MAT1332 Spring/Summer 2009 (Instructor: Michael A. Warren)

Handout: Improper integrals.

There are two "versions" of the improper integral. For the first we are given a function $f(x)$ which is defined and integrable in the interval consisting of all x such that $a < x \leq b$, but for which $\lim_{x \to a} f(x) = \infty$. For example, the function $f(x) := \frac{1}{x}$ has exhibits such behavior at the point $x = 0$ (cf. Figure 1).

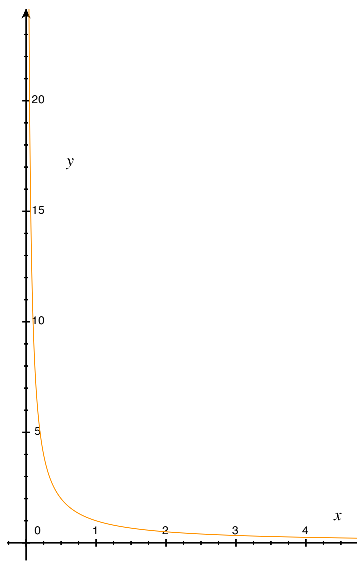

Figure 1: $f(x) = \frac{1}{x}$

In such cases, although it does not make sense to consider the definite integral $\int_a^b f(x)dx$ (defined as before as the limit of Riemann sums), nonetheless the area under $f(x)$ between a and b can be approximated by taking diminishing values of $\epsilon > 0$ and then calculating the quantity $\int_{a+\epsilon}^b f(x)dx$ (which is a genuine real number). This gives better and better (as ϵ becomes smaller and smaller) approximations of the area under the curve $f(x)$ between a and b. The *actual* area then is the limit (should it exist) of these approximations and we define the *improper integral of f between a and b* by

$$\int_a^b f(x)dx := \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^b f(x)dx,$$

where the superscript + indicates that we are taking the limit from the right. When $\int_a^b f(x)dx$ is finite we say that the integral *converges* and when it does not exist or is infinite we say that it *diverges*.

The second "version" of the improper integral is obtained by considering the area under a curve where now one of the bounds is allowed to be either ∞ or $-\infty$. I.e., the improper integral $\int_a^\infty f(x)dx$ should be the area under $f(x)$ starting at a and extending indefinitely to the right along the x-axis, for f a function which is integrable on $a \leq x < \infty$, for a a fixed real number. The idea for defining the improper integral is again the same. Namely, to approximate the area by $\int_a^{a+n} f(x)dx$ for increasing values of $n > 0$. Specifically, the improper integral of f from a to ∞ is defined to be the limit

$$\int_a^\infty f(x)dx := \lim_{n \to \infty} \int_a^{a+n} f(x)dx.$$

Again such an integral *converges* when $\int_a^\infty f(x)dx$ is finite and it *diverges* otherwise.

Example: $f(x) = \frac{1}{x^r}$

Consider a function of the form

$$f(x) := \frac{1}{x^r}$$

for $r > 0$ a fixed real number. Such a fucntion is undefined at $x = 0$ and as we take values of x which are closer and closer to 0 the values of $f(x)$ increase. In particular, $\lim_{x\to 0} f(x) = \infty$. On the other hand, as we take increasingly large values of x, $f(x)$ is closer and closer to 0 and, indeed, $\lim_{x\to\infty} f(x) = 0$. Consider the improper integrals

$$\int_0^1 \frac{1}{x^r} dx \qquad \text{and} \qquad \int_1^\infty \frac{1}{x^r} dx.$$

Do these integrals converge or diverge? It turns out that the answer to this question depends on the value of r. In particular, we may distinguish three cases: $r < 1$, $r = 1$ and $r > 1$. We consider each of these cases in turn (cf. Figure 2 below).

