International Journal of Mathematics, Vol. 13, No. 7 (2002) 1–18

A COMPLETE SET OF ISOMETRIC IMPLEMENTERS FOR A QUASI-FREE ENDOMORPHISM OF THE CAR ALGEBRA VIA UNITARY DILATIONS

MICHAEL J. GABRIEL
Department of Mathematics and Statistics, University of Ottawa
585 King Edward Avenue, Ottawa, Ontario, Canada K1N 6N5
mamjg@swansea.ac.uk

Received 8 November 2001

In this paper we concern ourselves with non-surjective quasi-free endomorphisms of the CAR algebra. More precisely we give a construction of a complete set of isometric implementers for such an endomorphism in a fixed pure quasi-free state, the approach to which is more natural than has previously been proposed. The key idea in this construction is the use of the unitary implementers of an associated quasi-free automorphism in a set of carefully chosen quasi-free states. Our treatment is also novel in that we provide a sequence of implementers within the framework of the *complex* formalism of the CAR algebra.

Keywords:

Mathematics Subject Classification 2000: 46L60

1. Introduction

The implementation of quasi-free automorphisms (or Bogoliubov automorphisms) of the CAR algebra in a given pure quasi-free state by unitary operators on anti-symmetric Fock space is well-established and thoroughly understood, both in the complex and self-dual formalisms [2, 10–13, 15–17].

In [3], adopting the self-dual approach, the author considers the more general notion of implementation of (non-surjective) quasi-free endomorphisms of the CAR algebra in a pure quasi-free state by sequences of isometric operators on anti-symmetric Fock space, and explicitly produces such implementing sequences when they exist. These implementing isometries by definition satisfy the Cuntz relations and so give rise to representations of the Cuntz algebras [4] on anti-symmetric Fock space.

Much earlier in [7], in the complex formalism the definition of quasi-free automorphisms and endomorphisms was extended to completely positive quasi-free maps, and in [8], the notion of unitary implementation of a quasi-free automorphism in a pure quasi-free state was generalized to that of producing a *dissipator* for the completely positive quasi-free map in the pure quasi-free state. Furthermore, an explicit construction for this dissipator was given.

The methods of construction of [3] differ greatly from those employed in [8], and in this paper we show how using the approach of [8] restricted to the case of quasi-free endomorphisms, one can obtain the isometric implementers for a quasi-free endomorphism ρ_T produced in [3]. Consequently our results unify these distinct approaches.

The paper is organized as follows. In Sec. 2, we give the definitions and set notation that will be used throughout. Further, we briefly discuss the implementation theory of quasi-free automorphisms, endomorphisms and completely positive maps of the CAR algebra. Although this leaves us with a rather long preliminary section, we feel it is necessary if our motivation is to be made clear.

In Sec. 3, we prove that in the case of isometric T, the contractions $\mathcal{F}$ of Theorem 2.1 below (i.e. [8, Theorem 3.1]), are indeed intertwiners from π_P to $\pi_P \circ \rho_T$, where π_P is the GNS-representation associated with the quasi-free state ω_P, and are consequently scalar multiples of isometries.

Finally, in Sec. 4, we show how one can explicitly produce the complete set of isometric implementers for the quasi-free endomorphism ρ_T in the pure quasi-free state ω_P constructed in [3].

2. Preliminaries

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space. We denote by $A^{\mathrm{CAR}}(\mathcal{H})$ the CAR algebra over $\mathcal{H}$, i.e. the unital C^*-algebra generated by the range of a conjugate linear map $c : \mathcal{H} \ni f \mapsto c(f) \in A^{\mathrm{CAR}}(\mathcal{H})$ that satisfies the *canonical anti-commutation relations* (CAR's):

$$c(f)^* c(g) + c(g) c(f)^* = \langle f, g \rangle 1 \,,$$

$$c(f) c(g) + c(g) c(f) = 0$$

for $f, g \in \mathcal{H}$ where 1 is the unit of $A^{\mathrm{CAR}}(\mathcal{H})$ and $\langle \ , \ \rangle$ is the inner product on $\mathcal{H}$.

