

MAT 1332: Calculus for Life Sciences

A course based on the book  
Modeling the dynamics of life  
by F.R. Adler

Supplementary material  
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## Complex numbers

### Introductory consideration

We can easily solve the equation  $x^2 - 4 = 0$ . The answer is  $x = \pm 2$ ; in particular,  $x$  is a rational number, even an integer. The equation  $x^2 - 2 = 0$  is a bit more tricky. The solution  $x = \pm\sqrt{2}$  is not a rational number. Instead, we have defined the square root of a positive number as the real number that gives the original number back when multiplied by itself. But what should we do with the equation  $x^2 + 1 = 0$ ? The answer cannot be a real number. (Why?) Can we do the same as above and define a number whose square equals -1? Indeed, this is what mathematicians did in the eighteenth century (it was a daring act and caused a lot of controversy), and they call that number 'i' or *imaginary*. (We will see that complex numbers are hardly more imaginary than  $\sqrt{2}$ .)

### Definition

A *complex number*  $z$  is a number of the form

$$z = a + bi$$

with real numbers  $a, b$  and the symbol  $i$  that satisfies  $i^2 = -1$ . We call  $a = \operatorname{Re}(z)$  the *real part* of  $z$  and  $b = \operatorname{Im}(z)$  the *imaginary part* of  $z$ . The real number  $a$  can be considered the complex number  $a + 0i$ . A complex number of the form  $z = bi$  is called *purely imaginary*.

### Addition, subtraction, and multiplication of complex numbers

Complex numbers are easily added, subtracted and multiplied, if we keep the rule  $i^2 = -1$  in mind and use the distributive laws.

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$$

$$(a + bi) \times (c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (ad + bc)i,$$

since  $i^2 = -1$ .

### Examples

1.  $(3 + 5i) + (2 - 7i) = 5 - 2i$
2.  $(0.5 + 1.7i) - (0.8 - 2.6i) = -0.3 + 4.3i$
3.  $(-3 + 2i) \times (4 - 5i) = (-12 - (-10)) + (15 + 8)i = -2 + 23i$
4.  $(2 - 0.5i) \times (3 + 4i) = (6 - (-2)) + (-1.5 + 8)i = 8 + 6.5i$
5.  $(9 + 2i) + 5 = (9 + 2i) + (5 + 0i) = 14 + 2i$
6.  $-3i + (2 + 3i) = (0 - 3i) + (2 + 3i) = 2 + 0i = 2$

$$7. 2 \times (3 - 5i) = 6 - 10i$$

$$8. 3i \times (-1 + 4i) = -12 - 3i$$

Before we look at inverses and division of complex numbers, we introduce the *complex conjugate* of a complex number.

### Definition and observation

The *complex conjugate* of  $z = a + bi$  is  $\bar{z} = a - bi$ , i.e., we simply change the sign of the imaginary part. Since the multiplication

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2$$

always produces a non-negative real number, we can take the square root. We define the *modulus* or *absolute value* of  $z = a + bi$  as

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}.$$

From the identity  $z\bar{z} = |z|^2$ , we find the inverse of  $z$  to be

$$\frac{1}{z} = z^{-1} = \bar{z}/|z|^2.$$

### Example 1

Start with  $z = 3 + 4i$  and  $w = 2 - i$ . The complex conjugates are  $\bar{z} = 3 - 4i$  and  $\bar{w} = 2 + i$ . The absolute values are  $|z| = 5$  and  $|w| = \sqrt{5}$ . The inverses are

$$z^{-1} = \frac{1}{25}(3 - 4i), \quad w^{-1} = \frac{1}{5}(2 + i).$$

Finally, we can divide

$$\frac{z}{w} = z \frac{\bar{w}}{|w|^2} = \frac{1}{5}(2 + 11i), \quad \frac{w}{z} = w \frac{\bar{z}}{|z|^2} = \frac{1}{25}(2 - 11i).$$

Another way to think about this: make the denominator real (similar to the way you'd rationalize the denominator) by multiplying top and bottom by the conjugate of the denominator (i.e., "real-ize" the denominator). Thus,

$$\begin{aligned} \frac{1}{z} &= \frac{1}{3 + 4i} \\ &= \frac{1}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} \\ &= \frac{3 - 4i}{3^2 - (4i)^2} \\ &= \frac{3 - 4i}{3^2 + 4^2} \\ &= \frac{3 - 4i}{25} \end{aligned}$$

Similarly,

$$\begin{aligned}
 \frac{1}{w} &= \frac{1}{2+i} \cdot \frac{2-i}{2-i} \\
 &= \frac{2-i}{2^2-i^2} \\
 &= \frac{2-i}{5} \\
 \frac{z}{w} &= \frac{3+4i}{2-i} \cdot \frac{2+i}{2+i} \\
 &= \frac{6+3i+8i+4i^2}{2^2+1^2} \\
 &= \frac{2+11i}{5} \\
 \frac{w}{z} &= \frac{2-i}{3+4i} \cdot \frac{3-4i}{3-4i} \\
 &= \frac{6-8i-3i+4i^2}{9+16} \\
 &= \frac{2-11i}{25}
 \end{aligned}$$

### Example 2

Start with  $z = 1 - 4i$  and  $w = 0.5 + 3i$ . The complex conjugates are  $\bar{z} = 1 + 4i$  and  $\bar{w} = 0.5 - 3i$ . The absolute values are  $|z| = \sqrt{17}$  and  $|w| = \sqrt{37/4}$ . The inverses are

$$z^{-1} = \frac{1}{17}(1 + 4i), \quad w^{-1} = \frac{4}{37}(0.5 - 3i).$$

Division gives

$$\frac{z}{w} = z \frac{\bar{w}}{|w|^2} = \frac{4}{37}(-11.5 - 5i), \quad \frac{w}{z} = w \frac{\bar{z}}{|z|^2} = \frac{1}{17}(-11.5 + 5i).$$