Case $r < 1$**:** In this case we have

$$\int f(x)dx = \int \frac{1}{x^r} dx = \frac{1}{1-r} x^{(1-r)} + C.$$

Thus, the approximating integrals $\int_\epsilon^1 f(x)dx$ are of the form

$$\int_\epsilon^1 f(x)dx = \frac{1}{1-r}(1 - \epsilon^{(1-r)}).$$

The limit is then

$$\int_0^1 f(x)dx = \frac{1}{1-r}\left(1 - \lim_{\epsilon\to 0^+} \epsilon^{(1-r)}\right) = \frac{1}{1-r}$$

since, for $r < 1$, $\lim_{\epsilon\to 0^+} \epsilon^{(1-r)} = 0$ as can be seen using the definition of the limit (since, for $p > 0$, $a < b$ implies $a^p < b^p$).

On the other hand, the improper integral $\int_1^\infty f(x)dx$ diverges in this case since, again using the definition of the limit, $\lim_{n\to\infty}(n+1)^{(1-r)} = \infty$.

Case $r = 1$**:** In this case $\int f(x)dx = \ln|x| + C$ and since $\lim_{x\to 0^+} \ln|x| = -\infty$ we have that $\int_0^1 f(x)dx = \infty$ and the integral diverges.

On the other hand, $\lim_{n\to\infty} \ln|n| = \infty$ and so $\int_1^\infty f(x)dx = \infty$. I.e., this integral also diverges.

Case $r > 1$**:** In this case the indefinite integral is the same as for $r < 1$ so that $(1-r) < 0$ and the first improper integral is

$$\int_0^1 f(x)dx = \frac{1}{1-r}\left(1 - \lim_{\epsilon\to 0^+} \epsilon^{(1-r)}\right) = \infty,$$

as is again a consequence of the definition (for $p < 0$, $a < b$ implies $b^p < a^p$).

On the other hand, the integral $\int_1^\infty f(x)dx$ now converges to $\frac{1}{r-1}$ by similar reasoning to that giving the convergence of $\int_0^1 \frac{1}{x^r} dx$ when $0 < r < 1$.

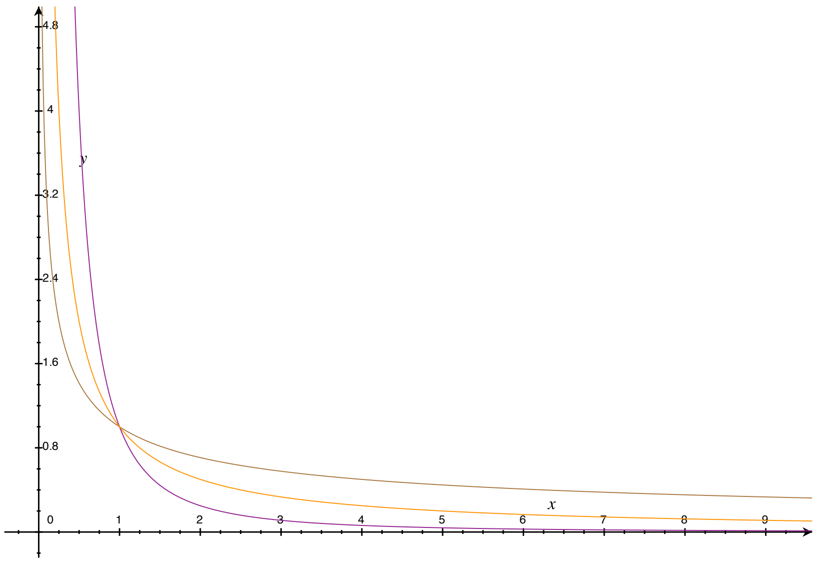

Figure 2: Graphs of $\frac{1}{x^r}$ for $r = \frac{1}{2}$, $r = 1$ and $r = 2$

Determining convergence and divergence by comparing rates

We will not recount all of the tests for convergence and divergence discussed in class (or in the textbook), but mention only the method of comparing the asymptotic behavior of the functions in question. Assume that f and g are non-negative functions which are integrable on the interval consisting of those x such that $a < x \leq b$. Assume moreover that $\lim_{x \to a^+} f(x) = \infty = \lim_{x \to a^+} g(x)$. Then we have the following test

Ratio test: If there exists a real number $\alpha > 0$ ($\alpha \neq \infty$) such that

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \alpha,$$

then $\int_a^b f(x)dx$ converges if and only if $\int_a^b g(x)dx$ converges.

