MAT1332 Spring/Summer 2009 (Instructor: Michael A. Warren)

Some more review problems for the final exam.

Disclaimer: The problems contained here are only on the topic of equilibria and stability of systems of differential equations. You should see the other notes posted on the course webpage for further practice problems. Additionally, I advise you to go over the problems from midterms 1 and 2 as well.

Practice problems

1. The following systems have only $(0,0)$ as an equilibrium. In each case determine the stability of the $(0,0)$. (I.e., is it asymptotically stable, asymptotically unstable, a saddle point, a stable spiral, an unstable spiral, or a center?)

   (a) $$\begin{cases} \frac{dv}{dt} = -3v - 3w \\ \frac{dw}{dt} = -6v + 4w \end{cases}$$

   (b) $$\begin{cases} \frac{dv}{dt} = 12v - 3w \\ \frac{dw}{dt} = 3v + 2w \end{cases}$$

   (c) $$\begin{cases} \frac{dv}{dt} = -4v + 5w \\ \frac{dw}{dt} = -7v - 2w \end{cases}$$

   (d) $$\begin{cases} \frac{dv}{dt} = 0 + 4w \\ \frac{dw}{dt} = -2v + 0 \end{cases}$$

   (e) $$\begin{cases} \frac{dv}{dt} = -2v - 3w \\ \frac{dw}{dt} = 4v - 10w \end{cases}$$

2. Find the equilibria of the following systems

   (a) $$\begin{cases} \frac{dx}{dt} = x^3 + 4x^2 - xy + 5x \\ \frac{dy}{dt} = y^2 + yx + y \end{cases}$$

   (b) The system corresponding to the equation $\vartheta'' = \cos(\vartheta)$.

3. Find the equilibria of the system

$$\begin{cases} \frac{dx}{dt} = yx^2 - 4xy - 5y \\ \frac{dy}{dt} = -x^2 + 2xy + x \end{cases}$$

   Describe the stability of these equilibria.

4. Find the equilibria of the system

$$\begin{cases} \frac{dv}{dt} = -v^2 w + 2vw + 3w \\ \frac{dw}{dt} = -4v - 2w \end{cases}$$

   Describe the stability of these equilibria.

Solutions to practice problems

1. (a) The eigenvalues are 6 and -5. It is a saddle point.
   (b) The eigenvalues are 11 and 3. It is asymptotically unstable.
   (c) The eigenvalues are $-3 \pm \sqrt{34}i$. It is a stable spiral.
   (d) The eigenvalues are $\pm 2\sqrt{2}i$. It is a center.
   (e) The eigenvalues are -8 and -4. It is asymptotically stable.

2. (a) For the x-nullcline, we have $x^3+4x^2-yx+5x = x(x^2+4x-y+5)$ which is 0 if and only if either $x=0$ or $y = x^2+4x+5$. On the other hand, the we have that $y^2+xy+y=0$ if and only if $y=0$ or $x=-y-1$. So we obtain equilibria at $(0,0)$, $(0,-1)$, and when $x=-y-1$ and $y=x^2+4x+5$ (note that the case where $y=0$ and $y=x^2+4x+5$ is ruled out since it has complex roots). In the latter case we may substitute to obtain that

$$y \;=\; (-y-1)^2-4y+1 \;=\; y^2-2y+2$$

   This is the case only when $y^2-3y+2=0$ which is the case for $y=1$ or $y=2$. When $y=1$, substituting back into the equation for x gives that $x=-2$. Similarly, when $y=2$ we have that $x=-3$. Thus, the equilibria are $(0,0)$, $(0,-1)$, $(-2,1)$ and $(-3,2)$.

   (b) The corresponding system is $\begin{cases} \frac{dx}{dt} &= & y \\ \frac{dy}{dt} &= & \cos(x) \end{cases}$. Therefore the x-nullcline consists of those pairs (x,y) such that $y=0$. On the other hand, the y-nullcline consists of those pairs (x,y) such that $\cos(x)=0$. Since $\cos(x)=0$ if and only if $x = n\frac{\pi}{2}$ for an odd integer n. Thus, the y-nullcline is all pairs of the form $(n\frac{\pi}{2}, y)$ for n an odd integer. Thus, the equilibria of the system are precisely the points $(n\frac{\pi}{2}, 0)$ for n an odd integer.

