MAT1332 Spring/Summer 2010

Assignment 2, Due June 1, 2010.

Give decimal expansions to four places. You must justify your answers to all of the questions below.

1. Suppose the quantity v of a radioactive isotope decays according to the law

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -kv \tag{1}$$

with $k > 0$ a constant, v measured in moles and t measured in years. Assume that at time $t = 0$ there are 100 moles of the isotope and at $t = 2$ there are 90 moles remaining.

   (a) Solve the differential equation (1), with the initial condition given above, using separation of variables.

   Solution: The solution is given by

$$v(t) = 100e^{-kt}.$$

   This is obtained by separating variables and integrating to obtain

$$\ln v = \int \frac{1}{v}\,\mathrm{d}v = -\int k\,\mathrm{d}t = -kt + C$$

   so that $v = Ce^{-kt}$ and, by the initial conditions, $C = 100$.

   (b) Find the value of k using your solution to (a).

   Solution: We have $2 = -\frac{1}{k}\int_{100}^{90}\frac{1}{v}\,\mathrm{d}v = -\frac{1}{k}\ln\frac{9}{10}$ so that $k = -\frac{1}{2}\ln\frac{9}{10} \approx 0.0527$.

   (c) Determine the half-life of the isotope using your solutions to (a) and (b). (The half-life is the time t at which $v(t) = \frac{1}{2}v(0)$.)

   Solution: The halflife t is given by $t = -\frac{1}{k}\int_{100}^{50}\frac{1}{v}\,\mathrm{d}v = \frac{\ln\frac{1}{4}}{\ln\frac{9}{10}} \approx 13.1576$ years.

   (d) How much of the isotope remains after 20 years?

   Solution: The amount of isotope after 20 years is $v(20) = 100e^{-20k} = 34.8678$ moles.

2. Do the following integrals converge? (If so, then give their values and if not then prove it.)

(a) $\int_1^\infty \frac{x}{e^x}\,\mathrm{d}x$.

Solution: The solution is given by

$$\int_1^\infty \frac{x}{e^x}\,\mathrm{d}x = \lim_{N\to\infty}\big(-\frac{N}{e^N} - \frac{1}{e^N} + \frac{1}{e} + \frac{1}{e}\big) = \frac{2}{e}.$$

Note that you should use L'Hôpital's rule in order to calculate the limit $\lim_{N\to\infty}\frac{N}{e^N}$.

(b) $\int_{\frac{1}{2}}^1 \frac{1}{t^2\sqrt{1-t^2}}\,\mathrm{d}t$.

Solution: We have (using integration by parts and the substitution $x = \sin(u)$ to integrate)

$$\int_{\frac{1}{2}}^1 \frac{1}{t^2\sqrt{1-t^2}}\,\mathrm{d}t \;=\; \lim_{\epsilon\to 1} -\frac{\sqrt{1-t^2}}{t}\Big|_{\frac{1}{2}}^{\epsilon} \;=\; \lim_{\epsilon\to 1}\big(-\frac{\sqrt{1-\epsilon^2}}{\epsilon} + \sqrt{3}\big) \;=\; \sqrt{3}.$$

(c) $\int_2^\infty \frac{5}{2x^2+4x-12}\,\mathrm{d}x$.

Solution: Using partial fractions we have

$$\begin{aligned}\int_2^\infty \frac{5}{2x^2+4x-12}\,\mathrm{d}x &= \lim_{N\to\infty}\frac{5}{4\sqrt{7}}\ln\frac{x+1-\sqrt{7}}{x+1+\sqrt{7}}\Big|_2^N \\ &= \lim_{N\to\infty}\big(\ln\frac{N+1-\sqrt{7}}{N+1+\sqrt{7}} - \frac{5}{4\sqrt{7}}\ln\frac{3-\sqrt{7}}{3+\sqrt{7}}\big) \\ &= -\frac{5}{4\sqrt{7}}\ln\frac{3-\sqrt{7}}{3+\sqrt{7}} \approx 1.3081.\end{aligned}$$

(d) $\int_0^1 \frac{1}{xe^x}\,\mathrm{d}x$.

Solution: There are various ways to see that this integral diverges. We observe that

$$\lim_{x\to 0}\frac{\frac{1}{x}}{\frac{1}{xe^x}} \;=\; \lim_{x\to 0} e^x \;=\; 1.$$

Thus, by the ratio text, $\int_0^1 \frac{1}{xe^x}\,\mathrm{d}x$ converges if and only if $\int_0^1 \frac{1}{x}\,\mathrm{d}x$ does. However, as we saw in class, the latter integral diverges.

3. To a barrel holding 500l of water is added 25kg of salt. A hole is made in the barrel so that 50l of water flows out of the barrel per minute. Simultaneously 50l per minute of fresh water is added to the barrel. Assume that the contents of the barrel are continuously stirred to ensure a uniform mixture.

   (a) Write the autonomous differential equation describing the rate of change (in kilograms per minute) of the quantity v of salt in the barrel.

   Solution: The rate of change of the quantity v of salt is given by the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{10}v.$$

   (b) Solve the differential equation from (a) using separation of variables.

   Solution: The solution is given by $v = Ce^{-0.1t}$. By the initial conditions as stated in the problem, we know $C = 25$ so that $v = 25e^{-0.1t}$.

   (c) How long before only 5kg of salt remains in the barrel?

   Solution: $t = -10\int_{25}^{5} \frac{1}{v}\,\mathrm{d}v = 10\ln 5 \approx 16.0944$ minutes.