Alternatively, we can “real-ize” the denominator as before. Thus

$$\begin{aligned}
 \frac{z}{w} &= \frac{1-4i}{0.5+3i} \cdot \frac{0.5-3i}{0.5-3i} \\
 &= \frac{0.5-3i-2i+12i^2}{0.25+9} \\
 &= \frac{-11.5-5i}{9.25} \\
 \frac{w}{z} &= \frac{2-i}{3+4i} \cdot \frac{3-4i}{3-4i} \\
 &= \frac{6-8i-3i+4i^2}{9+16} \\
 &= \frac{2-11i}{25}
 \end{aligned}$$

## Geometric interpretation

It is very helpful to think of a complex number as a point in the plane with the real part as the  $x$ -value and the imaginary part as the  $y$ -value. Hence, we identify the complex number  $z = a + bi$  with the point  $(a, b)$  or with the vector (arrow) from the origin to the point  $(a, b)$ . (We will talk about vectors in more detail shortly). Then the absolute value of the complex number is simply the distance of the corresponding point from the origin or the length of the vector (arrow). See Figure 1.

With this correspondence, the addition of complex numbers become the addition of vectors as it is known from the physics of forces. See Figure 1.

To interpret multiplication, we take a slightly different point of view. Instead of giving the coordinates of the vector as the endpoint  $(a, b)$ , we consider its length  $r \geq 0$  and the angle  $\phi$  it makes with the  $x$ -axis (counterclockwise) as  $(r \cos \phi, r \sin \phi)$ . This representation is called *polar coordinates*. Then multiplication of two numbers is simply multiplication of the lengths and addition of the angles. See Figure 1. We write

$$z = r(\cos \phi + i \sin \phi) \quad \text{and} \quad w = s(\cos \psi + i \sin \psi).$$

Then we multiply, using the trigonometric identities

$$\begin{aligned} zw &= r(\cos \phi + i \sin \phi) \times s(\cos \psi + i \sin \psi) \\ &= rs[\cos \phi \cos \psi - \sin \phi \sin \psi + i(\cos \phi \sin \psi + \cos \psi \sin \phi)] \\ &= rs[\cos(\phi + \psi) + i \sin(\phi + \psi)]. \end{aligned}$$

## Observation and definition

Every complex number of the form  $z = \cos \phi + i \sin \phi$  has absolute value one. We introduce the exponential notation (known as Euler's formula)

$$\exp(i\phi) = e^{i\phi} = \cos \phi + i \sin \phi.$$

It might look strange at first, but the same rules as for the real exponential function apply. In fact, if we denote

$$f(\phi) = \cos \phi + i \sin \phi$$

then

$$\begin{aligned} f'(\phi) &= -\sin \phi + i \cos \phi \\ \frac{f'(\phi)}{f(\phi)} &= \frac{-\sin \phi + i \cos \phi}{\cos \phi + i \sin \phi} \\ &= \frac{-\sin \phi + i \cos \phi}{\cos \phi + i \sin \phi} \cdot \frac{\cos \phi - i \sin \phi}{\cos \phi - i \sin \phi} \\ &= \frac{-\sin \phi \cos \phi + i \sin^2 \phi + i \cos^2 \phi + \cos \phi \sin \phi}{\cos^2 \phi + \sin^2 \phi} \\ &= i \end{aligned}$$

since  $\cos^2 \phi + \sin^2 \phi = 1$ . Now integrate:

$$\int \frac{f'(\phi)}{f(\phi)} d\phi = \int i d\phi$$

$$\ln f(\phi) = i\phi$$

$$f(\phi) = e^{i\phi}$$

and thus  $\cos \phi + i \sin \phi = e^{i\phi}$ .

This has many advantages. First of all, we can write any complex number in polar coordinates as  $z = re^{i\phi}$ . And we can easily multiply complex numbers in this form. For example, the calculation above becomes a single step (no need to look up the trig identities)

$$re^{i\phi} \times se^{i\psi} = rse^{i(\phi+\psi)}.$$

### Examples

1. The complex number  $z = 1 + i$  has modulus  $|z| = \sqrt{2}$  and angle  $\phi = \pi/4$ . Hence  $z = 1 + i = \sqrt{2}e^{i\pi/4}$ .
2. The complex number  $w = \sqrt{3} + i$  has modulus  $|w| = 2$  and angle  $\phi = \pi/6$ . Hence  $w = \sqrt{3} + i = 2e^{i\pi/6}$ .
3. Their product is  $zw = (\sqrt{3} - 1) + (\sqrt{3} + 1)i = 2\sqrt{2}e^{i5\pi/12}$ .
4. In general, if  $z = a + bi$  then  $r = |z| = \sqrt{a^2 + b^2}$ . The argument  $\phi$  is not uniquely defined. If we restrict it between  $-\pi$  and  $\pi$  we get

$$\begin{cases} \phi = \arctan(b/a) & \text{if } a > 0 \\ \phi = \arctan(b/a) + \pi & \text{if } a < 0, b > 0 \\ \phi = \arctan(b/a) - \pi & \text{if } a < 0, b < 0 \end{cases}$$