3. First, to find the x-nullcline we observe that $yx^2-4xy-5y=0$ if and only if either $y=0$ or $x^2-4x-5=0$. But the latter factors as $(x-5)(x+1)$ and therefore it is 0 if and only if $x=5$ or $x=-1$. To summarize the x-nullcline consists of all pairs (x,y) such that one of the following cases holds:

$$\begin{cases} y &= & 0 \\ x &= & 5 \\ x &= & -1 \end{cases}$$

   For the y-nullcline, we have that $-x^2+2yx+x=0$ if and only if $x=0$ or $-x+2y+1=0$. The latter holds if and only if $x=2y+1$. Thus, the y-nullcline consists of all (x,y) such that one of the following cases holds:

$$\begin{cases} x &= & 0 \\ x &= & 2y+1 \end{cases}$$

The equilibria are the points of intersection of the x- and y- nullclines. This holds then when both $x = 0$ and $y = 0$. It also holds when $x = 5$ and $y = 2$ (since $5 = 2(2) + 1$), and, finally, when $x = -1$ and $y = -1$ (since $-1 = 2(-1) + 1$). Thus, the equilibria are exactly $(0,0)$, $(5,2)$ and $(-1,-1)$.

For stability, first note that the matrix for the linearized systems is obtained by

$$\begin{bmatrix} \nabla(yx^2 - 4xy - 5y) \\ \nabla(-x^2 + 2xy + x) \end{bmatrix} = \begin{bmatrix} 2xy - 4y & x^2 - 4x - 5 \\ -2x + 2y + 1 & 2x \end{bmatrix}$$

At $(0,0)$ this becomes

$$\begin{bmatrix} 0 & -5 \\ 1 & 0 \end{bmatrix}$$

which has characteristic polynomial $\lambda^2 + 5$ which has roots $\pm i\sqrt{5}$ by the quadratic formula. Thus, the eigenvalues are $\pm i\sqrt{5}$ and therefore in the linearized system the origin is a center. Thus, we cannot determine the stability of $(0,0)$ in the original system from the eigenvalues alone.

At $(5,2)$, the linearization has coefficient matrix

$$\begin{bmatrix} 12 & 0 \\ -5 & 10 \end{bmatrix}$$

which has characteristic polynomial $(A - \lambda I) = \lambda^2 - 22\lambda + 120 = (\lambda - 12)(\lambda - 10)$. Thus, the eigenvalues are 12 and 10 which are both positive. As such, $(5,2)$ is unstable.

Finally, at $(-1,-1)$, the coefficient matrix of the linearization is

$$\begin{bmatrix} 6 & 0 \\ 1 & -2 \end{bmatrix}$$

which has characteristic polynomial $\lambda^2 - 4\lambda - 12 = (\lambda - 6)(\lambda + 2)$ so that the eigenvalues are 6 and -2. Since one of these is positive it follows that $(-1,-1)$ is also unstable.

The matrix for the linearized system is given by

$$\begin{bmatrix} 3x^2 + 8x - y + 5 & -x \\ y & 2y + x + 1 \end{bmatrix}$$

At $(0,0)$ this becomes

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

which has eigenvalues 5 and 1. Therefore, $(0,0)$ is unstable. At $(0,-1)$ we have

$$\begin{bmatrix} 6 & 0 \\ -1 & 0 \end{bmatrix}$$

which has eigenvalues 6 and 0 and is therefore unstable.

At $(-2,1)$ we have

$$\begin{bmatrix} -6 & 2 \\ 1 & 1 \end{bmatrix} \tag{1}$$

which has eigenvalues $\frac{15\pm\sqrt{177}}{2}$ which are both positive ($\sqrt{177} \approx 13.3$). So $(-2,1)$ is unstable.

4. For the x-nullcline we have that $-w(v^2-2v-3)=0$ if and only if $w=0$ or $v=-1$ or $v=3$. For the y-nullcline, we have $-4v-2w=0$ if and only if $-2v=w$. So we have equilibria at $(0,0)$, $(-1,2)$ and $(3,-6)$. For stability, observe that the coefficient matrix of the linearized system has the form

$$\begin{bmatrix} -2vw+2w & -v^2+2v+3 \\ -4 & -2 \end{bmatrix}$$

Thus, at $(0,0)$ it is

$$\begin{bmatrix} 0 & 3 \\ -4 & -2 \end{bmatrix}$$

which has eigenvalues $-1\pm\sqrt{11}i$. Therefore $(0,0)$ is stable.

At $(-1,2)$ it is

$$\begin{bmatrix} 8 & 0 \\ -4 & -2 \end{bmatrix}$$

which has eigenvalues 8 and -2 and therefore $(-1,2)$ is unstable (since one eigenvalues is positive).

Finally, at $(3,-6)$ it is

$$\begin{bmatrix} 24 & 0 \\ -4 & -2 \end{bmatrix}$$

which has eigenvalues 24 and -2. Therefore $(3,-6)$ is unstable.