A similar test also applies to the case of integrals of the form $\int_b^\infty f(x)dx$, et cetera.

An important fact to remember in trying to use the ratio text is *l'Hôpital's rule*, which can be stated for limits approaching a definite point a (from the right or left) or for limits approaching $-\infty$ or ∞. We state the rule for a limit approaching ∞. In this case, l'Hôpital's rule states that, for f and g are functions which are differentiable on the interval consisting of those x such that $x \geq b$, if $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x)$ are both 0, ∞ or $-\infty$ and

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

is a real number, ∞ or $-\infty$, then the limit of the ratio $\frac{f(x)}{g(x)}$ exists and is equal to this value:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$

This fact is quite useful since it is often more convenient to evaluate the limit involving the derivatives than to evaluate the limit of $\frac{f(x)}{g(x)}$ directly.

Examples of how to use the ratio test

1. *Does* $\int_0^1 \frac{1}{(1+x)\sqrt{x}}dx$ *converge or diverge?* We apply the ratio test to the functions $\frac{1}{(1+x)\sqrt{x}}$ and $\frac{1}{\sqrt{x}}$:

$$\lim_{x\to 0^+}\frac{\sqrt{x}}{(1+x)\sqrt{x}} = \lim_{x\to 0^+}\frac{1}{1+x} = 1.$$

   Thus, $\int_0^1 \frac{1}{(1+x)\sqrt{x}}dx$ converges if and only if $\int_0^1 \frac{1}{\sqrt{x}}dx$ does. By our earlier remarks (since $r=\frac{1}{2}$ in this case) this converges and therefore our integral also converges (it converges, in fact, to $\frac{\pi}{2}$).

2. *Does* $\int_1^\infty \frac{1}{x^2+3x+2}dx$ *converge or diverge?* We know by the discussion above (since $r=2$) that $\int_1^\infty \frac{1}{x^2}dx$ converges. We will show that $\frac{1}{x^2+3x+2}$ and $\frac{1}{x^2}$ satisfy the hypotheses of the ratio test (which in turn implies that $\int_1^\infty \frac{1}{x^2+3x+2}dx$ converges). First, we would like to apply l'Hôpital's rule to show that $\lim_{x\to\infty}\frac{x^2}{x^2+3x+2}=1$. To do this we first note that $\lim_{x\to\infty}x^2 = \lim_{x\to\infty}x^2+3x+2=\infty$ and we then consider the limit of the ratio of the derivatives:

$$\lim_{x\to\infty}\frac{2x}{2x+3}=1,$$

   where this is seen to be 1 by now applying l'Hôpital's rule to the functions $2x$ and $2x+3$. Thus,

$$\lim_{x\to\infty}\frac{\left(\frac{1}{x^2+3x+2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty}\frac{x^2}{x^2+3x+2} = \lim_{x\to\infty}\frac{2x}{2x+3} = 1.$$

   (The actual value of this integral is $\ln(\frac{3}{2})$.)

3. *Does* $\int_1^\infty \frac{e^x(x+1)}{x}dx$ *converge or diverge?* We will apply the ratio test to the functions $\frac{e^x(x+1)}{x}$ and e^x. In particular, we will use l'Hôpital's rule to show that

$$\lim_{x\to\infty}\frac{\left(\frac{e^x(x+1)}{x}\right)}{e^x} = \lim_{x\to\infty}\frac{(x+1)}{x} = 1. \tag{1}$$

To this end, observe that both $(x+1) \to \infty$ and $x \to \infty$ as $x \to \infty$, and $\frac{d}{dx}(x+1) = 1 = \frac{d}{dx}(x)$. Thus, $\lim_{x\to\infty} \frac{\frac{d}{dx}(x+1)}{\frac{d}{dx}(x)} = 1$ and by l'Hôpital's rule we have proved (1). Thus, by the ratio test and the fact that

$$\int_1^{\infty} e^x dx$$

diverges, it follows that $\int_1^{\infty} \frac{e^x(x+1)}{x} dx$ also diverges.