When R is a positive contraction on $\mathcal{H}$, there exists a unique state ω_R on $A^{\mathrm{CAR}}(\mathcal{H})$ satisfying

$$\omega_R(c(f_n)^* \cdots c(f_1)^* c(g_1) \cdots c(g_m)) = \delta_{nm} \det[\langle R f_i, g_j \rangle] \,,$$

for $f_1, \ldots, f_n, g_1, \ldots g_m \in \mathcal{H}$, and such a state is called a (gauge-invariant) *quasi-free* state on $A^{\mathrm{CAR}}(\mathcal{H})$. Note that ω_R is pure if and only if R is a projection, and that it is these pure quasi-free states that appear in this article.

For R a projection on $\mathcal{H}$, the GNS-decomposition of ω_R can then be identified with the triple $(F_a(\mathcal{H}), \pi_R, \Omega)$, where $F_a(\mathcal{H})$ is anti-symmetric Fock space over $\mathcal{H}$ with vacuum vector Ω, and with J a conjugation on $\mathcal{H}$ commuting with R, π_R is the representation defined by

$$\pi_R(c(f)) = a((1-R)f) + a(JRf)^* \,,$$

where $a(\cdot)$ and $a(\cdot)^*$ are respectively the annihilation and creation operators on $F_a(\mathcal{H})$.

If U is a unitary operator, respectively a non-unitary isometry, on $\mathcal{H}$, let α_U, respectively ρ_U, denote the *quasi-free automorphism*, respectively *endomorphism*, of $A^{\mathrm{CAR}}(\mathcal{H})$ determined by the maps,

$$c(f) \mapsto c(Uf) \quad \text{and} \quad c(f)^* \mapsto c(Uf)^* , \quad \text{for } f \in \mathcal{H} .$$

Then when P is a projection on $\mathcal{H}$, α_U is said to be *unitarily implementable* in the state ω_P if for some unitary $\mathcal{U} \in B(F_a(\mathcal{H}))$,

$$\pi_P(\alpha_U(x)) = \mathcal{U}\pi_P(x)\mathcal{U}^* , \quad \text{for all } x \in A^{\mathrm{CAR}}(\mathcal{H}) ,$$

and in [17], Shale and Stinespring showed that the existence of such a unitary $\mathcal{U}$ (necessarily unique up to a phase) is equivalent to the commutator $[U, P]$ being Hilbert–Schmidt class. However, more significant for us than existence, is the construction of $\mathcal{U}$ itself and the properties of this operator.

Many authors have given explicit constructions of these unitary implementers [2, 10–13, 15, 16] but it is the simple expression obtained by Ruijsenaars [15] that will be of greatest importantance as far as we are concerned.

Now the notion of unitary implementation of an automorphism of a C^*-algebra in a representation can be extended to the setting of completely positive maps on a C^*-algebra. With T a completely positive contraction on a C^*-algebra A, and π a representation of A on a Hilbert space $\mathcal{H}$, an operator F on $\mathcal{H}$ such that the map

$$A \ni x \mapsto \pi(T(x)) - F\pi(x)F^* \in B(\mathcal{H})$$

is completely positive, is said to be a *dissipator* for T in the representation π [9].

