MAT1332 Spring/Summer 2010 (Instructor: Michael A. Warren)

Further notes and practice problems on systems of differential equations.

These notes are meant to compliment Prof. Lutscher's notes on systems of differential equations and as such the two should be read in conjunction.

Basic idea

We would like to be able to solve first-order systems of linear homogeneous differential equations with constant coefficients. I.e., systems of the form

$$\begin{cases} \frac{dv}{dt} &= av + bw \\ \\ \frac{dw}{dt} &= cv + dw \end{cases}$$

where a, b, c and d are constants. As usual, a *solution* to such a system is given by a pair of functions f and g for which the equations are satisfied. I.e., such that $f' = af + bg$ and $g' = cf + dg$. We may also impose initial conditons $v(0) = v_0$ and $w(0) = w_0$ for v_0 and w_0 constants. In that case, f and g must also satisfy $f(0) = v_0$ and $g(0) = w_0$.

Rather than tackle all of this head-on, we will instead utilize some of the machinery (matrices) we have been adding to our toolbox. To begin with, we may rewrite the entire system of equations in vector form as

$$\frac{d}{dt}\mathbf{u} = A\mathbf{u} \tag{1}$$

where A denotes the coefficient matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and the column vector $\mathbf{u} = \begin{bmatrix} v \\ w \end{bmatrix}$ denotes the unknowns. In this setting we may see that the problem of solving (**??**) can be reduced to the problem of determining the eigenvalues of A and the corresponding eigenvectors. To see that this is the case, assume that λ is an eigenvalue of A with a corresponding eigenvector $\mathbf{x}$. Then, setting $\mathbf{u} = e^{\lambda t}\mathbf{x}$, we have

$$\frac{d}{dt}\mathbf{u} = e^{\lambda t}\lambda\mathbf{x} = e^{\lambda t}A\mathbf{x} = A(\mathbf{u})$$

where the second equation is by the fact that $\mathbf{x}$ is an eigenvector and third equation is by the fact that A is linear (and therefore commutes with scalar multiplication). Thus, $\mathbf{u}$ is a solution of the system.

Before turning to the question of initial conditions we first mention the following important fact: *all linear combinations of solutions of (**??**) are also solutions.* I.e., if $\mathbf{u}$ and $\mathbf{t}$ are solutions and α, β are scalars, then $\alpha\mathbf{u}+\beta\mathbf{t}$ is also a solution. (However, once we impose initial conditions we have restricted to a single solution.)

The idea behind solving for intial conditions is similar to what we did with Markov chains when we wanted to calculate the result of iteratively applying the transition matrix to intial conditions. Namely, given the initial conditions $v(0) = v_0$ and $w(0) = w_0$, we would like to write the vector $\mathbf{u}_0 = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}$ of initial conditions as a linear combination of the eigenvectors of the matrix A. I.e., where μ is the other eigenvalue of A with corresponding eigenvector $\mathbf{y}$, we would like to find scalars α and β such that

$$\mathbf{u}_0 = \alpha\mathbf{x}+\beta\mathbf{y}.$$

Given such α and β, we then have that the solution

$$\alpha\mathbf{x}e^{\lambda t}+\beta\mathbf{y}e^{\mu t}$$

also satisfies the initial conditions.

Example: Distinct real eigenvalues

Consider the system

$$\begin{cases} \frac{dv}{dt} = 2v - 2w \\ \frac{dw}{dt} = 3v + 7w \end{cases}$$

with initial conditions $v_0 = 2$ and $w_0 = -4$.

