MAT 1332: CALCULUS FOR LIFE SCIENCES

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1. Review: Function of several variables II: Vector-valued functions

- Definition
- Linear approximation and the Jacobian matrix

2. Systems of differential equations I: linear case, phase plane

2.1. The explicit general solution for linear systems. Recall that

- a single linear differential equation

$$\frac{dx(t)}{dt} = ax, \quad x(0) = x_0$$

has a solution $x(t) = e^{at}x_0$. In particular,

  – If $a > 0$, the solution grows to infinity.

  – If $a < 0$, the solution approaches zero.

- Newton's Law of Cooling

$$\begin{array}{rcccl} \frac{dH}{dt} & = & \alpha(A-H) & = & \alpha A - \alpha H \\ \frac{dA}{dt} & = & \alpha_2(H-A) & = & \alpha_2 H - \alpha_2 A \end{array}$$

where
$H(t)$: the object temperature at time t;
$A(t)$: the ambient temperature at time t;
α: constant of proportionality;
α_2 : constant of proportionality.
This is an example of linear systems of differential equations.

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More generally, we now want to generalize the result of the single linear differential equation to the more general linear systems of differential equations:

$$\begin{array}{rcl} \frac{d}{dt}x_1(t) & = & ax_1(t)+bx_2(t) \qquad x_1(0)=x_{10} \\ \frac{d}{dt}x_2(t) & = & cx_1(t)+dx_2(t) \qquad x_2(0)=x_{20} \end{array}$$

We can also write this in matrix notation (suppressing the argument t for the moment) as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

2.1.1. *Observation I: Eigenvalues and eigenvectors provide solutions.* Suppose we are looking for solutions of the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{\lambda t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Fact: If λ is an eigenvalue of A and v is the corresponding eigenvector, then

$$x(t) = e^{\lambda t} v$$

is a solution of the linear system of differential equations

$$\frac{d}{dt} x(t) = A x(t)$$

where $x = (x_1, \ldots, x_n)^T$, and A is an $n \times n$ matrix.

Example 1. *Consider the system*

$$\begin{aligned} \frac{d}{dt} x_1(t) &= x_1(t) + 2x_2(t) \\ \frac{d}{dt} x_2(t) &= 3x_1(t) - 4x_2(t) \end{aligned}$$

2.1.2. *Observation II: Sums and multiples of solutions are solutions.* Suppose we have two solutions $w(t)$ and $z(t)$ for the same system $\frac{d}{dt}x(t) = Ax(t)$, i.e.,

$$\frac{d}{dt}w(t) = Aw(t), \quad \frac{d}{dt}z(t) = Az(t)$$

Now pick two numbers C_1, C_2 and form

$$x(t) = C_1 w(t) + C_2 z(t).$$

Then $x(t)$ is also a solution since

$$\frac{d}{dt}x(t) = C_1 \frac{d}{dt}w(t) + C_2 \frac{d}{dt}z(t) = C_1 Aw(t) + C_2 Az(t) = A\big[C_1 w(t) + C_2 z(t)\big] = Ax(t)$$

Example 2. ***Example 1, continued*** *Consider the system*

$$\begin{aligned} \frac{d}{dt}x_1(t) &= x_1(t) + 2x_2(t) \\ \frac{d}{dt}x_2(t) &= 3x_1(t) - 4x_2(t) \end{aligned}$$

with initial values $x_1(0) = 5$ *and* $x_2(0) = 5$.

Example 3. *Fine the solution of Newton's Law of cooling*

$$\begin{aligned} \frac{d}{dt}x_1(t) &= 3\big(x_2(t) - x_1(t)\big) \\ \frac{d}{dt}x_2(t) &= x_1(t) - x_2(t) \end{aligned}$$

with initial values $x_1(0) = 5$ *and* $x_2(0) = 1$*.*

Explicit solutions in the case of distinct real eigenvalues:

To solve the system

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

do the following steps:

(1) Find the eigenvalues λ_1, λ_2 of A.

(2) If $\lambda_1 \neq \lambda_2$ are real numbers, then find the corresponding eigenvectors v_1, v_2.

(3) Find the constants C_1, C_2 such that

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = C_1 v_1 + c_2 v_2$$

(4) The solution is

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2.$$

Note: If the eigenvalues λ_1, λ_2 are not real numbers or if $\lambda_1 = \lambda_2$, then the procedure is similar but a bit more tricky. We will NOT consider these cases here.

Example 4. *Consider the system*

$$\begin{aligned} \frac{d}{dt}x_1(t) &= x_1(t) + 4x_2(t) \\ \frac{d}{dt}x_2(t) &= x_1(t) - 2x_2(t) \end{aligned}$$

with initial values $x_1(0) = 2$ *and* $x_2(0) = 5$.

2.1.3. *Observation III: Stability of zero.* Consider the system of differential equations

$$\frac{d}{dt}x(t) = Ax(t), \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

and assume that λ is an eigenvalue of A with corresponding eigenvector v. Then we know that

$$x(t) = e^{\lambda t}v$$

is a solution.

- If λ is a real number and
  - $\lambda > 0$, then this solution grows in time.
  - $\lambda < 0$, then this solution decays to zero.
- If λ is a complex number, then we use Euler's formula

$$e^{\lambda t} = e^{(a+bi)t} = e^{at}\big[\cos(bt) + \sin(bt)\big]$$

  - $Re(z) = a > 0$, then the solution grows.
  - $Re(z) = a < 0$, then the solution decays to zero.