The completely positive maps that concern us are the so-called completely positive quasi-free maps on $A^{\mathrm{CAR}}(\mathcal{H})$ introduced and studied by Evans in [7]. Suppose that $\mathcal{H}$ and $\mathcal{L}$ are Hilbert spaces on which we have projections R and S respectively, and let T be a contraction from $\mathcal{H}$ to $\mathcal{L}$ such that $TR = ST$. The completely positive unital map $A_R(T)$ from $A^{CAR}\mathcal{H}$ into $A^{\mathrm{CAR}}(\mathcal{L})$ given by

$$\begin{aligned} &A_R(T)(: a(f_1)^* \cdots a(f_n)^* a(g_1) \cdots a(g_m) :_R) \\ &\qquad =: a(Tf_1)^* \cdots a(Tf_n)^* a(Tg_1) \cdots a(Tg_m) :_S \end{aligned}$$

is defined to be a *completely positive quasi-free map* [7], where

$$: a(h_1)^* \cdots a(h_n)^* a(l_1) \cdots a(l_m) :_X$$

denotes the Wick ordered product, normally ordered with respect to the quasi-free state ω_X, for X a contraction [7].

For the explicit construction of a completely positive quasi-free map on $A^{\mathrm{CAR}}(\mathcal{H})$ and the details of the general theory of these maps, see [7, 9].

Note that when T is an isometry, $A_R(T)$, then denoted $A(T)$ due to its consequent lack of dependence on R [7], is a *-homomorphism from $A^{\mathrm{CAR}}(\mathcal{H})$ into

$A^{\mathrm{CAR}}(\mathcal{K})$, and that in particular, if T is an isometry on $\mathcal{H}$ then of course $A(T)$ is the quasi-free endomorphism ρ_T.

Suppose that P and R are projections on $\mathcal{H}$, and T is a contraction on $\mathcal{H}$ with $TR = RT$. Then in [7], for $T \in C(P,R)$, where $C(P,R)$ is defined as the *-semigroup consisting of those contractions T on $\mathcal{H}$ such that $[T,P]$ is Hilbert–Schmidt, $TR = RT$, and

$$P(1-T^*T)(1-R)P\,, \quad P(1-TT^*)(1-R)P\,,$$
$$(1-P)(1-T^*T)R(1-P)\,, \quad (1-P)(1-TT^*)R(1-P)$$

are trace-class, a dissipator for $A_R(T)$ and a dissipator for $A_R(T^*)$ in π_P is constructed. That is, with $D_{\mathcal{H}}$ defined to be the dense subspace of $F_a(\mathcal{H})$ consisting of finite linear combinations of vectors of the form $\prod_{i=1}^n a(h_i)^*\Omega$, with $h_i \in \mathcal{H}$ and $n \geq 0$.

Theorem 2.1. *There exists a non-zero contraction* $\mathcal{F}$ *on* $F_a(\mathcal{H})$, *given on* $D_{\mathcal{H}}$ *by*

$$\mathcal{F} = \det[P_- + \Lambda(U)^*_{+-}\Lambda(U)_{+-}]^{1/2} : \exp(b(\Lambda(T))) :$$

such that the maps

$$x \mapsto \pi_P(A_R(T)(x)) - \mathcal{F}\pi_P(x)\mathcal{F}^* \quad \text{and} \quad x \mapsto \pi_P(A_R(T^*)(x)) - \mathcal{F}^*\pi_P(x)\mathcal{F}$$

are completely positive from $A^{\mathrm{CAR}}(\mathcal{H})$ *into* $B(F_a(\mathcal{H}))$.

Here the dissipator $\mathcal{F}$ is actually the operator $F(W)^*\mathcal{U}F(W)$, where $\mathcal{U}$ is the unitary implementer of α_U in $\pi_{P\oplus R}$ given by Ruijsenaars in [15], with U the unitary,

$$\begin{pmatrix} T & -(1-TT^*)^{1/2} \\ (1-T^*T)^{1/2} & T^* \end{pmatrix} \tag{1}$$

on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$, and $F(W)$ is the second quantization of the isometry W defined by $W : \mathcal{H} \ni h \mapsto h \oplus 0 \in \mathcal{K}$.

The problem of generalizing the notion of unitary implementation of a quasi-free automorphism in a pure quasi-free representation to "isometric implementation of a quasi-free *endo*morphism" in such a representation has been considered in [3]. However for his study Binnenhei adopts an alternative formalism for the CAR algebra, the *self-dual* formalism, as opposed to the *complex* formalism, which is the one with which we have been dealing so far. Thus in [3], the author is in fact generalizing the work of [16], as opposed to [15], which was in essence the starting point for [8].

