**MAT1332** Spring/Summer 2009

Assignment 4 Solutions

1. First observe that the first equation and the third are the same. Thus, we are simply working with two equations

   $$2x+3y+v=0 \qquad \text{and} \qquad 2y+z=0$$

   in four unknowns. To obtain the non-trivial solutions we observe that the coefficient matrix of the system

   $$\begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

   is already in row-echelon form and the free variables are $z$ and $v$. First, we set $z := 2$, then the second equation yields $y = -1$. Setting $v := 1$ in the first equation yields $2x-3+1=0$ and therefore $x = 1$.

2. The expressions evaluate to

   (a) $\begin{bmatrix} -4 & -25 & -2 \\ -5 & -15 & 8 \\ -5 & 6 & 8 \end{bmatrix}$

   (b) $\begin{bmatrix} -63 & 67 \\ -75 & 11 \end{bmatrix}$

   (c) $\begin{bmatrix} -54 & -150 \\ -4 & -9 \\ 66 & 63 \end{bmatrix}$

   (d) $\begin{bmatrix} -12 & 44 \\ 9 & 5 \\ -42 & -36 \end{bmatrix}$

   (e) $\begin{bmatrix} 38 & 20 & -1 \\ -29 & -30 & 31 \\ 68 & -8 & 68 \end{bmatrix}$

   (f) $\begin{bmatrix} -9 & 1 & -6 \\ 11 & 2 & -3 \end{bmatrix}$

   (g) $\begin{bmatrix} 351 & -201 \\ 461 & 205 \end{bmatrix}$

3. We test the determinants to find $\det(A) = 1$, $\det(B) = 0$ and $\det(C) = -1$. Thus, $A$ and $C$ are invertible, but $B$ is not. To determine the inverses we augment the matrices and apply the elementary row operations as follows:

$$\left[\begin{array}{ccc|ccc} 6 & 1 & 4 & 1 & 0 & 0 \\ 1 & -2 & 3 & 0 & 1 & 0 \\ 2 & 8 & -7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R3-2*R2} \left[\begin{array}{ccc|ccc} 6 & 1 & 4 & 1 & 0 & 0 \\ 1 & -2 & 3 & 0 & 1 & 0 \\ 0 & 12 & -13 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R2-\frac{1}{6}R1} \left[\begin{array}{ccc|ccc} 6 & 1 & 4 & 1 & 0 & 0 \\ 0 & -\frac{13}{6} & \frac{7}{3} & -\frac{1}{6} & 1 & 0 \\ 0 & 12 & -13 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R3+\frac{72}{13}R2} \left[\begin{array}{ccc|ccc} 6 & 1 & 4 & 1 & 0 & 0 \\ 0 & -\frac{13}{6} & \frac{7}{3} & -\frac{1}{6} & 1 & 0 \\ 0 & 0 & -\frac{1}{13} & -\frac{12}{13} & \frac{46}{13} & 1 \end{array}\right]$$

$$\xrightarrow{R2+\frac{91}{3}*R3} \left[\begin{array}{ccc|ccc} 6 & 1 & 4 & 1 & 0 & 0 \\ 0 & -\frac{13}{6} & 0 & -\frac{169}{6} & \frac{325}{3} & \frac{91}{3} \\ 0 & 0 & -\frac{1}{13} & -\frac{12}{13} & \frac{46}{13} & 1 \end{array}\right]$$

$$\xrightarrow{R1+52*R3} \left[\begin{array}{ccc|ccc} 6 & 1 & 0 & -47 & 184 & 52 \\ 0 & -\frac{13}{6} & 0 & -\frac{169}{6} & \frac{325}{3} & \frac{91}{3} \\ 0 & 0 & -\frac{1}{13} & -\frac{12}{13} & \frac{46}{13} & 1 \end{array}\right]$$

$$\xrightarrow{R1+\frac{6}{13}*R2} \left[\begin{array}{ccc|ccc} 6 & 0 & 0 & -60 & 234 & 66 \\ 0 & -\frac{13}{6} & 0 & -\frac{169}{6} & \frac{325}{3} & \frac{91}{3} \\ 0 & 0 & -\frac{1}{13} & -\frac{12}{13} & \frac{46}{13} & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{6}*R1,-\frac{6}{13}*R2,-13*R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -10 & 39 & 11 \\ 0 & 1 & 0 & 13 & -50 & -14 \\ 0 & 0 & 1 & 12 & -46 & -13 \end{array}\right]$$

