MAT1332 Spring/Summer 2009 (Instructor: Michael A. Warren)

Further notes and practice problems on multivariate calculus.

In these notes we provide some further detail regarding some of the material covered in class on multivariate calculus. You sould also consult the notes by Prof. Lutscher posted on the course website for further details.

Basics

In single-variable calculus we consider functions $f(x)$ which take as input a single real number and output another real number. So $f(x) := x^2$ takes as input a number and returns as output its square. There are immediate ways in which this approach can be generalized:

1. We may consider functions $f(x_1, x_2, \ldots, x_n)$ which take as input many (in this case n) real numbers and give as output a single real number.
2. We may consider functions $f(x)$ which take a single real number as input and give as output a *tuple* or *vector* of real numbers.

Denoting by $\mathbb{R}$ the set of real numbers, we indicate that f is a function of type (1) by writing $f : \mathbb{R}^n \to \mathbb{R}$ (f has domain the set of n-tuples of real numbers and as codomain the set of real numbers). Similarly, we write $f : \mathbb{R} \to \mathbb{R}^m$ to indicate that f is a function of type (2). Here are some examples:

- As an example of (1), we might have $f(x, y) := x^2 + y^2$. Then, say, $f(3,4) = 25$, $f(1,7) = 15$, et cetera.
- As an example of (2), we may consider the function

  $$g(x) := \begin{bmatrix} x^2 \\ 2x \end{bmatrix}$$

  Before giving an example of what happens when we apply g to a real number it is worth emphasizing that we may also think of the output of $g(x)$ as being a pair of real numbers $g(x) := (x^2, 2x)$ and we will sometimes switch between these two views (vector versus tuple) at our convenience. So, in particular,

  $$g(2) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \quad g(3) = \begin{bmatrix} 9 \\ 6 \end{bmatrix}, \quad g(4) = \begin{bmatrix} 16 \\ 8 \end{bmatrix}, \quad \text{et cetera.}$$

Finally, we may combine these generalizations to consider functions $f(x_1, \ldots, x_n)$ which take *both* multiple inputs (in this case n many) and return as their output a vector:

$$f(x_1, \ldots, x_n) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ f_2(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{bmatrix}$$

Here each $f_i(x_1, \ldots, x_n)$, for $i = 1, \ldots, m$, is itself a function of n variables which returns a single real number (i.e., it is a function of type (1) from above).

Directional derivatives

We assume that the reader is already familiar with partial derivatives (see Prof. Lutscher's notes). Recall that, given a function $f(x,y)$ of two variables (which returns just a single real number as output), the geometric interpretation of $\frac{\partial f}{\partial x}$ is as the slope of the line tangent to the surface $f(x,y)$ in the direction of the x-axis. I.e., at a point (a,b) in the domain of f, $f_x(a,b)$ is the slope of the line tangent to $f(x,y)$ at (a,b) in the direction parallel to the x-axis. Similarly, $f_y(a,b)$ is the corresponding slope in the direction parallel to the y-axis. In terms of this picture, *directional derivatives* tell us the slope of f at (a,b) in the direction of an arbitrary unit vector $\mathbf{u} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in the plane. By definition, the *directional derivative of f at (a,b) in the direction of* $\mathbf{u}$ (if it exists) is

$$\lim_{h\to 0} \frac{f(a+hv_1, b+hv_2) - f(a,b)}{h}.$$

The directional derivative of f at (a,b) in direction $\mathbf{u}$ is denoted $\frac{d}{d\mathbf{u}} f(a,b)$ or $D_{\mathbf{u}} f(a,b)$. Given f, (a,b) and $\mathbf{u}$ as above, the directional derivative has a more elegant presentation as

$$D_{\mathbf{u}} f(a,b) = f_x(a,b)v_1 + f_y(a,b)v_2,$$

(note that the gradient of f is lurking in this expression). To see that this is the case, we observe that $D_{\mathbf{u}} f(a,b)$ is actually the usual derivative $g'(t)$, at $t=0$, of a functions $g(t)$ of a single variable. Namely, $g(t) := f(a+tv_1, b+tv_2)$. Thus, applying to the chain rule to calculate $g'(0)$ gives the desired expression for the directional derivative (see below for the general form of the chain rule).