Fact:
- If all the eigenvalues of the matrix A have negative real part, then all the solutions to the system of differential equation

$$\frac{d}{dt}x(t) = Ax(t), \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

  decay to zero as $t \to \infty$.
- If one eigenvalue has positive real part, then there is at least one solutions that does not decay to zero. (In fact, there is a solution that grows to infinity in an appropriate norm)

Note: We make no statement about the case of eigenvalues with zero real part.

Example 5. *Examples 1, 3 & 4 revisited*

(1) Example 1:

(2) Example 3:

(3) Example 4:

Example 6. *Consider the system*

$$\begin{aligned} \frac{d}{dt}x_1(t) &= -4x_1(t) + 2x_2(t) \\ \frac{d}{dt}x_2(t) &= -3x_1(t) + x_2(t) \end{aligned}$$

Example 7. *Consider the system*

$$\begin{aligned}\frac{d}{dt}x_1(t) &= x_1(t) - x_2(t)\\ \frac{d}{dt}x_2(t) &= 8x_1(t) - 5x_2(t)\end{aligned}$$

2.2. **The phase plane.** Phase plane for system of autonomous DEs is an extension of the phase-line diagram for single equations.

A system of two autonomous differential equations can be written in the form

$$\begin{aligned}\frac{d}{dt}x(t) &= f(x,y)\\ \frac{d}{dt}y(t) &= g(x,y)\end{aligned}$$

where $x(t), y(t)$ are the two functions that we are looking for.

- If both $f(x,y)$, and $g(x,y)$ are linear functions, then we can use the method we just learned in Section 2.1 to obtain the explicit solutions for the case where the coefficient matrix has two distinct eigenvalues.
- If either/both of $f(x,y)$ and $g(x,y)$ are nonlinear functions, this system is called nonlinear. In general, there is no explicit solution available. i.e., the functions $x(t)$ and $y(t)$ cannot be written down in term of t. Nonetheless, we can find out the general shape of solutions. This will be done in two parts:
  (1) Phase plane (as explained in the textbook in Sections 5.6-5.8)
  (2) Linear stability analysis (Linearization) (the topic for next subsection)

To draw the phase plane, we need equilibria, nullclines, direction arrows.

Recalling that for a single DE,

$$\frac{d}{dt}x(t) = f(x),$$

The equilibrium x^*: $f(x^*) = 0$ is the point where the rate of change of state variable is 0.

For a two-dimensional system of autonomous DEs:

$$\begin{aligned} \frac{d}{dt}x(t) &= f(x,y) \\ \frac{d}{dt}y(t) &= g(x,y) \end{aligned}$$

An equilibrium of this system: (x^*, y^*) such that

$$\begin{aligned} f(x^*, y^*) &= 0 \\ g(x^*, y^*) &= 0 \end{aligned}$$

a point where the rate of change of each state variable is 0. Let us look at the following example:

Example 8. *Consider the system of predator-prey*

$$\begin{aligned} \frac{db}{dt} &= \big(1-p\big)b \\ \frac{dp}{dt} &= \big(-1+\frac{1}{2}b\big)p \end{aligned}$$

Definitions:

- The $x-$nullcline is the set of all points (x, y) where $x(t)$ does not change, i.e.,

$$\frac{d}{dt}x(t) = 0, \quad f(x,y) = 0$$

- The $y-$nullcline is the set of all points (x, y) where $y(t)$ does not change, i.e.,

$$\frac{d}{dt}y(t) = 0, \quad g(x,y) = 0$$

- A steady state or equilibrium is a point where neither x nor y change, i.e.,

$$\frac{d}{dt}x(t) = 0 \;\;\&\;\; \frac{d}{dt}y(t) = 0 \quad \text{or, equivalently} \quad f(x,y) = 0 \;\;\&\;\; g(x,y) = 0$$

- A direction arrow is a vector that indicates in which the direction the solution will go from a given point. The direction arrow wt the point (x, y) has the coordinates $(f(x, y), g(x, y))$.

Example 9. *Consider the system of competition*

$$\begin{aligned} \frac{da}{dt} &= 2\big(1 - \frac{a+b}{2}\big)a \\ \frac{db}{dt} &= 2\big(1 - (a+b)\big)b \end{aligned}$$

Solution:

The algorithm of finding the nullclines and equilibria of system of autonomous DEs is summarized as follows:

Algorithm: Finding the nullclines and equilibria of system of autonomous DEs

(1) Pick one of the variables to act as the vertical variable in the phase plane.

(2) Write the equation for the first nullcline by setting the rate of change of the first state variable equal to 0.
   (a) Factor.
   (b) Solve each factor of the vertical variable. If there is no vertical variable in the factor. Solve for the horizontal variable.
   (c) Graph each solution in the phase plane. If the equation contains no vertical variable, the graph is a vertical line.

(3) Follow the same steps to find the second nullcline, using the equation for the second state variable. Graph in the phase plane, using a different color or style.

(4) Find the intersections, which are the equilibria of the system. Solutions that begin at an equilibrium remain there.

Remark: Some examples in this note are cited from the book titled "Calculus for Biology and Medicine" by Claudia Neuhauser.