Thus, the inverse $A^{-1}$ of $A$ is the matrix

$$\begin{bmatrix} -10 & 39 & 11 \\ 13 & -50 & -14 \\ 12 & -46 & -13 \end{bmatrix}$$

The inverse $C^{-1}$ is calculated in a similar way as follows:

$$\left[\begin{array}{ccc|ccc} 7 & -2 & 4 & 1 & 0 & 0 \\ -3 & 5 & -1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R3+R2} \left[\begin{array}{ccc|ccc} 7 & -2 & 4 & 1 & 0 & 0 \\ -3 & 5 & -1 & 0 & 1 & 0 \\ 0 & 6 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R2+\frac{3}{7}R1} \left[\begin{array}{ccc|ccc} 7 & -2 & 4 & 1 & 0 & 0 \\ 0 & \frac{29}{7} & \frac{5}{7} & \frac{3}{7} & 1 & 0 \\ 0 & 6 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R3-\frac{42}{29}R2} \left[\begin{array}{ccc|ccc} 7 & -2 & 4 & 1 & 0 & 0 \\ 0 & \frac{29}{7} & \frac{5}{7} & \frac{3}{7} & 1 & 0 \\ 0 & 0 & -\frac{1}{29} & -\frac{18}{29} & -\frac{13}{29} & 1 \end{array}\right]$$

$$\xrightarrow{R2+\frac{145}{7}R3, R1+116R3} \left[\begin{array}{ccc|ccc} 7 & -2 & 0 & -71 & -52 & 116 \\ 0 & \frac{29}{7} & 0 & -\frac{87}{7} & -\frac{58}{7} & \frac{145}{7} \\ 0 & 0 & -\frac{1}{29} & -\frac{18}{29} & -\frac{13}{29} & 1 \end{array}\right]$$

$$\xrightarrow{R1+\frac{14}{29}R2} \left[\begin{array}{ccc|ccc} 7 & 0 & 0 & -77 & -56 & 126 \\ 0 & \frac{29}{7} & 0 & -\frac{87}{7} & -29 & \frac{145}{7} \\ 0 & 0 & -\frac{1}{29} & -\frac{18}{29} & -\frac{13}{29} & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{7}R1, \frac{7}{29}R2, -29R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & -8 & 18 \\ 0 & 1 & 0 & -3 & -2 & 5 \\ 0 & 0 & 1 & 18 & 13 & -29 \end{array}\right]$$

so that

$$C^{-1} = \begin{bmatrix} -11 & -8 & 18 \\ -3 & -2 & 5 \\ 18 & 13 & -29 \end{bmatrix}$$

4. To find the eigenvalues of $A$ we solve for the roots $\lambda$ of the polynomial

$$\det(A-\lambda I) \;=\; (18-\lambda)(16-\lambda)-(-2)(-4) \;=\; \lambda^2-34\lambda+280.$$

E.g., we may find the roots using the quadratic formula which tells us that here the roots are

$$\frac{34\pm\sqrt{1156-4(280)}}{2} \;=\; \frac{34\pm 2\sqrt{289-280}}{2} \;=\; 17\pm 3.$$

I.e., the eigenvalues of $A$ are $\lambda = 14$ and $\lambda = 20$. In the first case, where $\lambda = 14$, we find the eigenvectors by

$$A-\lambda I \;=\; \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & -4 \\ 0 & 0 \end{bmatrix}$$

The solutions of the corresponding equation $4x-4y=0$ are then $x=y=1$ ($y$ is the free variable). Thus, the eigenvectors for $\lambda=14$ are the non-zero scalar multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For $\lambda=20$, we have

$$A-\lambda I = \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & -4 \\ 0 & 0 \end{bmatrix}$$

In this case we have that $-2x-4y=0$ and a solution is given by $x=-2, y=1$ (again, $y$ is the free variable). Thus, the eigenvectors for $\lambda=20$ are the non-zero scalar multiples of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

For $B$ we have that

$$\det(B-\lambda I) = (3-\lambda)(7-\lambda)(2-\lambda)-(2-\lambda)(9)(5) = (2-\lambda)(\lambda^2-10\lambda-24).$$