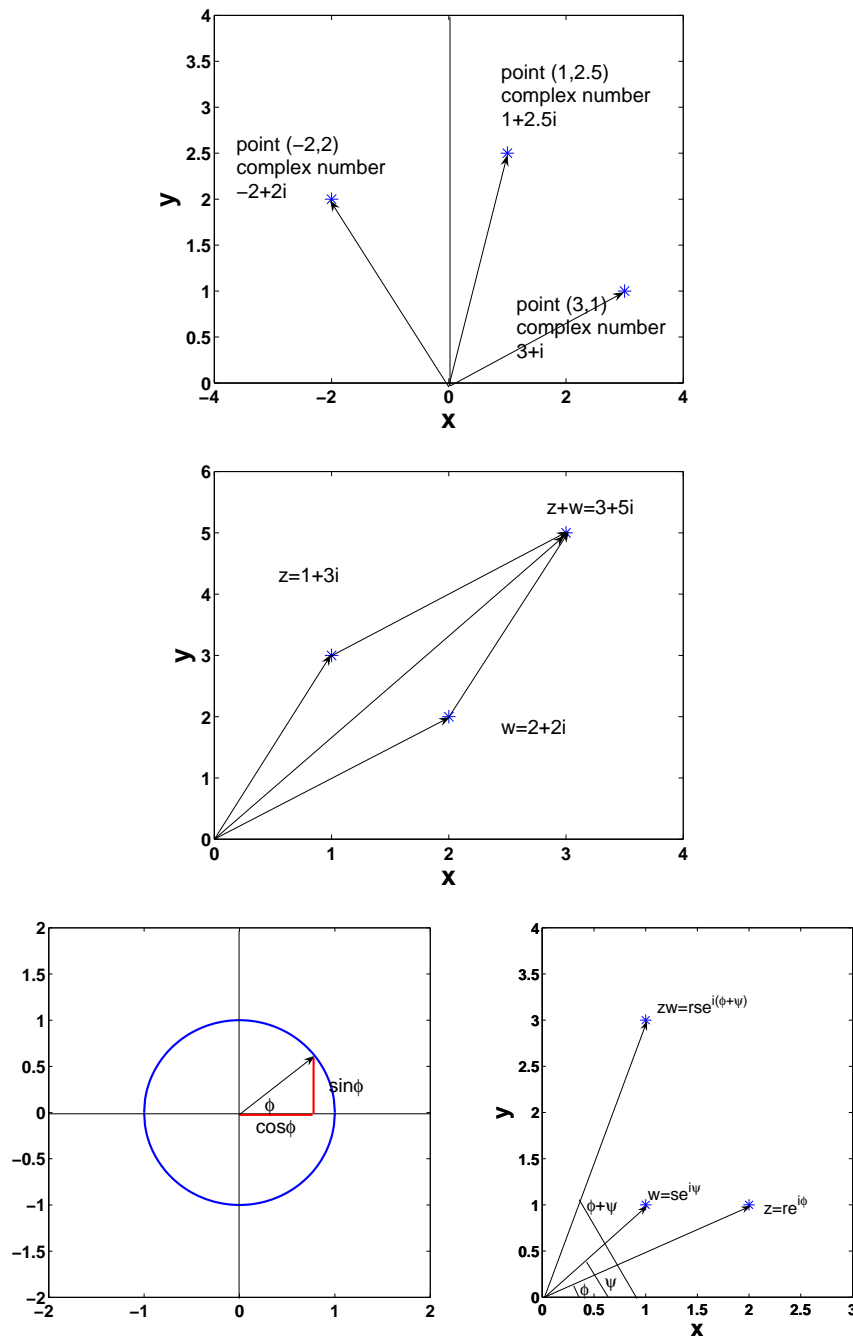


Figure 1: Top panel: correspondence between complex numbers and points in the plane. Middle panel: Addition of complex numbers. Bottom panel: polar coordinates on the unit circle (left) and multiplication of complex numbers using polar coordinates (right)

## Linear Algebra I - Linear systems of equations

### Introductory example

Suppose there are two types of food. Type I contains 10g of protein and 5g of carbohydrates per 100g, type II contains 8g of protein and 12g of carbohydrates per 100g.

**Easy question** Suppose you take 50g of type I and 75g of type II, how much protein and carbohydrates do you get? Answer: First we have to choose units, let's say 100g. Then I take 0.5 units of type I and 0.75 units of type II. Now let  $x_1, x_2$  denote the respective units of food type I and II. Then the amount of protein and carbohydrates are given by

$$\begin{aligned} b_1 &= 10x_1 + 8x_2 = 10 \times 0.5 + 8 \times 0.75 = 11, \\ b_2 &= 5x_1 + 12x_2 = 5 \times 0.5 + 12 \times 0.75 = 11.5, \end{aligned}$$

grams respectively. NOTE: we are given  $x_1$  and  $x_2$ , and want to find  $b_1$  and  $b_2$ .

**Harder question** Suppose you want to take 16g of protein and 20g of carbohydrates. How much of each food type to you have to take? The equations are just as above.

$$\begin{aligned} 10x_1 + 8x_2 &= b_1 = 16 \\ 5x_1 + 12x_2 &= b_2 = 20. \end{aligned}$$

HOWEVER: this time, with the same notation as above, we are given  $b_1$  and  $b_2$ , and want to find  $x_1$  and  $x_2$ .

SOLUTION: We multiply the second equation by -2:

$$\begin{aligned} 10x_1 + 8x_2 &= 16 \\ -10x_1 - 24x_2 &= -40. \end{aligned}$$

Then we add the two equations:

$$-16x_2 = -24,$$

which gives  $x_2 = 3/2$ . This we put back into one of the original equations to get  $10x_1 + 12 = 16$  or  $x_1 = 2/5$ .

The example above is a special case of a *linear system of equations*. More generally, we write a *linear system of  $m$  equations with  $n$  unknowns* as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m, \end{aligned}$$

where

1.  $x_1, \dots, x_n$  are the variables or unknowns,
2.  $a_{11}, \dots, a_{mn}$  are the coefficients,
3.  $b_1, \dots, b_m$  are the right hand side.

A *solution* of such a system is a set of numbers  $(s_1, \dots, s_n)$  that makes *all* the equations true when we substitute the  $s_i$  for the  $x_i$ . The equations are called *linear* since each of the variables  $x_i$  appears only linearly (as opposed to higher powers or other nonlinear functions). The subject of linear algebra is to study such linear systems of equations.

### Goal of this chapter

In this chapter we learn how to solve linear systems of equations. In particular, we answer the three questions:

1. Is there always a solution?
2. Can there be more than one solution?
3. How can we compute all solutions?

**Fact:** A linear system of equations has either

1. exactly one solution,
2. infinitely many solutions, or
3. no solution.

