MAT1332 Spring/Summer 2010

Midterm Exam 2 Solutions

1. [5 pts] The growth in number of individuals N of a certain population is described by the equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right),$$

where r and K are positive constants. Without solving this equation, find the equilibria and their stability, and draw the phase line diagram. Explain the meaning of the parameters r and K and the interpretation of the equilibria.

Solution: $rN(1-\frac{N}{K}) = 0$ if and only if $N = 0$ or $1 - \frac{N}{K} = 0$. Therefore the equilibria are $N_* = 0, K$. If $F(N) = rN(1-\frac{N}{K})$, then $F'(N) = r - \frac{2rN}{K}$. So $F'(0) = r > 0$ and $F'(K) = r - 2r = -r < 0$. As such, 0 is unstable and K is stable. The arrows in the phase line diagram should have the following orientation: $\leftarrow 0 \rightarrow K \leftarrow$. r indicates the rate of reproduction and K determines the position of the stable equilibrium (i.e., the biologically ideal state of the system).

2. [3 pts] We consider the complex numbers $z = i(3-4i)$ and $w = i^3$. Calculate zw, $|z|$ and $z^4\overline{z}^4$.

Solution: $zw = 3 - 4i$, $|z| = 5$, $z^4\bar{z}^4 = (|z|^2)^4 = 25^4 = 390625$.

3. [5 pts] Solve the following system. Check your answer.

$$\begin{array}{rcrcrcl} 2x_1 & - & x_2 & + & x_3 & = & 10 \\ -x_1 & + & 3x_2 & + & x_3 & = & 0 \\ & & x_2 & + & 2x_3 & = & 9. \end{array}$$

Solution: We reduce the matrix to reduced row echelon form as follows:

$$
\left[\begin{array}{ccc|c} 2 & -1 & 1 & 10 \\ -1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 9 \end{array}\right]
\xrightarrow{\frac{1}{2}R_1 \to R_1}
\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 5 \\ -1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 9 \end{array}\right]
\xrightarrow{R_1+R_2 \to R_2}
\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 5 \\ 0 & \frac{5}{2} & \frac{3}{2} & 5 \\ 0 & 1 & 2 & 9 \end{array}\right]
$$

$$
\xrightarrow{\frac{2}{5}R_2 \to R_2}
\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 5 \\ 0 & 1 & \frac{3}{5} & 2 \\ 0 & 1 & 2 & 9 \end{array}\right]
\xrightarrow{-R_2+R_3 \to R_3}
\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 5 \\ 0 & 1 & \frac{3}{5} & 2 \\ 0 & 0 & \frac{7}{5} & 7 \end{array}\right]
$$

$$
\xrightarrow{\frac{5}{7}R_3 \to R_3}
\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 5 \\ 0 & 1 & \frac{3}{5} & 2 \\ 0 & 0 & 1 & 5 \end{array}\right]
\xrightarrow{-\frac{3}{5}R_3+R_2 \to R_2}
\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right]
$$

$$
\xrightarrow{-\frac{1}{2}R_3+R_1 \to R_1}
\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & \frac{5}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right]
\xrightarrow{\frac{1}{2}R_2+R_1 \to R_1}
\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right].
$$

Switching back to equation notation gives

$$
\begin{aligned}
x_1 &= 2 \\
x_2 &= -1 \\
x_3 &= 5
\end{aligned}
$$

and so the system has the unique solution $(2,-1,5)$. To check our answer, we substitute this solution back into the original equations:

$$
\begin{array}{ccccccc}
2\cdot 2 & - & (-1) & + & 5 & = & 10 \\
-2 & + & 3\cdot(-1) & + & 5 & = & 0 \\
 & & (-1) & + & 2\cdot 5 & = & 9.
\end{array}
$$

Since each substitution yields a true statement, we have verified that $(2,-1,5)$ is indeed a solution.