For example, if $f(x,y) := 5x + \sqrt{3y}$ and $\mathbf{u}$ is the vector $\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$, then

$$\begin{aligned} D_{\mathbf{u}} f(x,y) &= 5 \cdot \frac{3}{5} + \frac{\sqrt{3}}{2\sqrt{y}} \cdot \frac{4}{5} \\ &= 3 + \frac{2\sqrt{3}}{5\sqrt{y}}. \end{aligned}$$

So, e.g., $D_{\mathbf{u}} f(5,4) = 3 + \frac{\sqrt{3}}{5}$.

Chain rules

We here state several forms of the chain rule. Throughout we assume that the functions involved possess the required partial derivatives.

1. Given a real-valued function $f(x,y)$ of two variables, together with two real-valued functions $g(x)$ and $h(x)$ of a single variable, we may form the function $h(x) := f(g(x), h(x))$ of a single variable. The chain rule for this case states that

$$h'(x) = f_x(g(x),h(x))g'(x) + f_y(g(x),h(x))h'(x).$$

2. Given a real-valued function $f(x,y)$ of two variables, together with two real-valued functions $g(x,y)$ and $h(x,y)$ of two variables, we may form the real-valued function $h(x,y) := f(g(x,y),h(x,y))$ of two variables. The chain rule for this case states that

$$\begin{aligned} h_x(x,y) &= f_x(g(x,y),h(x,y))g_x(x,y) + f_y(g(x,y),h(x,y))h_x(x,y), \text{ and} \\ h_y(x,y) &= f_x(g(x,y),h(x,y))g_y(x,y) + f_y(g(x,y),h(x,y))h_y(x,y). \end{aligned}$$

Practice problems

For problems (1)-(3) let $f(x,y) := y\tan(3x) - x^2\sec(y) + 4x^3y$.

1. Calculate $\frac{\partial f}{\partial x}$.
2. Calculuate $\frac{\partial f}{\partial xy}$.
3. Calculate $\frac{\partial f}{\partial yyx}$.
4. Let $g(x,y,z) := 3xyz + 2x^2\sqrt{z} - \frac{2\sqrt{x^2-z^3}}{y^2}$. Calculate $\frac{\partial f}{\partial xyz}$.
5. Let $f(x,y) := y - x$, $h(x,y) := \sin(x+y)y$ and let $k(x,y) := x^2 - xy$. If $f(x,y)$ is as above, calculate $\frac{\partial}{\partial x}h(f(x,y),k(x,y))$ and $\frac{\partial}{\partial y}k(h(x,y),f(x,y)$.

For problems (6)-(8) let $f(x,y) := x^2y - y^2x + 3xy + 5$.

6. Let $\mathbf{u} := \begin{bmatrix} \frac{7}{\sqrt{130}} \\ \frac{9}{\sqrt{130}} \end{bmatrix}$, calculate $D_{\mathbf{u}}f(3,6)$.
7. Let $\mathbf{u} := \begin{bmatrix} \frac{1}{5\sqrt{2}} \\ \frac{7}{5\sqrt{2}} \end{bmatrix}$, calculate $D_{\mathbf{u}}f(1,1)$.
8. Let $\mathbf{u} := \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, calculate $D_{\mathbf{u}}f(4,8)$.

Solutions to practice problems

1. $3y\sec(3x)^2 - 2x\sec(y) + 12x^2y$.
2. $3\sec(3x)^2 - 2x\sec(y)\tan(y) + 12x^2$.
3. $-2x\sec(y)^3 - 2x\sec(y)\tan(y)^2$.
4. $3 + \frac{6xz^2}{y^3(x^2-z^3)^{\frac{3}{2}}}$.
5. We have that $\frac{\partial}{\partial x}h(f(x,y),k(x,y))$ is equal to

$$-(2x-y)\sin(-x^2+x-y+xy) - (1-2x+y)(x^2-xy)\cos(-x^2+x-y+xy).$$

6. $-18\sqrt{\frac{2}{65}}$.
7. $\frac{9\sqrt{2}}{5}$.
8. $-6\sqrt{2}$.