MAT1332 Spring/Summer 2010

Additional Practice Problems (possibly for discussion in DGD 8)

1. Find the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{bmatrix} 18 & -4 \\ -2 & 16 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 5 & -3 \\ 9 & 7 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

2. Let $x_{n+1} := Ax_n$ be a discrete dynamical system with

$$x_0 := \begin{bmatrix} .25 \\ .75 \end{bmatrix} \qquad A := \begin{bmatrix} .9 & .4 \\ .1 & .6 \end{bmatrix}$$

   (a) Determine the eigenvalues and eigenvectors of A.

   (b) Find the equilibrium point.

   (c) Write the solution x_n by finding $x_n = A^n x_0$.

3. Find the eigenvalues and eigenvectors of the following matrix:

$$\begin{bmatrix} -6 & 0 & 2 \\ 1 & 5 & -1 \\ -2 & 0 & -6 \end{bmatrix}$$

Solutions

1. To find the eigenvalues of A we solve for the roots λ of the polynomial

$$\det(A-\lambda I) \;=\; (18-\lambda)(16-\lambda)-(-2)(-4) \;=\; \lambda^2-34\lambda+280.$$

E.g., we may find the roots using the quadratic formula which tells us that here the roots are

$$\frac{34\pm\sqrt{1156-4(280)}}{2} \;=\; \frac{34\pm 2\sqrt{289-280}}{2} \;=\; 17\pm 3.$$

I.e., the eigenvalues of A are $\lambda = 14$ and $\lambda = 20$. In the first case, where $\lambda = 14$, we find the eigenvectors by

$$A-\lambda I \;=\; \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix} \to \begin{bmatrix} 4 & -4 \\ 0 & 0 \end{bmatrix}$$

The solutions of the corresponding equation $4x-4y=0$ are then $x = y = 1$ (y is the free variable). Thus, the eigenvectors for $\lambda = 14$ are the scalar multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For $\lambda = 20$, we have

$$A-\lambda I \;=\; \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix} \to \begin{bmatrix} -2 & -4 \\ 0 & 0 \end{bmatrix}$$

In this case we have that $-2x-4y=0$ and a solution is given by $x=-2$, $y=1$ (again, y is the free variable). Thus, the eigenvectors for $\lambda = 20$ are the scalar multiples of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

For B we have that

$$\det(B-\lambda I) \;=\; (3-\lambda)(7-\lambda)(2-\lambda)-(2-\lambda)(9)(5) \;=\; (2-\lambda)(\lambda^2-10\lambda-24).$$

Thus the eigenvalues of B are $\lambda = 2$ as well as the roots of $\lambda^2-10\lambda-24$ which are given by

$$\frac{10\pm\sqrt{100+96}}{2} \;=\; \frac{10\pm\sqrt{(4)(49)}}{2} \;=\; 5\pm 7.$$

So the eigenvalues of B are $\lambda = 2, -2$ and 12. For $\lambda = 2$ we have

$$B-\lambda I \;=\; \begin{bmatrix} 1 & 5 & -3 \\ 9 & 5 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 \\ 0 & -40 & 25 \\ 0 & 0 & 0 \end{bmatrix}$$

In the corresponding system of equations ($x+5y-3z=0$, $-40y+25z=0$) the variable z is free and it is convenient to take $z := 8$. Then, by the second equation, $y = 5$ and, by the first equation, $x = -1$. So the eigenvectors corresponding to eigenvalue $\lambda = 2$ are scalar multiples of $\begin{bmatrix} -1 \\ 5 \\ 8 \end{bmatrix}$.

When $\lambda = -2$ we have

$$B - \lambda I = \begin{bmatrix} 5 & 5 & -3 \\ 9 & 9 & -2 \\ 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 5 & 5 & -3 \\ 0 & 0 & \frac{17}{5} \\ 0 & 0 & 4 \end{bmatrix}$$

So, $z = 0$, y is the free variable and taking $y = 1$ gives $x = -1$. As such, for $\lambda = -2$, the eigenvectors are scalar multiples of $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Finally, when $\lambda = 12$,

$$B - \lambda I = \begin{bmatrix} -9 & 5 & -3 \\ 9 & -5 & -2 \\ 0 & 0 & -10 \end{bmatrix} \to \begin{bmatrix} -9 & 5 & -3 \\ 0 & 0 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

Therefore, $z = 0$ and y is the free variable. Taking $y := 9$ gives $x = 5$. So, the eigenvectors, for $\lambda = 12$, are scalar multiples of $\begin{bmatrix} 5 \\ 9 \\ 0 \end{bmatrix}$.