Cases (1) and (2) are called *consistent* whereas case (3) is called *inconsistent*.

### Examples

Consider the three systems of equations:

$$\left\{ \begin{array}{l} x_1 - x_2 = 1 \\ x_1 - x_2 = 0 \end{array} \right\}, \quad \left\{ \begin{array}{l} x_1 - x_2 = 1 \\ x_1 - 2x_2 = 0 \end{array} \right\}, \quad \left\{ \begin{array}{l} x_1 - x_2 = 1 \\ -x_1 + x_2 = -1 \end{array} \right\}.$$

The first system has no solution (since  $x_1 - x_2$  cannot simultaneously be both 1 and 0), the second has exactly one solution ( $x_1 = 2, x_2 = 1$ ) and the last system has infinitely many solutions (since the two equations are essentially the same equation): for all real numbers  $t$ , the pair  $x_1 = t, x_2 = t - 1$  is a solution. We can see graphically why there are these three cases. Each equation is the equation of a line in  $x_1$ - $x_2$  space. In the first case, the two lines are parallel and have no point in common. In the second case, there is one point of intersection. In the third case, the lines are identical, and all points are part of both lines, see Figure 2.

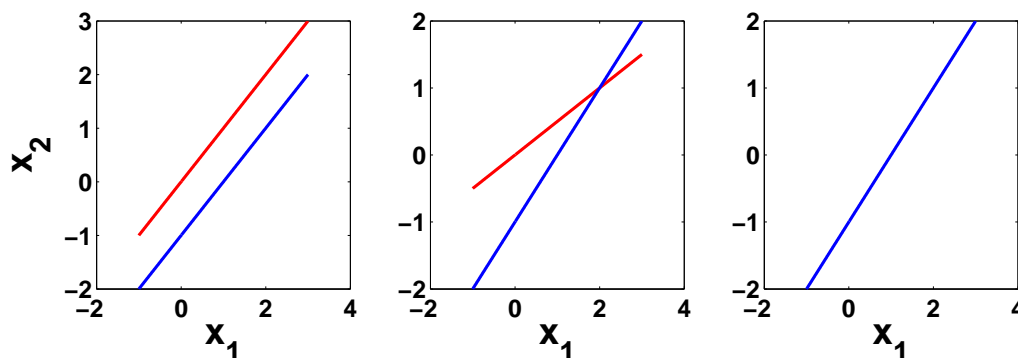


Figure 2: Graphical interpretation of systems of linear equations. Left: no solution, Middle: unique solution, Right: infinitely many solutions.

### The Gaussian Elimination Algorithm

**Elementary row operations:** The solution of a linear system of equations does not change under the following three elementary row operations:

1. multiply a row by a nonzero number,
2. add a multiple of one row to another,
3. exchange the order of two rows.

#### Example 1

Find the solution of

$$\begin{cases} 2x_1 - x_2 = 1 \\ 4x_1 + 2x_2 = 10 \end{cases}$$

Answer:

$$\begin{aligned} \begin{cases} 2x_1 - x_2 = 1 \\ 4x_1 + 2x_2 = 10 \end{cases} &\xrightarrow{(-0.5) \times R2} \begin{cases} 2x_1 - x_2 = 1 \\ -2x_1 - x_2 = -5 \end{cases} \xrightarrow{R2+R1} \begin{cases} 2x_1 - x_2 = 1 \\ -2x_2 = -4 \end{cases} \\ &\xrightarrow{-2 \times R1 + R2} \begin{cases} -4x_1 = -6 \\ -2x_2 = -4 \end{cases} \xrightarrow{\begin{smallmatrix} (-1/4) \times R1 \\ (-1/2) \times R2 \end{smallmatrix}} \begin{cases} x_1 = 3/2 \\ x_2 = 2 \end{cases} \end{aligned}$$

Now, the solution is obvious:  $x_1 = 3/2$  and  $x_2 = 2$ , or  $(3/2, 2)$ .

#### Example 2

Find the solution of

$$\begin{cases} 2x_2 + 3x_3 = 1 \\ x_1 + x_2 + x_3 = 2 \\ x_1 - x_2 = -1 \end{cases}$$

Answer:

$$\begin{aligned} \left\{ \begin{array}{l} 2x_2 + 3x_3 = 1 \\ x_1 + x_2 + x_3 = 2 \\ x_1 - x_2 = -1 \end{array} \right. & \xrightarrow{R1 \leftrightarrow R2} \left\{ \begin{array}{l} x_1 + x_2 + x_3 = 2 \\ 2x_2 + 3x_3 = 1 \\ x_1 - x_2 = -1 \end{array} \right. & \xrightarrow{(-1) \times R3 + R1} \left\{ \begin{array}{l} x_1 + x_2 + x_3 = 2 \\ 2x_2 + 3x_3 = 1 \\ 2x_2 + x_3 = 3 \end{array} \right. \\ & \xrightarrow{(-1) \times R3 + R2} \left\{ \begin{array}{l} x_1 + x_2 + x_3 = 2 \\ 2x_2 + 3x_3 = 1 \\ 2x_3 = -2 \end{array} \right. & \xrightarrow{(1/2) \times R3} \xrightarrow{(-3) \times R3 + R2} \xrightarrow{R1 - R3} \left\{ \begin{array}{l} x_1 + x_2 = 3 \\ 2x_2 = 4 \\ x_3 = -1 \end{array} \right. \end{aligned}$$

and finally

$$\xrightarrow{(1/2) \times R2} \xrightarrow{R1 - R2} \left\{ \begin{array}{l} x_1 = 1 \\ x_2 = 2 \\ x_3 = -1 \end{array} \right.$$

So the solution is  $(1, 2, -1)$ .