The *self-dual* CAR algebra over a Hilbert space $\mathcal{K}$, with distinguished conjugation Γ, denoted by $A^{\mathrm{SDC}}(\mathcal{K},\Gamma)$, is the unital C^*-algebra generated by the range of a complex linear map B, on $\mathcal{K}$, satisfying the *self-dual canonical anti-commutation relations*:

$$B(f)B(g)^* + B(g)^*B(f) = \langle g, f\rangle 1\,, \quad B(f)^* = B(\Gamma f)$$

for $f, g \in \mathcal{K}$. If P is a projection on $\mathcal{K}$ satisfying $\Gamma P \Gamma = 1 - P$, then for $f \in P\mathcal{K}$, one can respectively identify $B(f)$ and $B(\Gamma f)$ with $a(f)^*$ and $a(f)$ in $A^{\mathrm{CAR}}(P\mathcal{K})$ so that $A^{\mathrm{SDC}}(\mathcal{K}, \Gamma) \cong A^{\mathrm{CAR}}(P\mathcal{K})$. Such a P is called a *basis projection* on $\mathcal{K}$.

The following definition, for arbitrary C^*-algebras, *-endomorphisms and representations, appears in [3], though its motivation lies in the work of [5, 6] and [14]:

Definition 2.2. A *-endomorphism ϱ of a C^*-algebra A is *isometrically implementable* in a representation $(\pi, \mathcal{H})$ if there exists a (possibly finite) sequence $\{\Psi_n\}_{n \in I}$ in $B(\mathcal{H})$ with relations

$$\Psi_m^* \Psi_n = \delta_{mn} 1 \,, \quad \sum_{n \in I} \Psi_n \Psi_n^* = 1$$

which implements ϱ by

$$\pi \circ \varrho = \sum_{n \in I} \Psi_n \pi(\cdot) \Psi_n^* \,,$$

with convergence of the sums with respect to the strong operator topology if I is infinite.

The meaning of the terms, quasi-free endomorphism and automorphism in this, the self-dual formalism, now needs to be clarified. Thus, if V is an isometry in $B(\mathcal{K})$, commuting with γ, then it is said to be a *Bogoliubov operator* and it induces a unital isometric *-endomorphism ϱ_V of $A^{\mathrm{SDC}}(\mathcal{K}, \Gamma)$, i.e. a *quasi-free endomorphism*, via the map

$$B(k) \mapsto B(Vk) \,, \quad k \in \mathcal{K} \,.$$

Then ϱ_V is a *-automorphism if and only if V is unitary.

Now the notion of a quasi-free state extends to the self-dual setting. For each $S \in B(\mathcal{K})$ with $0 \le S \le 1$ and $\Gamma S \Gamma = 1 - S$, there exists such a state, φ_S, on $A^{\mathrm{SDC}}(\mathcal{K}, \Gamma)$ and furthermore it is pure if and only if S is a (basis) projection. We choose to omit its definition here, but it suffices to say that for P_1 a basis projection on $\mathcal{K}$, the GNS-decomposition of φ_{P_1} can be identified with the triple $(\pi_{P_1}, F_a(\mathcal{K}_1), \Omega_{P_1})$, where Ω_{P_1} is the usual Fock vacuum vector, and π_{P_1} is the representation given by

$$\pi_{P_1}(B(k)) = a(P_1 k)^* + a(P_1 \Gamma k) \,, \quad \text{for } k \in \mathcal{K} \,,$$

with $a(\cdot)^*$ and $a(\cdot)$ the creation and annihilation operators on $F_a(\mathcal{K}_1)$ and $\mathcal{K}_1 := P_1 \mathcal{K}$.

