MAT1332 Spring/Summer 2010 (Instructor: Michael A. Warren)

Partial Fractions.

Recall that a *rational function* is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials ($Q(x) \neq 0$). So a rational function has the form

$$\frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \cdots + b_m^m}$$

for some $n, m \geq 0$ and the coefficients $a_0, \ldots, a_n, b_0, \ldots, b_m$ real numbers. So, for example, the following are rational functions:

$$\frac{x^2 + 2x + 1}{2x^3 - 3x - 2}, \quad \frac{1}{x^2 + 1}, \quad \frac{x}{2x - 1}, \ldots$$

In order to be able to integrate general rational functions we will need to first perform some algebraic manipulations to ensure that the rational function we are dealing with may be rewritten in a particular form. In particular, our goal, given a rational function $f(x)$, is to reduce the problem of integrating $f(x)$ to the problem of integrating polynomials and rational functions of one of the following simple forms (or their powers):

$$\begin{aligned} \frac{1}{x}, &\quad \text{which has } \int \frac{1}{x} dx = \ln|x| + C, \\ \frac{x}{x^2+1} dx, &\quad \text{which has } \int \frac{x}{x^2+1} dx = \frac{\ln|x^2+1|}{2} + C, \text{ and} \\ \frac{1}{x^2+1}, &\quad \text{which has } \int \frac{1}{x^2+1} dx = \arctan(x) + C. \end{aligned}$$

However, prior to considering integration of rational functions, we will first describe the particular form into which a rational function should be rewritten prior to integration.

Rational Functions in the Appropriate Form

Given a rational function $f(x) = \frac{P(x)}{Q(x)}$ as above, the first thing we want to observe is that, by a theorem of algebra (which, alas, will not be studied in this course), $Q(x)$ can be written (in a unique way) as a product of linear and irreducible quadratic factors.[1] (Recall that a quadratic polynomial is *irreducible* whenever that it has negative discriminant.) I.e.,

$$Q(x) = (a_0 x + b_0)^{i_0} (a_1 x + b_1)^{i_1} \cdots (a_k x + b_k)^{i_k} (\tilde{a}_0 x^2 + \tilde{b}_0 x + \tilde{c}_0)^{j_0} \cdots (\tilde{a}_l x^2 + \tilde{b}_l x + \tilde{c}_l)^{j_l} \tag{1}$$

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[1] The relevant analogy to have in mind is the case of positive integers n, each of which has a canonical factorization as a product of powers of prime numbers. Here the linear and irreducible quadratic polynomials are playing the same rôle as prime numbers.

where the coefficients $a_0, b_0, \ldots$ are real numbers.[2] Here each linear factor $(a_q x + b_q)$ corresponds to one real root of the polynomial $Q(x)$ and the power i_q to which this factor is raised indicates that this is a repeated root of the polynomial. Similarly, each irreducible quadratic factor corresponds to a pair of immaginary roots and again the power j_q indicates to what extent these are repeated roots.

Now, assume that $f(x)$ is as above, that the degree of the polynomial $P(x)$ is strictly below the degree of $Q(x)$ and that $Q(x)$ has been factored as indicated in (1) above. Then we rewrite $f(x)$ in the following general form

$$\begin{aligned} f(x) = \Big(\sum_{i=0}^{i_0-1} \frac{A_{0,i}}{(a_0x+b_0)^{i+1}}\Big) + \Big(\sum_{i=0}^{i_1-1} \frac{A_{1,i}}{(a_1x+b_1)^{i+1}}\Big) + \cdots \qquad (2) \\ \cdots + \Big(\sum_{i=0}^{i_k-1} \frac{A_{k,i}}{(a_kx+b_k)^{i+1}}\Big) + \Big(\sum_{i=0}^{j_0-1} \frac{\tilde{A}_{0,i}x+\tilde{B}_{0,i}}{(\tilde{a}_0x^2+\tilde{b}_0x+\tilde{c}_0)^{i+1}}\Big) + \cdots + \Big(\sum_{i=0}^{j_l-1} \frac{\tilde{A}_{l,i}x+\tilde{B}_{l,i}}{(\tilde{a}_lx^2+\tilde{b}_lx+\tilde{c}_l)^{i+1}}\Big), \end{aligned}$$

where the $A_{m,n}, B_{m,n}, \tilde{A}_{m,n}, \tilde{B}_{m,n}$ are fixed real numbers. We will now give several examples of the procedure by which $f(x)$ may be rewritten in the form (2). (Note that the intuition here is that since the denominator factors as in (1) we should have obtained $f(x)$ by addition of simpler rational function.)