4. **[5 pts]** Does the following system have a solution or not? If it does, then what find it.

$$\begin{aligned} x_1 + 2x_3 + 5x_4 + 7x_5 &= x_2 + 1 \\ -x_1 + 2x_2 &= x_2 + x_4 + x_5 - 3 \\ x_1 - x_2 + x_3 + 6x_4 + 5x_5 - 5 &= -x_1 + x_2 + 2x_4 \end{aligned}$$

Solution: We first write the system in standard form:

$$\begin{array}{rcrcrcrcrcr} x_1 & - & x_2 & + & 2x_3 & + & 5x_4 & + & 7x_5 & = & 1 \\ -x_1 & + & x_2 & & & - & x_4 & - & x_5 & = & -3 \\ 2x_1 & - & 2x_2 & + & x_3 & + & 4x_4 & + & 5x_5 & = & 5 \end{array}$$

Then write down the augmented matrix and row reduce to RREF:

$$\left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 5 & 7 & 1 \\ -1 & 1 & 0 & -1 & -1 & -3 \\ 2 & -2 & 1 & 4 & 5 & 5 \end{array}\right] \xrightarrow[-2R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 5 & 7 & 1 \\ 0 & 0 & 2 & 4 & 6 & -2 \\ 0 & 0 & -3 & -6 & -9 & 3 \end{array}\right]$$

$$\xrightarrow{\frac{3}{2}R_2+R_3 \to R_3} \left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 5 & 7 & 1 \\ 0 & 0 & 2 & 4 & 6 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 5 & 7 & 1 \\ 0 & 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_1 \to R_1} \left[\begin{array}{rrrrr|r} 1 & -1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Switching back to equation notation gives:

$$\begin{array}{rcrcrcrcrcr} x_1 & - & x_2 & & & + & x_4 & + & x_5 & = & 3 \\ & & & & x_3 & + & 2x_4 & + & 3x_5 & = & -1 \end{array}$$

Solving for the basic variables in terms of the free variables, we obtain the general solution:

$$\begin{aligned} x_1 &= x_2 - x_4 - x_5 + 3 \\ x_3 &= -2x_4 - 3x_5 - 1 \\ x_2, x_4, x_5 & \quad \text{free} \end{aligned}$$

5. [4 pts] Suppose a linear system has augmented matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & p+5 & q+9 \end{array}\right]$$

for some real numbers p and q. For which values of p and q does the system have:

(a) No solution?

(b) Exactly one solution?

(c) Infinitely many solutions?

Your answers should include all possibilities for the values of p and q. Write your final answer in the spaces at the bottom of the page.

Solution: We row reduce the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & p+5 & q+9 \end{array}\right] \xrightarrow{-3R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & p-4 & q+3 \end{array}\right]$$

This has a pivot in the last column (and hence no solution) when $p-4=0$ and $q+3 \neq 0$. In other words, when $p=4$ and $q \neq -3$.

The reduced matrix has pivots in both columns (and hence a unique solution) when $p-4 \neq 0$. In other words, when $p \neq 4$ (q can be any real number).

The system is consistent with a free variable (and hence has infinitely many solutions) when $p-4=0$ and $q+3=0$. In other words, when $p=4$ and $q=-3$.

6. [6 pts] Calculate AB, $-2A$ and A^T+B where

$$A = \begin{bmatrix} 3 & 0 \\ 2 & -1 \\ 1 & 4 \end{bmatrix}, \quad \text{et} \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 1 \\ 1 & 8 & -13 \end{bmatrix}, -2A = \begin{bmatrix} -6 & 0 \\ -4 & 2 \\ -2 & -8 \end{bmatrix} \text{ and } A^T+B = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

7. **[4 points]** Let M be the following matrix:

$$M = \begin{bmatrix} -1 & 2 & -3 \\ 3 & -2 & 5 \\ 1 & 0 & 2 \end{bmatrix}.$$

Is M invertible? If yes, find M^{-1}.

Solution: We form the superaugmented matrix and row reduce:

$$\left[\ M \mid I\ \right] = \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 3 & -2 & 5 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_1\to R_3]{R_2+3R_1\to R_2} \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 4 & -4 & 3 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-\frac{1}{2}R_2\to R_3} \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 4 & -4 & 3 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1 \end{array}\right] \xrightarrow[R_1+3R_3\to R_1]{R_2+4R_3\to R_2} \left[\begin{array}{ccc|ccc} -1 & 2 & 0 & -1/2 & -3/2 & 3 \\ 0 & 4 & 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2\to R_2} \left[\begin{array}{ccc|ccc} -1 & 2 & 0 & -1/2 & -3/2 & 3 \\ 0 & 1 & 0 & 1/4 & -1/4 & 1 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1 \end{array}\right] \xrightarrow{R_1-2R_2\to R_1} \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1/4 & -1/4 & 1 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1 \end{array}\right]$$

$$\xrightarrow{(-1)R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1/4 & -1/4 & 1 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1 \end{array}\right]$$

Since M is row equivalent to the identity matrix, it is invertible. The inverse is

$$M^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1/4 & -1/4 & 1 \\ -1/2 & -1/2 & 1 \end{bmatrix}.$$