MAT1332 Spring/Summer 2010

Assignment 1 Solutions.

1. Evaluate the following definite integrals.

(a) $\int_{-2}^{2}(u^2-2u^4)\,\mathrm{d}u.$

$$\int_{-2}^{2}(u^2-2u^4)\,\mathrm{d}u = \frac{u^3}{3}-2\frac{u^5}{5}\Big|_{-2}^{2} = -\frac{304}{15}.$$

(b) $\int_{-\pi/2}^{\pi/2}(2x-5\cos x)\,\mathrm{d}x.$

$$\int_{-\pi/2}^{\pi/2}(2x-5\cos x)\,\mathrm{d}x = x^2-5\sin x\Big|_{-\pi/2}^{\pi/2} = -10$$

(c) $\int_{0}^{1} 2\sin(\pi(4-3x))\,\mathrm{d}x.$

We use the substitution $u=\pi(4-3x)$ so that $\mathrm{d}u=-3\pi\,\mathrm{d}x$. Therefore

$$\int_{0}^{1} 2\sin(\pi(4-3x))\,\mathrm{d}x = \int_{4\pi}^{\pi} 2\sin y\frac{\mathrm{d}y}{-3\pi} = \frac{2}{3\pi}(\cos y\Big|_{4\pi}^{\pi}) = -\frac{4}{3}\pi.$$

(d) $\int_{-2}^{2} x^2e^x\,\mathrm{d}x.$

We will use successive applications of integration by parts. Setting $f(x)=x^2$, $g'(x)=e^x$ we have $f'(x)=2x$, $g(x)=e^x$ and therefore:

$$\int_{-2}^{2} x^2e^x\,\mathrm{d}x = x^2e^x\Big|_{-2}^{2} - \int_{-2}^{2} 2xe^x\,\mathrm{d}x.$$

Another integration by parts gives

$$\begin{aligned}\int_{-2}^{2} x^2e^x\,\mathrm{d}x &= x^2e^x\Big|_{-2}^{2} - 2(xe^x\Big|_{-2}^{2} - \int_{-2}^{2} e^x\,\mathrm{d}x)\\ &= x^2e^x-2xe^x\Big|_{-2}^{2}+2(e^x\Big|_{-2}^{2})\\ &= (x^2-2x+2)e^x\Big|_{-2}^{2}\\ &= -10e^{-2}+2e^2.\end{aligned}$$

(e) $\displaystyle\int_{1/5}^{2/5} \frac{2}{1-2t}\,\mathrm{d}t.$

We use the substitution $u = 1-2t$ which gives $\mathrm{d}u = -2\,\mathrm{d}t$. Therefore

$$\int_{1/5}^{2/5} \frac{2}{1-2t}\,\mathrm{d}t = \int_{3/5}^{1/5} \frac{2}{u} \cdot \frac{\mathrm{d}u}{-2} = -\ln u\Big|_{3/5}^{1/5} = \ln 3.$$

2. Calculate the following integrals using the method of partial fractions.

(a) $\displaystyle\int_{3}^{4} \frac{2x-1}{x^2-3x+2}\,\mathrm{d}x.$

We factor the denominator as $x^2-3x+2 = (x-2)(x-1)$ and then solve for A, B to find $A = 3$, $B = -1$ and therefore

$$\frac{2x-1}{x^2-3x+2} = \frac{3}{x-2} - \frac{1}{x-1}.$$

Integrating gives:

$$\int_{3}^{4} \frac{3}{x-2} - \frac{1}{x-1}\,\mathrm{d}x = 3\ln(x-2) - \ln(x-1)\Big|_{3}^{4} = 4\ln 2 - \ln 3.$$

(b) $\displaystyle\int_{-2}^{-1} -\frac{8x^2-10x-1}{x(2x-1)^2}\,\mathrm{d}x.$

Developing the partial fraction as usual we find that

$$-\frac{8x^2-10x-1}{x(2x-1)^2} = \frac{1}{x} + \frac{2}{(x-\frac{1}{2})^2} - \frac{3}{x-\frac{1}{2}}.$$

Integrating gives:

$$\begin{aligned}\int_{-2}^{-1} -\frac{8x^2-10x-1}{x(2x-1)^2}\,\mathrm{d}x &= \ln|x| - \frac{2}{x-\frac{1}{2}} - 3\ln\left|x-\frac{1}{2}\right|\,\Big|_{-2}^{-1} \\ &= \frac{8}{15} - \ln 2 + 3\ln 5 - 3\ln 3.\end{aligned}$$

(c) $\displaystyle\int_{2}^{5} \frac{1}{2x^2-8x+12}\,\mathrm{d}x.$

The discriminant of the denominator is negative and therefore it cannot be factored. We divide and complete the square to obtain $2x^2-8x+12 = 2[(x-2)^2+2]$. (Note that in this case solving for A, B so that the numerator of our fraction is equal to $Ax+B$ is trivial: $A = 0$ and $B = 1$.) So

$$\begin{aligned}\int_2^5 \frac{1}{2x^2-8x+12}\,\mathrm{d}x &= \frac{1}{2}\int_2^5 \frac{1}{(x-2)^2+2}\,\mathrm{d}x \\ &= \frac{1}{4}\int_2^5 \frac{1}{\left((\frac{1}{\sqrt{2}})(x-2)\right)^2+1}\,\mathrm{d}x \\ &= \frac{1}{2\sqrt{2}}\left(\arctan \frac{x-2}{\sqrt{2}}\Big|_2^5\right) \\ &= \frac{1}{2\sqrt{2}}\arctan\frac{3}{\sqrt{2}}.\end{aligned}$$

3. In each of the following cases, find the area of the region bounded by the graphs of the functions f and g between $x = a$ and $x = b$, and draw a graph of the figure.