Example 1

Suppose

$$f(x) = \frac{10x+2}{x^2+2x-15}.$$

We first observe that the denominator can be factored as $x^2+2x-15 = (x-3)(x+5)$ and in order to put $f(x)$ in the form (2) we must fine A_0 and A_1 such that

$$\frac{10x+2}{x^2+2x-15} = \frac{A_0}{x-3} + \frac{A_1}{x+5}.$$

In order for such an equation to hold we must have

$$10x+2 \quad = \quad A_0(x+5) + A_1(x-3) \quad = \quad (A_0+A_1)x + (5A_0 - 3A_1),$$

and, in particular, the terms (in this case 10 and $(A_0 + A_1)$) which occur as multiples of x on each side of the equation must be identical. Similarly, the constant terms on each side must be equal. I.e., we obtain the equations

$$\begin{aligned} 10 &= A_0 + A_1 \\ 2 &= 5A_0 - 3A_1. \end{aligned}$$

We now solve this system of equations for A_0 and A_1 to obtain $A_0 = 4$ and $A_1 = 6$. Thus,

$$f(x) = \frac{4}{x-3} + \frac{6}{x+5}.$$

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[2] Here I simply write $\tilde{a}_0$ to denote some other coefficients: there is no connection at all between a_0 and $\tilde{a}_0$.

Example 2

Suppose

$$f(x) = \frac{7x^3 + 11x^2 + 7x + 1}{(x^2 + 2x + 2)(3x^2 + 3x + 1)}.$$

In this case the denominator is already the product of two irreducible quadratic factors and therefore (as indicated in 2) we must find A_0, B_0 and A_1, B_1 such that

$$\frac{7x^3 + 11x^2 + 7x + 1}{(x^2 + 2x + 2)(3x^2 + 3x + 1)} = \frac{A_0 x + B_0}{x^2 + 2x + 2} + \frac{A_1 x + B_1}{3x^2 + 3x + 1}.$$

This equation holds if and only if

$$\begin{aligned} 7x^3 + 11x^2 + 7x + 1 &= (A_0 x + B_0)(3x^2 + 3x + 1) + (A_1 x + B_1)(x^2 + 2x + 2) \\ &= (3A_0 + A_1)x^3 + (3A_0 + 3B_0 + 2A_1 + B_1)x^2 + (A_0 + 3B_0 + 2A_1 + 2B_1)x + (B_0 + 2B_1). \end{aligned}$$

Again, multiples of x^n on both sides must be equal ($n = 0, 1, 2, 3$) and therefore we obtain the following system of equations:

$$\begin{aligned} 7 &= 3A_0 + A_1 \\ 11 &= 3A_0 + 3B_0 + 2A_1 + B_1 \\ 7 &= A_0 + 3B_0 + 2A_1 + 2B_1 \\ 1 &= B_0 + 2B_1. \end{aligned}$$

We solve this system of equations to obtain $A_0 = 2, A_1 = B_0 = 1$ and $B_1 = 0$. So

$$f(x) = \frac{2x + 1}{x^2 + 2x + 2} + \frac{x}{3x^2 + 3x + 1}.$$

Example 3

Suppose

$$f(x) = \frac{2x^2 + 11x + 14}{x^3 + 6x^2 + 12x + 8}.$$

We factor the denominator by observing that $x = -2$ is a root of the polynomial $x^3 + 6x^2 + 12x + 8$ and therefore

$$x^3 + 6x^2 + 12x + 8 \quad = \quad (x + 2)(x^2 + 4x + 4) \quad = \quad (x + 2)(x + 2)^2.$$

As such, we must find A_0 and A_1 such that

$$\frac{2x^2 + 11x + 14}{x^3 + 6x^2 + 12x + 8} = \frac{A_0}{x + 2} + \frac{A_1}{(x + 2)^2}.$$