### Simplify Notation!

Looking at the two examples above, we realize that we do not need to write the variables  $x_i$  all the time, provided we agree and stick to a particular order. Similarly, we can do away with the '+' and the '=' signs. The only things that matters are the coefficients in front of the  $x_i$  and the right hand side. Hence, we collect these into an array of numbers, which we call *matrix*. We can then simply perform the row operations on the rows of the matrix.

For the example above,

$$\left\{ \begin{array}{l} 2x_2 + 3x_3 = 1 \\ x_1 + x_2 + x_3 = 2 \\ x_1 - x_2 = -1 \end{array} \right\} \longleftrightarrow \left[ \begin{array}{ccc|c} 0 & 2 & 3 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & -1 \end{array} \right]$$

We call the matrix above the *augmented matrix*, where we have the coefficients,  $a_{ij}$  of the linear system together with the right hand side  $b_i$ , separated by the vertical lines. We also call the matrix

$$\left[ \begin{array}{ccc} 0 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{array} \right]$$

the *coefficient matrix*, i.e., this matrix contains only the coefficients  $a_{ij}$ .

The same steps as in example 2 above, but in matrix notation are:

$$\begin{aligned} \left[ \begin{array}{ccc|c} 0 & 2 & 3 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & -1 \end{array} \right] & \xrightarrow{R1 \leftrightarrow R2} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 1 \\ 1 & -1 & 0 & -1 \end{array} \right] & \xrightarrow{(-1) \times R3 + R1} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 1 \\ 0 & 2 & 1 & 3 \end{array} \right] \\ & \xrightarrow{(-1) \times R3 + R2} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 2 & -2 \end{array} \right] & \longrightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array} \right] & \longrightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array} \right] \end{aligned}$$

Now, we remember that the first column corresponds to  $x_1$ , the second to  $x_2$ , and the third to  $x_3$ . Then the solution is  $x_1 = 1, x_2 = 2, x_3 = -1$  or simply  $(1, 2, -1)$ .

## Reduced row-echelon form

Now that we have seen several examples of ‘simple’ forms of systems where the solutions could easily be read off, we will formalize this process a bit.

**Definition:** The *leading entry* of a row in a matrix is the leftmost nonzero coefficient in that row.

A matrix is in *row-echelon form* if the following three rules are true

1. Rows of zeros are below any nonzero row.
2. The leading entry of any row is to the right of any leading entry in any row above it.
3. All entries in the column below a leading entry are zero.

A matrix is in *reduced row-echelon form* if it is in row-echelon form and *in addition*

4. Each leading entry is 1.
5. All entries in the column above a leading entry are zero.

## Examples

The first two systems are in row-echelon form but not reduced. The third and fourth are in reduced row-echelon form, the last one is neither.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 2 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix},$$

### Example 3

Find the solution of

$$\begin{cases} x_1 + x_2 + x_3 = 5 \\ 2x_1 - x_3 = 0 \\ x_2 + 4x_3 = 15 \end{cases}$$

Answer: Use the matrix notation to do the row operations.

$$\begin{bmatrix} 1 & 1 & 1 & | & 5 \\ 2 & 0 & -1 & | & 0 \\ 0 & 1 & 4 & | & 15 \end{bmatrix} \xrightarrow{(-2)R1+R2} \begin{bmatrix} 1 & 1 & 1 & | & 5 \\ 0 & -2 & -3 & | & -10 \\ 0 & 1 & 4 & | & 15 \end{bmatrix} \xrightarrow{\begin{matrix} R2+2R3 \\ (-1)\times R2 \end{matrix}} \begin{bmatrix} 1 & 1 & 1 & | & 5 \\ 0 & 2 & 3 & | & 10 \\ 0 & 0 & 5 & | & 20 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 0 & | & 1 \\ 0 & 2 & 0 & | & -2 \\ 0 & 0 & 1 & | & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 4 \end{bmatrix}$$

Hence, we have the solution  $x_1 = 2, x_2 = -1, x_3 = 4$  or  $(2, -1, 4)$ .

**Example 4**

Find the solution of

$$\begin{cases} 2x_1 + x_2 + 2x_3 & = 1 \\ -4x_1 & - x_3 & = 2 \\ 2x_1 + 5x_2 + 8x_3 & = 11 \end{cases}$$

Answer in matrix notation:

$$\left[ \begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ -4 & 0 & -1 & 2 \\ 2 & 5 & 8 & 11 \end{array} \right] \xrightarrow[-R1+R3]{2 \times R1+R2} \left[ \begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 4 & 6 & 10 \end{array} \right] \xrightarrow{(-1/2) \times R3+R2} \left[ \begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 2 \end{array} \right]$$

At this point, we pause and look at the last row of zeros. In the original notation with variables  $x_i$ , this row reads:  $0x_1 + 0x_2 + 0x_3 = 2$ . This is clearly impossible. Hence, this system is inconsistent, it has no solution.

**Example 5**

Find the solution of

$$\begin{cases} 2x_1 - 4x_2 - 8x_3 & = -18 \\ 3x_1 + 3x_2 + 15x_3 & = 18 \end{cases}$$

Answer in matrix notation:

$$\left[ \begin{array}{ccc|c} 2 & -4 & -8 & -18 \\ 3 & 3 & 15 & 18 \end{array} \right] \xrightarrow[1/3 \times R2]{0.5 \times R1} \left[ \begin{array}{ccc|c} 1 & -2 & -4 & -9 \\ 1 & 1 & 5 & 6 \end{array} \right] \xrightarrow{R2-R1} \left[ \begin{array}{ccc|c} 1 & -2 & -4 & -9 \\ 0 & 3 & 9 & 15 \end{array} \right]$$

$$\rightarrow \left[ \begin{array}{ccc|c} 1 & -2 & -4 & -9 \\ 0 & 1 & 3 & 5 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 5 \end{array} \right]$$

This time, the last row reads

$$x_2 + 3x_3 = 5.$$

Whatever we choose for  $x_3$ , we can always find an  $x_2$  to make the equation true. Hence, this equation has infinitely many solutions. We set  $x_3 = t$  as a *free variable*, then we get  $x_2 = 5 - 3t$ . We plug this into the first equation and solve for  $x_1$  as

$$x_1 + 2t = 1, \quad \text{or} \quad x_1 = 1 - 2t.$$

Hence, the infinitely many solutions can be written as the set  $\{(1 - 2t, 5 - 3t, t) : t \in \mathbb{R}\}$ .

