(2)$ as before.

Next, taking t in (1) gives

$$2.5 = \int_1^{b(2.5)} \frac{1}{kb} db = \frac{1}{\ln(2)}\Big(\ln|b(2.5)| - \ln(1)\Big) = \frac{1}{\ln(2)}\ln|b(2.5)|.$$

As such

$$b(2.5) = e^{2.5\ln(2)} = 2^{2.5} \approx 5.6569,$$

where we may dispense with the absolute value since $b(2.5)$ is non-negative.

Example: Freezing a metal rod

Suppose we place a metal rod which has initial temperature $H(0) = 80°C$ in a freezer which is constantly at $-20°C$. After one minute the rod has temperature $70°C$. *When will the rod have temperature* $40°C$*?* (Assume that Newton's Law of Cooling applies in this situation so that $\frac{dH}{dt} = \alpha(A - H)$.)

Again we use the equation (1) in order to find the constant α as follows:

$$1 = \int_{80}^{70} \frac{1}{\alpha(-20 - H)} dH = \frac{1}{\alpha}\big(-\ln|-90| + \ln|-100|\big)$$

so that $\alpha = \ln(\frac{100}{90}) \approx -0.1054$.

We may now solve for t with $H(t) = 40$ using (1):

$$t = \frac{1}{\ln(\frac{100}{90})}\int_{80}^{40} \frac{1}{-20 - H} dH = \frac{1}{\ln(\frac{100}{90})}\big(-\ln|-20 - 40| + \ln|-20 - 80|\big)$$

This is ≈ 4.8484. Thus, in 4.8484 minutes the rod has reaches a temperature of $40°C$.