In particular, we must have

$$2x^2+11x+14 \quad = \quad A_0(x+2)^2+A_1(x+2) \quad = \quad A_0x^2+(4A_0+A_1)x+(4A_0+2A_1)$$

which gives the following system of equations:

$$\begin{aligned} 2 &= A_0 \\ 11 &= 4A_0+A_1 \\ 14 &= 4A_0+2A_1. \end{aligned}$$

So, substituting 2 for A_0 into either of the other equations we obtain $A_1 = 3$. Thus,

$$f(x) = \frac{2}{x+2} + \frac{3}{(x+2)^2}.$$

Integrating Rational Functions

We now turn to the question of integrating rational functions. The procedure for integrating rational functions consists roughly of the following steps:

1. Ensure that, when $f(x) = \frac{P(x)}{Q(x)}$, the degree of $P(x)$ is strictly lower than that of $Q(x)$ by carrying out long division (if necessary). This step gives $f(x) = S(x) + \frac{R(x)}{Q(x)}$ where $S(x)$ and $R(x)$ are polynomials and the degree of $R(x)$ is strictly lower than the degree of $Q(x)$ (note that $S(x)$ could be 0, in which case $R(x) = P(x)$).

2. Factorize the denominator $Q(x)$ of the remaining rational function $Q(x)$ as indicated in (1).

3. Put the rational function $\frac{R(x)}{Q(x)}$ in the form (2).

4. Integrate, the resulting expression of $f(x)$.

Before going on to completely worked examples it will be useful to first consider what the summands of the resulting expression for $f(x)$ will look like when integrated.

First, observe that a summand of the form $\frac{A}{ax+b}$ with linear denominator can be easily integrated as

$$\int \frac{A}{ax+b}dx \quad = \quad \frac{A}{a}\int \frac{1}{u}du \quad = \quad \frac{A}{a}\ln|ax+b|+c,$$

where we have used the substitution $u = ax+b$ so that $\frac{du}{dx} = a$. Next, a summand of the form $\frac{A}{(ax+b)^n}$ can be integrated as

$$\int \frac{A}{(ax+b)^n} \quad = \quad \frac{A}{a}\int \frac{1}{u^n}du \quad = \quad \frac{A}{a(1-n)(ax+b)^{n-1}}+c.$$

Next, to integrate a summand of the form $\frac{Ax+B}{ax^2+bx+c}$ requires a little bit more work. In particular, we begin by completing the square to obtain

$$ax^2+bx+c = a(x+\frac{b}{2a})^2+(c-\frac{b^2}{4a}).$$

set $h = c-\frac{b^2}{4a}$ and note that $h>0$ since ax^2+bx+c is irreducible. So (using $x = x+\frac{b}{2a}-\frac{b}{2a}$)

$$\frac{Ax+B}{ax^2+bx+c} \quad = \quad \frac{Ax+B}{a(x+\frac{b}{2a})^2+h} \quad = \quad \frac{A(x+\frac{b}{2})-\frac{Ab}{2}+B}{a(x+\frac{b}{2})^2+h}.$$

therefore

$$\int \frac{Ax+B}{ax^2+bx+c}dx \quad = \quad A\int \frac{x+\frac{b}{2}}{a(x+\frac{b}{2})^2+h}dx+(B-\frac{Ab}{2})\int \frac{1}{a(x+\frac{b}{2})^2+h}dx \tag{3}$$

now we may separately integrate the two summands of (3). First,

$$A\int \frac{x+\frac{b}{2}}{a(x+\frac{b}{2})^2+h}dx \;=\; A\int \frac{u}{au^2+h}du \;=\; \frac{A}{2a}\ln|au^2+h|+C \;=\; \frac{A}{2a}\ln|ax^2+bx+c|+C,$$

where we use the substitution $u = x+\frac{b}{2a}$. Next,

$$\begin{aligned}(B-\frac{Ab}{2})\int \frac{1}{a(x+\frac{b}{2})^2+h}dx &= (B-\frac{Ab}{2})\int \frac{1}{h\big(\frac{a}{h}(x+\frac{b}{2})^2+1\big)}dx\\ &= (\frac{B}{h}-\frac{Ab}{2h})\int \frac{1}{\big(\frac{\sqrt{a}}{\sqrt{h}}(x+\frac{b}{2})\big)^2+1}dx\\ &= (\frac{B}{h}-\frac{Ab}{2h})\int \frac{1}{u^2+1}\cdot\frac{\sqrt{h}}{\sqrt{a}}du\\ &= (\frac{B}{h}-\frac{Ab}{2h})\frac{\sqrt{h}}{\sqrt{a}}\arctan(u)+C\\ &= \frac{2B-Ab}{\sqrt{4ac-b^2}}\arctan\Big(\frac{a(2x+b)}{\sqrt{4ac-b^2}}\Big)+C,\end{aligned}$$