**Example 6**

Find the solution of

$$\begin{cases} 2x_1 + 2x_2 + 2x_3 & = 0 \\ -2x_1 + 5x_2 + 2x_3 & = 1 \\ 8x_1 + x_2 + 4x_3 & = -1 \end{cases}$$

Answer in matrix notation:

$$\left[ \begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array} \right] \xrightarrow[-4 \times R1 + R3]{\begin{array}{l} 0.5 \times R1 \\ R1 + R2 \end{array}} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array} \right] \xrightarrow{R2 + R3} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Again, we have a row of zeros at the bottom. However, this time the last row reads:  $0x_1 + 0x_2 + 0x_3 = 0$ . This equation is satisfied for all values of  $x_1, x_2, x_3$ . We do not run into the same problem as in example 4. We simply continue as in example 5.

$$\longrightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array} \right] \longrightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 3/7 & -1/7 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The second row

$$x_2 + (4/7)x_3 = 1/7.$$

has again infinitely many solutions. We denote  $x_3 = t$  as the *free variable* and compute  $x_2 = 1/7 - (4/7)t$ . In the first row, we get

$$x_1 + (3/7)t = -1/7 \quad \text{or} \quad x_1 = -1/7 - 3t/7.$$

Hence, the solution set is  $\{(-\frac{1}{7} - \frac{3t}{7}, \frac{1}{7} - \frac{4t}{7}, t) : t \in \mathbb{R}\}$ .

### Example 7

For which values of  $h$  does the system

$$\begin{cases} x_1 + hx_2 & = -3 \\ -2x_1 + 4x_2 & = 6 \end{cases}$$

have (a) a unique solution, (b) infinitely many solutions, and (c) no solution?

Answer:

$$\left[ \begin{array}{cc|c} 1 & h & -3 \\ -2 & 4 & 6 \end{array} \right] \xrightarrow{2 \times R1 + R2} \left[ \begin{array}{cc|c} 1 & h & -3 \\ 0 & 2h - 4 & 0 \end{array} \right]$$

The last row gives the equation

$$(2h - 4)x_2 = 0.$$

If  $2h - 4 = 0$ , (i.e.,  $h = 2$ ) then  $x_2 = t$  is a free variable, and  $x_1 = -3 - hx_2 = -3 - 2t$ . If, on the other hand,  $2h - 4 \neq 0$  then the only way to satisfy the second row is  $x_2 = 0$ . In this case, the first row gives  $x_1 = -3$ . Hence, if  $h \neq 2$  then there is a unique solution, if  $h = 2$  then there are infinitely many solutions. For no value of  $h$  is there no solution.

## Practice Problems

1. Solve the following systems of equations by bringing them into reduced row-echelon form

$$(a) \begin{cases} x_1 + x_2 + 2x_3 = 9 \\ 2x_1 + 4x_2 - 3x_3 = 1 \\ 3x_1 + 6x_2 - 5x_3 = 0 \end{cases} \quad (b) \begin{cases} 5x_1 - 2x_2 + 6x_3 = 0 \\ -2x_1 + x_2 + 3x_3 = 1 \end{cases}$$

$$(c) \begin{cases} 2x_1 + 2x_3 = 1 \\ 3x_1 - x_2 + 4x_3 = 7 \\ 6x_1 + x_2 - x_3 = 0 \end{cases} \quad (d) \begin{cases} 7x_1 + 2x_2 + x_3 - 3x_4 = 5 \\ x_1 + 2x_2 + 4x_3 = 1 \end{cases}$$

$$(e) \begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases} \quad (f) \begin{cases} 2x_1 - x_2 - 3x_3 = 0 \\ -x_1 + 2x_2 - 3x_3 = 0 \\ x_1 + x_2 + 4x_3 = 0 \end{cases}$$

## 2. Problems with parameters

(a) For which values of  $a, b$  does the system

$$\begin{cases} x_1 + ax_2 = 1 \\ 2x_1 + 3x_2 = b \end{cases}$$

have (i) a unique solution, (ii) infinitely many solutions, or (iii) no solutions?

(b) Explain why the system

$$\begin{cases} x_1 + x_2 + 2x_3 = a \\ x_1 + x_3 = b \\ 2x_1 + x_2 + 3x_3 = c \end{cases}$$

is consistent if  $c = a + b$  but inconsistent in all other cases.

## 3. Application

Insects of two species are reared on two types of food. Species 1 consumes 5 units of food A and 3 units of food B per day. Species 2 consumes 2 units of A and 4 units of B, respectively. Every day, 900 units of food A and 960 units of food B are provided. How many individuals of each species are reared?

**Solutions to practice problems****1.**

$$(a) \quad (1, 2, 3) \quad (b) \quad \{(2 - 12t, 5 - 27t, t) : t \in \mathbb{R}\} \quad (c) \quad \left(\frac{11}{12}, -\frac{71}{12}, -\frac{5}{12}\right)$$
$$(d) \quad \left\{ \left( \frac{4}{3} + \frac{1}{2}s + \frac{1}{2}t, -\frac{1}{6} - \frac{27}{12}s - \frac{1}{4}t, s, t \right) : s, t \in \mathbb{R} \right\} \quad (e) \quad \left\{ \left( \frac{37}{3} - \frac{7}{3}t, 4t - 26, t \right) : t \in \mathbb{R} \right\}$$
$$(f) \quad (0, 0, 0)$$

**2.**

(a) Unique solution if  $a \neq 3/2$ . infinitely many solutions if  $a = 3/2$  and  $b = 2$ . No solution if  $a = 3/2$  and  $b \neq 2$ .