The coefficient matrix $A = \begin{bmatrix} 2 & -2 \\ 3 & 7 \end{bmatrix}$ has characteristic polynomial $\det(A-\lambda I) = \lambda^2 - 9\lambda + 20$ which factors as $(\lambda-5)(\lambda-4)$. So the eigenvalues are $\lambda = 5$ and $\mu = 4$. We may solve for the eigenvectors as usual to find that the eigenvectors corresponding to λ are scalar multiples of $\mathbf{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and the eigenvectors corresponding to μ are scalar multiples of $\mathbf{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Finally, in order to find a solution also for the initial condition we must solve the equation $\mathbf{u}_0 = \alpha\mathbf{x}+\beta\mathbf{y}$ for α and β, where $\mathbf{u}_0 = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ is the initial condition vector. We may solve this we would usually solve a system of equations:

$$\left[\begin{array}{cc|c} -2 & -1 & 2 \\ 3 & 1 & -4 \end{array}\right] \xrightarrow{\frac{3}{2}R1+R2} \left[\begin{array}{cc|c} -2 & -1 & 2 \\ 0 & -\frac{1}{2} & -1 \end{array}\right]$$

So $-\frac{1}{2}\beta = -1$ and therefore $\beta = 2$. By the top row $-2\alpha - \beta = 2$ so substituting 2 for β we find $\alpha = -2$. Therefore, the solution of the system of differential equations with these intitial conditions is given by

$$\mathbf{u} = -2\mathbf{x}e^{5t} + 2\mathbf{y}e^{4t}.$$

Explicitly, this means that $v = 4e^{5t} - 2e^{4t}$ and $w = -6e^{5t} + 2e^{4t}$.

Case: A has complex eigenvalues

When A has complex eigenvalues they are necessarily of the form $\gamma \pm \delta i$ and the corresponding eigenvectors are of the form $\mathbf{x} \pm \mathbf{y}i$. In this case, the solution sketched above is a complex solution and we would like to find the corresponding real solutions. For this, we consider one eigenvalue $\lambda = \gamma + \delta i$ and the corresponding eigenvector $\mathbf{z} = \mathbf{x} + \mathbf{y}i$. Then the complex solution is $\mathbf{z}e^{\lambda t} = \mathbf{z}e^{\gamma t + \delta t i} = \mathbf{z}e^{\gamma t}e^{\delta t i}$ which is equal by Euler's formula to

$$\mathbf{z}e^{\gamma t}(\cos(\delta t) + i\sin(\delta t)) \;=\; e^{\gamma t}(\mathbf{x}\cos(\delta t) - \mathbf{y}\sin(\delta t)) + ie^{\gamma t}(\mathbf{x}\sin(\delta t) + \mathbf{y}\cos(\delta t)).$$

Setting $\mathbf{u}_1 = e^{\gamma t}(\mathbf{x}\cos(\delta t) - \mathbf{y}\sin(\delta t))$ and $\mathbf{u}_2 = e^{\gamma t}(\mathbf{x}\sin(\delta t) + \mathbf{y}\cos(\delta t))$ we see that these are both solutions of the system (**??**) since

$$0 \;=\; \frac{d}{dt}\mathbf{z}e^{\lambda t} \;=\; (\frac{d}{dt}\mathbf{u}_1 - A\mathbf{u}_1) + i(\frac{d}{dt}\mathbf{u}_2 - A\mathbf{u}_2)$$

by linearity of $\frac{d}{dt}$ and A.

Now, if we impose initial conditions as above, then the corresponding solution is given by $\alpha\mathbf{u}_1 + \beta\mathbf{u}_2$ where α and β are constants such that $\mathbf{u}_0 = \alpha\mathbf{x} + \beta\mathbf{y}$. Note that in this case, where the eigenvalues are complex, we do not need to consider the other eigenvalue at all (both give the same solution).

Example: Complex case

Consider the system

$$\begin{cases} \frac{dv}{dt} &=& 2v &-& 4w \\ \frac{dw}{dt} &=& 3v &-& 2w \end{cases}$$

with initial conditions $v_0 = 2$ and $w_0 = 9$.