where we have used the substitution

$$u = \frac{\sqrt{a}}{\sqrt{h}}(x+\frac{b}{2})$$

so that $\frac{du}{dx} = \frac{\sqrt{a}}{\sqrt{h}}$, and we have simplified the expression (the reader is encouraged to go through the details of this simplification).[3] So, coming back to (3) we see that

$$\int \frac{Ax+B}{ax^2+bx+c}dx = \frac{A}{2a}\ln|ax^2+bx+c|+\frac{2B-Ab}{\sqrt{4ac-b^2}}\arctan\Big(\frac{a(2x+b)}{\sqrt{4ac-b^2}}\Big)+C$$

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[3] Note that we may assume without loss of generality that $a>0$, since the case where $a<0$ can be obtained from this case in the obvious way.

In the general case of powers of irreducible quadratic factors we proceed in roughly the same manner described above and then must say how to integrate rational functions of the forms

$$\frac{x+\frac{b}{2}}{(a(x+\frac{b}{2})^2+h)^k} \quad \text{and} \quad \frac{1}{(a(x+\frac{b}{2})^2+h)^k}.$$

Observe though that integration of these function can be reduced to the problem of integrating functions of the forms $\frac{x}{(x^2+1)^k}$ and $\frac{1}{(x^2+1)^k}$. For these observe that

$$\int \frac{x}{(x^2+1)^k}dx = \frac{1}{2(1-k)(1+x^2)^{k-1}} + C$$

by the substitution $u = x^2+1$, and, similarly,

$$\int \frac{1}{(x^2+1)^k}dx \ = \int \frac{\sec^2(u)}{(\tan^2(u)+1)^k}du \ =; \int \frac{sec^2(u)}{\sec^{2k}(u)}du \ = \int \frac{1}{\sec^{2k-2}(u)}du,$$

using the substitution $x = \tan(u)$ so that $u = \arctan(x)$. Thus,

$$\int \frac{1}{(x^2+1)^k}dx \ = \int \cos^{2k-2}(u)du$$

which can be integrated using the recurrence formula for $\int \cos^{2k-2}(u)du$ for given values of k.

Example 4

We will integrate the rational function

$$f(x) = \frac{x^4+4x^3-2x^2+19x+2}{x^2+4x-5}.$$

First, observe that we must use long division to reduce the degree of the numerator. This gives

$$f(x) \quad = \quad x^2+3+\frac{7x+17}{x^2+4x-5} \quad = \quad x^2+3+\frac{7x+17}{(x-1)(x+5)}.$$

We must now find A_0 and A_1 such that $\frac{7x+17}{(x-1)(x+5)} = \frac{A_0}{x-1} + \frac{A_1}{x+5}$. Using the method described above these are seen to be $A_0 = 4$ and $A_1 = 3$. Finally, we integrate as follows:

$$\begin{aligned}\int f(x)dx &= \frac{x^3}{3}+3x+4\int \frac{1}{x-1}dx+3\int 1x+5dx+C \\ &= \frac{x^3}{3}+3x+4\ln|x-1|+3\ln|x+5|+C.\end{aligned}$$

Example 5

We will integrate the function

$$f(x) = \frac{10x^4 - 30x^3 + 39x^2 - 17x + 15}{2x^3 - 6x^2 + 6x}.$$

Again, we must use long division to obtain the following expression for $f(x)$:

$$f(x) = 5x + \frac{9x^2 - 17x + 15}{2x^3 - 6x^2 + 6x}.$$

Note that the denominator of the remaining fractional summand can be factored as $2x(x^2 - 3x + 3)$ and we therefore must find A_0, B_0 and A_1 such that

$$\frac{9x^2 - 17x + 15}{2x(x^2 - 3x + 3)} = \frac{A_0 x + B_0}{x^2 - 3x + 3} + \frac{A_1}{2x}.$$

These are easily seen to be given by $A_0 = 2, A_1 = 5$ and $B_0 = -1$. Thus, we integrate

$$\begin{aligned} \int f(x)dx &= \frac{5}{2}x^2 + \int \frac{2x-1}{x^2 - 3x + 3}dx + 5\int \frac{1}{2x}dx + C \\ &= \frac{5}{2}x^2 + \frac{5}{2}\ln|x| + \ln|x^2 - 3x + 3| + \frac{4}{\sqrt{3}}\arctan\Big(\frac{2x-3}{\sqrt{3}}\Big) + C, \end{aligned}$$

where we have used the procedure describe above in order to integrate the summand with denominator $x^2 - 3x + 3$.