(b) Add the first and second row.

**3.**

Let  $x_i$  be the number of individuals of species  $i$ . Then the system is

$$5x_1 + 2x_2 = 900, \quad 3x_1 + 4x_2 = 960,$$

and the solution is  $(x_1, x_2) = (120, 150)$ .

## Linear Algebra II - Vectors and matrices

In the last section, we introduced matrices to simplify our life, as a short hand notation for linear systems of equations. In this section, we study matrices as objects in their own right. We learn when it is possible to add and multiply them and how to do it. While there is no direct biological application in this section, the content presented here is the foundation of everything to come. It is like learning the grammar of a language so that one can speak it properly later.

**Definition:** An  $m \times n$ -matrix  $A$  is a rectangular array of numbers with  $m$  rows and  $n$  columns, i.e.,

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

The numbers  $a_{ij}$  are called *entries*.

If  $m = n$  then  $A$  is a *square matrix*.

A  $1 \times n$ -matrix is called a row vector:  $[c_1, c_2, \dots, c_n]$ .

An  $m \times 1$ -matrix is called a column vector:  $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ .

Two matrices  $A = [a_{ij}]$ ,  $B = [b_{ij}]$  are said to be *equal* if they have the same dimension and if for all  $i, j$  we have  $a_{ij} = b_{ij}$ .

We note two special (classes of) matrices, the *zero matrix*

$$0 = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix},$$

and the *identity matrix*

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

The number of rows and columns of these two matrices is usually clear from the context.

If  $A$  is a square matrix, then the elements  $a_{ii}$  are called the *diagonal elements* and their sum of called the *trace* of  $A$ , i.e.,

$$A = \begin{bmatrix} 5 & 2 & -1 \\ 0 & 1 & 5 \\ 8 & 2 & -4 \end{bmatrix}$$

has the diagonal elements 5, 1, -4 and the trace is  $\text{tr}(A) = 5 + 1 - 4 = 2$ .

### Matrix addition and scalar multiplication

If two matrices,  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are both of the format  $m \times n$ , then we can form the sum of the two by entrywise addition to obtain a matrix of the same format

$$C = A + B = [c_{ij}] = [a_{ij} + b_{ij}].$$

If  $k$  is a number and  $A = [a_{ij}]$  an  $m \times n$ -matrix, then we define the entrywise product

$$kA = [ka_{ij}],$$

which is again an  $m \times n$ -matrix.

### Examples

Consider the two  $2 \times 3$ -matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then we have

$$\begin{aligned} A + B &= \begin{bmatrix} 2 & 2 & 4 \\ -3 & -1 & -1 \end{bmatrix} & 5A &= \begin{bmatrix} 5 & 10 & 15 \\ -15 & -10 & -5 \end{bmatrix}, \\ 3A + 2B &= \begin{bmatrix} 3 & 6 & 9 \\ -9 & -6 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 11 \\ -9 & -4 & -3 \end{bmatrix}. \end{aligned}$$

### The transpose of a matrix

The *transpose* of a matrix  $A$  is obtained from  $A$  by interchanging rows and columns, or, loosely speaking, by flipping the matrix along its diagonal. More formally, if  $A = [a_{ij}]$ , then the transpose is

$$A^T = [a_{ji}].$$

If  $A$  is of the format  $m \times n$  then  $A^T$  is of the format  $n \times m$ . In particular, the transpose of a column vector is a row vector and vice versa.

### Examples

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \end{bmatrix} & A^T &= \begin{bmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{bmatrix} \\ B &= \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} & B^T &= \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix} \\ v &= \begin{bmatrix} -4 \\ 7 \\ 0 \end{bmatrix} & v^T &= [ -4 \quad 7 \quad 0 ]. \end{aligned}$$

### Matrix-vector multiplication

If the matrix  $A$  has  $n$  columns and the column vector  $x$  has  $n$  rows, then we can define the product  $Ax$  as the following column vector:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}.$$

To remember this definition, simply think about linear systems of equations!

Note that the resulting vector has the same number of rows as the matrix. In general, the dimensions work as follows:

$$(m \times n) * (n \times 1) = (m \times 1).$$

### Examples

1.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \times 5 + 2 \times 6 \\ 3 \times 5 + 4 \times 6 \end{bmatrix} = \begin{bmatrix} 5 + 8 \\ 12 + 24 \end{bmatrix} = \begin{bmatrix} 13 \\ 39 \end{bmatrix}$$

2.

$$\begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times (-1) + 3 \times 2 \\ -3 \times 1 + (-2) \times (-1) + (-1) \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 2 + 6 \\ -3 + 2 - 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

3.

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 + 2 - 3 - 4 \\ 0 + 2 - 0 + 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

4.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - 3 + 2 \end{bmatrix} = 0$$

5.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1-2 \\ -3-4 \end{bmatrix} = \begin{bmatrix} -3 \\ -7 \end{bmatrix}$$

6.

$$\begin{bmatrix} -3 & 2 \\ -7 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -15-6 \\ -35-12 \\ 5-18 \end{bmatrix} = \begin{bmatrix} -21 \\ -47 \\ -13 \end{bmatrix}$$

7.

$$\begin{bmatrix} 30 & 25 \end{bmatrix} \begin{bmatrix} 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 60 + 225 \end{bmatrix} = 285$$

### Matrix-matrix multiplication

We use the definition of the matrix-vector product to define a product of two matrices. Consider a matrix  $A$  with  $n$  columns and a matrix  $B$  with  $n$  rows. We may think of each column of  $B$  as a column vector of length  $n$ . We know how to multiply each of these with the matrix  $A$  and then we put the resulting vectors into one matrix. More formally, the product of