2. We reason as follows:

   (a) We have

$$\det(A - \lambda I) = (0.9 - \lambda)(0.6 - \lambda) - 0.04 = \lambda^2 - 1.5\lambda + 0.5 = (\lambda - 0.5)(\lambda - 1)$$

   and so the eigenvalues are $\lambda = 1$ and $\lambda = 0.5$. For $\lambda = 1$ we find

$$A - \lambda I = \begin{bmatrix} -0.1 & 0.4 \\ 0.1 & -0.4 \end{bmatrix} \to \begin{bmatrix} -0.1 & 0.4 \\ 0 & 0 \end{bmatrix}$$

   so that, taking the free variable $y := 1$, $x = 4$ and the eigenvectors are scalar multiples of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. For $\lambda = 0.5$ on the other hand

$$A - \lambda I = \begin{bmatrix} 0.4 & 0.4 \\ 0.1 & 0.1 \end{bmatrix}$$

   and we find $x = -1$, $y = 1$ (y is again the free variable). So scalar multiples of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvectors.

(b) In order to find the equilibrium point v_* with $Av_* = v_*$ and such that the entries of v_* represent percentages (i.e., are between 0 and 1 and sum to 1), we take $r = \frac{1}{5}$ and multiply the eigenvector $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ for $\lambda = 1$ by r. I.e.,

$$v_* := r\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

(c) To find x_n we first write our initial condition x_0 as a linear combination

$$x_0 = \alpha v_* + \beta w,$$

where w is the eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. For this we look at the coefficient matrix of the corresponding system of equations:

$$\left[\begin{array}{cc|c} 0.8 & -1 & 0.25 \\ 0.2 & 1 & 0.75 \end{array}\right] \to \left[\begin{array}{cc|c} 0.2 & 1 & 0.75 \\ 0 & -5 & -2.75 \end{array}\right]$$

So $\beta = 0.55$ and $\alpha = 1$. Thus,

$$A^n x_0 \;=\; A^n(v_* + \beta w) \;=\; A^n(v_*) + A^n(\beta w) \;=\; v_* + \beta A^n(w) \;=\; v_* + (0.55)(0.5)^n w$$

3. We have that

$$\det(A - \lambda I) \;=\; (-6-\lambda)(5-\lambda)(-6-\lambda) + 4(5-\lambda) \;=\; (5-\lambda)(\lambda^2 + 12\lambda + 40)$$

The polynomial $\lambda^2 + 12\lambda + 40$ has roots given by

$$\frac{-12 \pm \sqrt{144-160}}{2} = \frac{-12 \pm \sqrt{-16}}{2} = \frac{-12 \pm 4i}{2} = -6 \pm 2i.$$

Thus, the eigenvalues are $\lambda = 5, -6+2i$ and $-6-2i$.

For $\lambda = 5$ in order to find the eigenvectors we have

$$\begin{bmatrix} -11 & 0 & 2 \\ 1 & 0 & -1 \\ -2 & 0 & -11 \end{bmatrix} \to_{11*R2+R1} \begin{bmatrix} 0 & 0 & -9 \\ 1 & 0 & -1 \\ -2 & 0 & -11 \end{bmatrix}$$

$$\to_{2*R2+R3} \begin{bmatrix} 0 & 0 & -9 \\ 1 & 0 & -1 \\ 0 & 0 & -13 \end{bmatrix}$$

So we have equations $x - z = 0$, $-9z = 0$ and $-13z = 0$ with variable y free and therefore the eigenvectors are scalar multiples of $\mathbf{v} := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

For $\lambda = -6+2i$ we have

$$\begin{bmatrix} -2i & 0 & 2 \\ 1 & 11-2i & -1 \\ -2 & 0 & -2i \end{bmatrix} \to_{\frac{1}{2i}R1+R2} \begin{bmatrix} -2i & 0 & 2 \\ 0 & 11-2i & -1-i \\ -2 & 0 & -2i \end{bmatrix}$$

$$\to_{-\frac{1}{i}R1+R3} \begin{bmatrix} -2i & 0 & 2 \\ 0 & 11-2i & -1-i \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore we have equations $-2ix+2z=0$ and $(11-2i)y+(-1-i)z=0$ with z a free variable. Taking $z := (11-2i)$ we have $y = 1+i$. So $-2ix = -2(11-2i)$ and

$$x = \frac{(11-2i)}{i} = (11-2i)(-i) = -2-11i.$$

Thus, the eigenvectors are scalar multiples of $\mathbf{w} := \begin{bmatrix} -2-11i \\ 1+i \\ 11-2i \end{bmatrix}$.

Finally, the eigenvectors when $\lambda = -6-2i$ are the complex conjugate of those for the case when $\lambda = -6-2i$. I.e., they are scalar multiples of $\bar{\mathbf{w}} = \begin{bmatrix} -2+11i \\ 1-i \\ 11+2i \end{bmatrix}$.