In this case, the characteristic polynomial $\det(A - \lambda I)$ of the coefficient matrix A of the system is $\lambda^2 + 8$ and so the eigenvalues are $\pm 2i\sqrt{2}$. We will consider just the case of $\lambda = 2i\sqrt{2}$ (remember, as mentioned above, the other choice of eigenvalue gives the same solution). We solve for the eigenvectors as follows:

$$\begin{bmatrix} 2-2i\sqrt{2} & -4 \\ 3 & -2-2i\sqrt{2} \end{bmatrix} \xrightarrow{-\frac{3}{2-2i\sqrt{2}}R1+R2} \begin{bmatrix} 2-2i\sqrt{2} & -4 \\ 0 & 0 \end{bmatrix}$$

where $-2-2i\sqrt{2} + \frac{12}{2-2i\sqrt{2}} = 0$ since

$$-2-2i\sqrt{2} \;=\; \frac{(2-2i\sqrt{2})(-2-2i\sqrt{2})}{2-2i\sqrt{2}} \;=\; \frac{-4-4i\sqrt{2}+4i\sqrt{2}-8}{2-2i\sqrt{2}} \;=\; \frac{-12}{2-2i\sqrt{2}}.$$

So we have an equation $(2-2i\sqrt{2})x-4y=0$ and the variable y is free. We set $y=3$ and obtain $x=2+2i\sqrt{2}$ since $(2+2i\sqrt{2})(2-2i\sqrt{2})=12$. Thus, the eigenvectors corresponding to $\lambda=2i\sqrt{2}$ are scalar multiples of $\mathbf{z}=\begin{bmatrix} 2+2i\sqrt{2} \\ 3 \end{bmatrix}=\begin{bmatrix} 2 \\ 3 \end{bmatrix}+i\begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}$.

Now, to solve for the initial conditions we must find α and β such that

$$\begin{bmatrix} 2 \\ 9 \end{bmatrix}=\alpha\begin{bmatrix} 2 \\ 3 \end{bmatrix}+\beta\begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}$$

This is solved as usual and we find that $\alpha=3$ and $\beta=-\sqrt{2}$ since $2\sqrt{2}\beta=-4$ and $-\frac{2}{\sqrt{2}}=-\sqrt{2}$. Thus, the solution of this system with these initial conditions is given by

$$\begin{aligned}
\mathbf{u} &= 3\Big(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\cos(2\sqrt{2}t)-\begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}\sin(2\sqrt{2}t)\Big)-\sqrt{2}\Big(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\sin(2\sqrt{2}t)+\begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}\cos(2\sqrt{2}t)\Big) \\
&= \begin{bmatrix} 6 \\ 9 \end{bmatrix}\cos(2\sqrt{2}t)-\begin{bmatrix} 6\sqrt{2} \\ 0 \end{bmatrix}\sin(2\sqrt{2}t)-\begin{bmatrix} 2\sqrt{2} \\ 3\sqrt{2} \end{bmatrix}\sin(2\sqrt{2}t)-\begin{bmatrix} 4 \\ 0 \end{bmatrix}\cos(2\sqrt{2}t) \\
&= \begin{bmatrix} 2 \\ 9 \end{bmatrix}\cos(2\sqrt{2}t)-\begin{bmatrix} 8\sqrt{2} \\ 3\sqrt{2} \end{bmatrix}\sin(2\sqrt{2}t).
\end{aligned}$$

Explicitly, $v=2\cos(2\sqrt{2}t)-8\sqrt{2}\sin(2\sqrt{2}t)$ and $w=9\cos(2\sqrt{2}t)-3\sqrt{2}\sin(2\sqrt{2}t)$.

Case: A has a repeated real eigenvalue

In the case where the coefficient matrix A has a repeated real eigenvalue λ (i.e., both eigenvalues are the same) we first find the corresponding eigenvector $\mathbf{x}$ for this eigenvalue. We then solve $(A-\lambda I)\mathbf{y}=\mathbf{x}$ for the vector $\mathbf{y}$. Then when the initial condition vector $\mathbf{u}_0$ is a linear combination $\alpha\mathbf{x}+\beta\mathbf{y}$ the solution of the system is given by

$$\mathbf{u}=\alpha\mathbf{x}e^{\lambda t}+\beta(\mathbf{x}t+\mathbf{y})e^{\lambda t}.$$

Example: Repeated real root

Consider the system

$$\begin{cases} \frac{dv}{dt} = -v + w \\ \frac{dw}{dt} = -v - 3w \end{cases}$$

with initial conditions $v_0=3$ and $w_0=6$.