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nk} \end{bmatrix} = \left[ \begin{array}{c|c|c|c} B_1 & B_2 & \cdots & B_k \end{array} \right]$$

is given by

$$AB = \left[ \begin{array}{c|c|c|c} AB_1 & AB_2 & \cdots & AB_k \end{array} \right].$$

Note that the dimensions multiply as follows:

$$(m \times n) * (n \times k) = (m \times k).$$

### Examples

Take the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ 4 & 2 \\ 0 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}.$$

The product  $AB$  is defined and it is

$$\begin{aligned} AB &= \begin{bmatrix} 1 \times 1 + 2 \times (-1) & 1 \times 2 + 2 \times 0 & 1 \times 1 + 2 \times 4 \\ 3 \times 1 + 4 \times (-1) & 3 \times 2 + 4 \times 0 & 3 \times 1 + 4 \times 4 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 2 & 9 \\ -1 & 6 & 19 \end{bmatrix}. \end{aligned}$$

The product  $BA$  is not defined since the number of columns of  $B$  does not equal the number of rows of  $A$  (try it and see!). However, if we transpose  $B$  first, then we can multiply  $B^T A$  :

$$\begin{bmatrix} -2 & -2 \\ 2 & 4 \\ 13 & 18 \end{bmatrix}.$$

The product  $CA$  is defined, but not vice versa (check the format!):

$$CA = \begin{bmatrix} -2 & -2 \\ 10 & 16 \\ 9 & 12 \end{bmatrix}.$$

Again, if we transpose  $C$ , then we can multiply  $AC^T$

$$AC^T = \begin{bmatrix} -1 & 8 & 6 \\ -1 & 20 & 12 \end{bmatrix}.$$

Since  $A$  is a square matrix, we can form the power

$$A^2 = AA = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}.$$

Since  $A$  and  $D$  are square of the same size, we can multiply them both ways:

$$AD = \begin{bmatrix} -1 & 0 \\ -3 & -2 \end{bmatrix}, \quad DA = \begin{bmatrix} -7 & -10 \\ 3 & 4 \end{bmatrix}.$$

Thus  $AD \neq DA$ , even though both are defined.

**NOTE:** The order of the product matters! Matrix multiplication is NOT commutative, even if both products are defined.

Multiplication of the identity matrix and the zero matrix are just as easy as multiplying the real numbers zero and one:

$$AI = IA = A, \quad A0 = 0A = 0.$$

As a final, not so obvious example for matrix multiplication, we note (check the formats!)

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ -1 & -3 & -2 \\ 2 & 6 & 4 \end{bmatrix}.$$

**Note:** We have seen that many operations with real numbers (addition, multiplication) also work with (some) matrices. Two of several differences are

1. The commutative law does not hold.
2. The cancellation law does not hold.

### Example

The matrix  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  is an example where  $A^2 = 0$  but  $A \neq 0$ .

**Practice Problems**

Consider the following matrices:

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}.$$

**1. Compute the following if possible. If not, explain why.**

- (a)  $D + E$       (b)  $D - E$       (c)  $5A$       (d)  $-7C$   
(e)  $2B - C$       (f)  $4E - 2D$       (g)  $-3(D + 2E)$       (h)  $A - A$   
(i)  $\text{tr}(D)$       (j)  $\text{tr}(D - 3E)$       (k)  $\text{tr}(A)$       (l)  $\text{tr}(B)$   
(m)  $2A^T + C$       (n)  $D^T - E^T$       (o)  $(D - E)^T$       (p)  $B - B^T$

**2. Compute the following if possible. If not, explain why.**

- (a)  $AB$       (b)  $BA$       (c)  $3ED$       (d)  $(AB)C$       (e)  $A(BC)$   
(f)  $CC^T$       (g)  $(DA)^T$       (h)  $(C^T B)A^T$       (i)  $\text{tr}(DD^T)$       (j)  $B^3$

## Solutions to Practice Problems

1.

$$(a) D + E = \begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}, \quad (b) D - E = \begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}, \quad (c) 5A = \begin{bmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{bmatrix},$$

$$(d) -7C = \begin{bmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{bmatrix}, \quad (e) 2B - C \text{ not defined}, \quad (f) 4E - 2D = \begin{bmatrix} 22 & -6 & 8 \\ -2 & 4 & 6 \\ 10 & 0 & 4 \end{bmatrix},$$

$$(g) -3(D + 2E) = \begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}, \quad (h) A - A = 0, \text{ the zero matrix}, \quad (i) \operatorname{tr}(D) = 1 + 0 + 4 = 5,$$

$$(j) \operatorname{tr}(D - 3E) = -17 - 3 - 5 = -25, \quad (k) \operatorname{tr}(A) \text{ not defined}, \quad (l) \operatorname{tr}(B) = 4 + 2 = 6.$$

$$(m) 2AT + C = \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix}, \quad (n) D^T - E^T = \begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}, \quad (o) (D_E)^T = D^T - E^T,$$

$$(p) B - B^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

2.

$$(a) AB = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}, \quad (b) BA \text{ not defined}, \quad (c) 3ED = \begin{bmatrix} 42 & 108 & 75 \\ 12 & -3 & 21 \\ 36 & 78 & 63 \end{bmatrix},$$

$$(d) (AB)C = \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}, \quad (e) A(BC) = (AB)C, \quad (f) CC^T = \begin{bmatrix} 21 & 17 \\ 17 & 35 \end{bmatrix},$$

$$(g) (DA)^T = \begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}, \quad (h) (C^T B)A^T = \begin{bmatrix} 12 & 6 & 9 \\ 48 & -20 & 14 \\ 24 & 8 & 16 \end{bmatrix}, \quad (i) \operatorname{tr}(DD^T) = 30 + 2 + 29 = 61,$$

$$(j) B^3 = \begin{bmatrix} 64 & -28 \\ 0 & 8 \end{bmatrix}.$$