(a) $f(x) = x^2$, $g(x) = 4$, $a = -2$, $b = 2$.
On the interval $[-2,2]$, $x^2 \le 4$ and therefore

$$\int_{-2}^{2} \left|x^2-4\right|\,\mathrm{d}x = \int_{-2}^{2} 4-x^2\,\mathrm{d}x = 4x - \frac{x^3}{3}\Big|_{-2}^{2} = \frac{32}{3}.$$

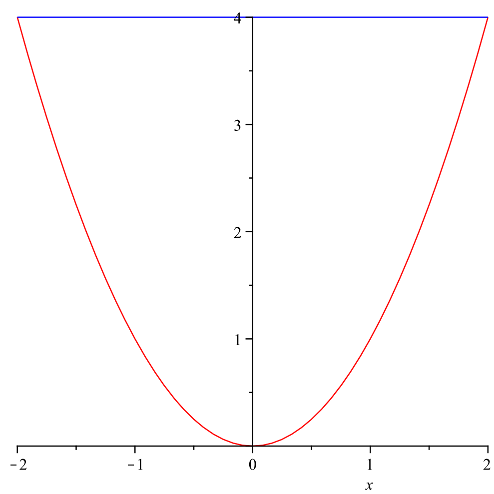

Figure 1: $f(x) = x^2$, $g(x) = 4$, $a = -2$, $b = 2$.

(b) $f(x) = e^x$, $g(x) = 1+2x$, $a = 0$, $b = 1$.
On the interval $[0,1]$, $e^x \le 1+2x$ and therefore

$$\int_0^1 \left|e^x-1-2x\right|\,\mathrm{d}x = \int_0^1 1+2x-e^x\,\mathrm{d}x = x+x^2-e^x\Big|_0^1 = 3-e.$$

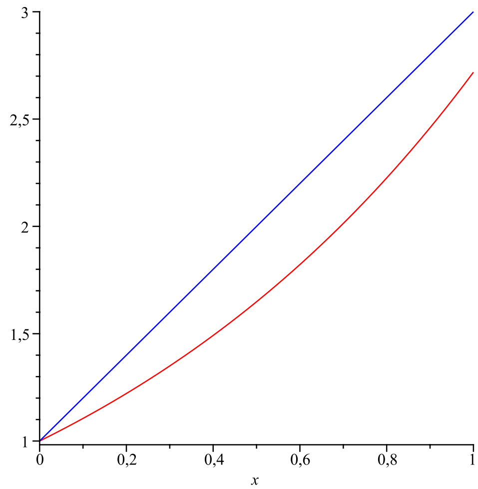

Figure 2: $f(x) = e^x$, $g(x) = 1 + 2x$, $a = 0$, $b = 1$.

(c) $f(x) = \cos x$, $g(x) = 1/2$, $a = 0$, $b = 2\pi$.
We begin by finding the points of intersection of f and g in the interval $[0, 2\pi]$:

$$\cos x = \frac{1}{2} \qquad \text{si et seulement si} \qquad x = \frac{\pi}{3} \text{ ou } x = \frac{5\pi}{3}.$$

It is easy to see that $\cos x \leq 1/2$ if and only if $x \in [\frac{\pi}{3}, \frac{5\pi}{3}]$. Therefore we have

$$\int_0^{2\pi} \left|\cos x - \frac{1}{2}\right| \mathrm{d}x = \int_0^{\frac{\pi}{3}} \cos x - \frac{1}{2} \mathrm{d}x + \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \frac{1}{2} - \cos x \,\mathrm{d}x + \int_{\frac{5\pi}{3}}^{2\pi} \cos x - \frac{1}{2} \mathrm{d}x.$$

We calculate separately that

$$\int_0^{\frac{\pi}{3}} \cos x - \frac{1}{2} \mathrm{d}x = \sin x - \frac{x}{2}\Big|_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{2} - \frac{\pi}{6},$$
$$\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \frac{1}{2} - \cos x \,\mathrm{d}x = \frac{x}{2} - \sin x\Big|_{\frac{\pi}{3}}^{\frac{5\pi}{3}} = \frac{2\pi}{3} + \sqrt{3}, \text{ and}$$
$$\int_{\frac{5\pi}{3}}^{2\pi} \cos x - \frac{1}{2} \mathrm{d}x = \int_0^{\frac{\pi}{3}} \cos x - \frac{1}{2} \mathrm{d}x = \frac{\sqrt{3}}{2} - \frac{\pi}{6}.$$

Therefore we conclude that

$$\int_0^{2\pi} \left|\cos x - \frac{1}{2}\right| \mathrm{d}x = 2\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) + \frac{2\pi}{3} + \sqrt{3} = \frac{\pi}{3} + 2\sqrt{3}.$$

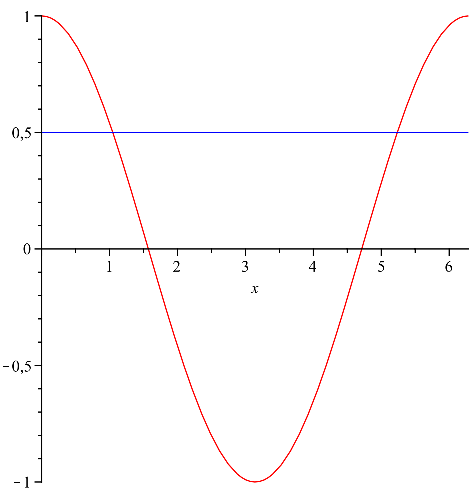

Figure 3: $f(x) = \cos x$, $g(x) = 1/2$, $a = 0$, $b = 2\pi$.