The characteristic polynomial of the coefficient matrix A is $\det(A-\lambda I)=\lambda^2+4\lambda+4=(\lambda+2)^2$ and therefore $\lambda=-2$ is a repeated real eigenvalue of the matrix. The corresponding eigenvector is given by scalar multiples of $\mathbf{x}=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as you should convince yourself. We would now like to solve for $\mathbf{y}$ with $(A-\lambda I)\mathbf{y}=\mathbf{x}$. We work with the corresponding augmented matrix as follows:

$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ -1 & -1 & 1 \end{array}\right]\xrightarrow{R1+R2}\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, the corresponding equation is $x+y=-1$ with y a free variable and we obtain therefore, by setting $y=1$, that $\mathbf{y}=\begin{bmatrix}-2\\1\end{bmatrix}$.[1] Thus all that remains is to solve for α and β so that $\mathbf{u}_0=\alpha\mathbf{x}+\beta\mathbf{y}$. You should convince yourself that $\alpha=15$ and $\beta=-9$ so that the solution is given by

$$\begin{aligned}\mathbf{u} &= 15\mathbf{x}e^{-2t}-9(\mathbf{x}t+\mathbf{y})e^{-2t}\\ &= \begin{bmatrix}-15\\15\end{bmatrix}e^{-2t}-\begin{bmatrix}-9\\9\end{bmatrix}te^{2t}+\begin{bmatrix}18\\-9\end{bmatrix}e^{-2t}\\ &= \begin{bmatrix}3\\6\end{bmatrix}e^{-2t}-\begin{bmatrix}-9\\9\end{bmatrix}te^{-2t}\end{aligned}$$

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[1] Note that here the x and y have nothing to do with the vectors $\mathbf{x}$ and $\mathbf{y}$ and are only functioning as “local” variables to use when solving $(A-\lambda I)\mathbf{y}=\mathbf{x}$.

Practice problems

1. Solve

$$\left\{\begin{array}{rcrcr} \frac{dv}{dt} & = & -2v & - & w \\ \frac{dw}{dt} & = & 5v & - & 4w \end{array}\right.$$

with initial conditions $v_0 = -1$ and $w_0 = 5$.

2. Solve

$$\left\{\begin{array}{rcrcr} \frac{dv}{dt} & = & 2v & - & 6w \\ \frac{dw}{dt} & = & -6v & - & 3w \end{array}\right.$$

with initial conditions $v_0 = 1$ and $w_0 = -5$.

3. Solve

$$\left\{\begin{array}{rcrcr} \frac{dv}{dt} & = & v & - & 2w \\ \frac{dw}{dt} & = & 2v & - & 3w \end{array}\right.$$

with initial conditions $v_0 = 0$ and $w_0 = 3$.

4. Solve

$$\left\{\begin{array}{rcrcr} \frac{dv}{dt} & = & -2v & + & 3w \\ \frac{dw}{dt} & = & -4v & + & 4w \end{array}\right.$$

with initial conditions $v_0 = -4$ and $w_0 = 2$.

5. Solve

$$\left\{\begin{array}{rcrcr} \frac{dv}{dt} & = & 4v & - & 7w \\ \frac{dw}{dt} & = & -1v & - & 2w \end{array}\right.$$

with initial conditions $v_0 = 5$ and $w_0 = -3$.

Solutions to practice problems

1. $v = -e^{-3t}\big(\cos(2t) + 3\sin(2t)\big)$, $w = 5e^{-3t}\big(\cos(2t) - \sin(2t)\big)$.
2. $v = -2e^{-7t} + 3e^{6t}$, $w = -3e^{-7t} - 2e^{6t}$.
3. $v = -6e^{-t}t$, $w = 3e^{-t} - 6e^{-t}t$.
4. $v = 2e^{t}\big(-2\cos(\sqrt{3}t) + 3\sqrt{3}\sin(\sqrt{3}t)\big)$, $w = \frac{2}{3}e^{t}\big(3\cos(\sqrt{3}t) + 11\sqrt{3}\sin(\sqrt{3}t)\big)$.
5. $v = -\frac{15}{2}e^{-3t} + \frac{21}{2}e^{5t}$, $w = -\frac{15}{2}e^{-3t} - \frac{3}{2}e^{5t}$.