Practice Problems

1. Integrate the rational function from Example 1.
2. Integrate the rational function from Example 2.
3. Integrate the rational function from Example 3.
4. Calculate

$$\int_1^2 \frac{6x^3 - 4x + 8}{3x^2 - 9x}dx$$

5. Calculate

$$\int \frac{1}{3x^2 - x + 2}dx$$

Solutions to Practice Problems

1. We already know that $f(x) = \frac{4}{x-3} + \frac{6}{x+5}$ and therefore

$$\int f(x) \;=\; 4\int \frac{1}{x-3}dx + 6\int \frac{1}{x+5}dx \;=\; 4\ln|x-3| + 6\ln|x+5| + C.$$

2. We already know that $f(x) = \frac{2x+1}{x^2+2x+2} + \frac{x}{3x^2+3x+1}$ and therefore

$$\begin{aligned}\int f(x) &= \int \frac{2x+1}{x^2+2x+2}dx + \int \frac{x}{3x^2+3x+1}dx \\ &= \ln|x^2+2x+2| - \arctan(x+1) + \frac{1}{6}\ln|3x^2+3x+1| - \frac{1}{\sqrt{3}}\arctan(\sqrt{3}(2x+1)) + C.\end{aligned}$$

3. We already know that $f(x) = \frac{2}{x+2} + \frac{3}{(x+2)^2}$ and therefore

$$\int f(x)dx \quad = \quad 2\int \frac{1}{x+2}dx + 3\int \frac{1}{(x+2)^2}dx \quad = \quad 2\ln|x+2| - \frac{3}{x+2} + C.$$

4. Long division gives

$$\frac{6x^3 - 4x + 8}{3x^2 - 9x} = 2x + 6 + \frac{50x + 8}{3x^2 - 9x}.$$

The denominator has two distinct real roots and in particular $3x^2 - 9x = 3x(x-3)$. We would like to find A_0 and A_1 such that $\frac{50x+8}{3x^2-9x} = \frac{A_0}{3x} + \frac{A_1}{(x-3)}$. So we must solve the equations

$$\begin{aligned}A_0 + 3A_1 &= 50 \\ -3A_0 &= 8\end{aligned}$$

for A_0 and A_1. Threfore, $A_0 = -\frac{8}{3}$ and $A_1 = \frac{158}{9}$. Thus,

$$\begin{aligned}\int \frac{6x^3 - 4x + 8}{3x^2 - 9x}dx &= x^2 + 6x - \frac{8}{9}\int \frac{1}{x}dx + \frac{158}{9}\int \frac{1}{x-3}dx + C \\ &= x^2 + 6x - \frac{8}{9}\ln|x| + \frac{158}{9}\ln|x-3| + C,\end{aligned}$$

and we conclude that

$$\begin{aligned}\int_1^2 \frac{6x^3 - 4x + 8}{3x^2 - 9x}dx &= 9 - \frac{1}{9}\big(8\ln|2| + 158\ln|2|\big) \\ &= 9 - \frac{166}{9}\ln|2|\end{aligned}$$

which is ≈ -3.7847.

5. The polynomial $3x^2 - x + 2$ has only imaginary roots (the discriminant is negative). As such, we complete the square to obtain $3x^2 - x + 2 = 3(x - \frac{1}{6})^2 + \frac{23}{12}$ and note that

$$\int \frac{1}{3x^2 - x + 2} dx = \int \frac{1}{3(x - \frac{1}{6})^2 + \frac{23}{12}} dx = \frac{12}{23} \int \frac{1}{\left(\frac{6}{\sqrt{23}}\right)^2 (x - \frac{1}{6})^2 + 1} dx.$$

Substitution with $u := \frac{6}{\sqrt{23}}(x - \frac{1}{6})$ gives

$$\int \frac{1}{3x^2 - x + 2} dx = \frac{2\sqrt{23}}{23} \int \frac{1}{u^2 + 1} du = \frac{2}{\sqrt{23}} \arctan\left(\frac{6x - 1}{\sqrt{23}}\right) + C.$$