Thus the eigenvalues of $B$ are $\lambda=2$ as well as the roots of $\lambda^2-10\lambda-24$ which are given by

$$\frac{10\pm\sqrt{100+96}}{2} = \frac{10\pm\sqrt{(4)(49)}}{2} = 5\pm 7.$$

So the eigenvalues of $B$ are $\lambda=2,-2$ and $12$. For $\lambda=2$ we have

$$B-\lambda I = \begin{bmatrix} 1 & 5 & -3 \\ 9 & 5 & -2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & -3 \\ 0 & -40 & 25 \\ 0 & 0 & 0 \end{bmatrix}$$

In the corresponding system of equations ($x+5y-3z=0$, $-40y+25z=0$) the variable $z$ is free and it is convenient to take $z:=8$. Then, by the second equation, $y=5$ and, by the first equation, $x=-1$. So the eigenvectors corresponding to eigenvalue $\lambda=2$ are non-zero scalar multiples of $\begin{bmatrix} -1 \\ 5 \\ 8 \end{bmatrix}$.

When $\lambda=-2$ we have

$$B-\lambda I = \begin{bmatrix} 5 & 5 & -3 \\ 9 & 9 & -2 \\ 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 & 5 & -3 \\ 0 & 0 & \frac{17}{5} \\ 0 & 0 & 4 \end{bmatrix}$$

So, $z=0$, $y$ is the free variable and taking $y=1$ gives $x=-1$. As such, for $\lambda=-2$, the eigenvectors are non-zero scalar multiples of $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Finally, when $\lambda=12$,

$$B-\lambda I = \begin{bmatrix} -9 & 5 & -3 \\ 9 & -5 & -2 \\ 0 & 0 & -10 \end{bmatrix} \longrightarrow \begin{bmatrix} -9 & 5 & -3 \\ 0 & 0 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

Therefore, $z = 0$ and $y$ is the free variable. Taking $y := 9$ gives $x = 5$. So, the eigenvectors, for $\lambda = 12$, are non-zero scalar multiples of $\begin{bmatrix} 5 \\ 9 \\ 0 \end{bmatrix}$.

5. We reason as follows:
   (a) We have

   $$\det(A - \lambda I) = (0.9 - \lambda)(0.6 - \lambda) - 0.04 = \lambda^2 - 1.5\lambda + 0.5 = (\lambda - 0.5)(\lambda - 1)$$

   and so the eigenvalues are $\lambda = 1$ and $\lambda = 0.5$. For $\lambda = 1$ we find

   $$A - \lambda I = \begin{bmatrix} -0.1 & 0.4 \\ 0.1 & -0.4 \end{bmatrix} \longrightarrow \begin{bmatrix} -0.1 & 0.4 \\ 0 & 0 \end{bmatrix}$$

   so that, taking the free variable $y := 1$, $x = 4$ and the eigenvectors are non-zero scalar multiples of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. For $\lambda = 0.5$ on the other hand

   $$A - \lambda I = \begin{bmatrix} 0.4 & 0.4 \\ 0.1 & 0.1 \end{bmatrix}$$

   and we find $x = -1$, $y = 1$ ($y$ is again the free variable). So non-zero scalar multiples of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvectors.

   (b) In order to find the equilibrium point $v_*$ with $Av_* = v_*$ and such that the entries of $v_*$ represent percentages (i.e., are between 0 and 1 and sum to 1), we take $r = \frac{1}{5}$ and multiply the eigenvector $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ for $\lambda = 1$ by $r$. I.e.,

   $$v_* := r \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

   (c) To find $x_n$ we first write our initial condition $x_0$ as a linear combination

   $$x_0 = \alpha v_* + \beta w,$$

   where $w$ is the eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. For this we look at the coefficient matrix of the corresponding system of equations:

   $$\left[\begin{array}{cc|c} 0.8 & -1 & 0.25 \\ 0.2 & 1 & 0.75 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 0.2 & 1 & 0.75 \\ 0 & -5 & -2.75 \end{array}\right]$$

   So $\beta = 0.55$ and $\alpha = 1$. Thus,

   $$A^n x_0 = A^n(v_* + \beta w) = A^n(v_*) + A^n(\beta w) = v_* + \beta A^n(w) = v_* + (0.55)(0.5)^n w$$