Further, we define here a dense subspace $\mathcal{D}$ of $F_a(\mathcal{K}_1)$ which will be important throughout by $\mathcal{D} := \pi_{P_1}(A^{\mathrm{SDC}}(\mathcal{K}, \Gamma))\Omega_{P_1}$, and for $P_2 := 1 - P_1$, when $A \in B(\mathcal{K})$ we denote by $A_{\alpha,\beta}$ the operator $P_\alpha A P_\beta$ where $\alpha, \beta = 1, 2$.

For the definition of a quasi-free state on $A^{\mathrm{SDC}}(\mathcal{K}, \Gamma)$, and for details on the self-dual CAR algebra itself, see for example [1, 9]. Furthermore, for the relation between the S of φ_S and the R of ω_R on identified self-dual and complex CAR algebras respectively, and similarly for the relation between the T of ρ_T and the V of ϱ_V, see [9].

Theorem 2.3. *A quasi-free endomorphism* ϱ_V *of* $A^{\mathrm{SDC}}(\mathcal{K},\Gamma)$ *is isometrically implementable in a Fock repectively* π_{P_1} *if and only if* V_{12} *is Hilbert–Schmidt.*

This is [3, Theorem 3.3]. Note that V_{12} being Hilbert–Schmidt class is equivalent to the Shale–Stinespring condition, $[V, P_1]$ being Hilbert–Schmidt class.

The principal idea behind the method of construction of isometric implementers for ϱ_V in π_{P_1} of [3] is to first construct a single intertwining isometry $\Psi_0(V)$, then build a complete sequence of implementing isometries by multiplying the original one by certain partial isometries in

$$\pi_{P_1}(\varrho_V(A^{\mathrm{SDC}}(\mathcal{K},\Gamma)))' \subset B(F_a(\mathcal{K}_1))\,.$$

3. An Intertwining Contraction

Suppose T is an isometry on a Hilbert space $\mathcal{H}$, on which P and R are projections. Let J_1 and J_2 be conjugations on $\mathcal{H}$, commuting respectively with P and R, and take $J = J_1 \oplus J_2$ to be our conjugation commuting with the projection $P \oplus R$ on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$. Assume that $T \in C(P, R)$ and as in (1), define the unitary,

$$U = \begin{pmatrix} T & TT^* - 1 \\ 0 & T^* \end{pmatrix}$$

on $\mathcal{K}$. Let $P_+ = 1 - P$, $P_- = P$, $Q_+ = (1-P) \oplus (1-R)$, $Q_- = P \oplus R$, $\mathcal{H}_\varepsilon = P_\varepsilon \mathcal{H}$, $\mathcal{K}_\varepsilon = Q_\varepsilon \mathcal{K}$, and for $X \in B(\mathcal{H})$, respectively $B(\mathcal{K})$, let $X_{\alpha,\beta}$ denote $P_\alpha X P_\beta$, respectively $Q_\alpha X Q_\beta$, where $\varepsilon, \alpha, \beta = +, -$. Furthermore, let $a_X(\cdot)^{(*)}$, $b_X(\cdot)^{(*)}$ be the creation and annihilation operators on $F_a(X)$, for $X = \mathcal{H}, \mathcal{K}$. Then we have that

$$\pi_{P\oplus R}(\alpha_U(x)) = \mathcal{U}\pi_{P\oplus R}(x)\mathcal{U}^*\,, \quad \text{for all } x \in A^{\mathrm{CAR}}(\mathcal{K})\,,$$

where $\mathcal{U}$ is the unitary implementer given in [15]. Equivalently,

$$\mathcal{U} a_{\mathcal{K}}(f)^* = a^U_{\mathcal{K}}(f)^* \mathcal{U}\,, \tag{2}$$

$$\mathcal{U} a_{\mathcal{K}}(f) = a^U_{\mathcal{K}}(f) \mathcal{U}\,, \quad \text{for } f \in \mathcal{K}_+\,, \quad \text{and} \tag{3}$$

$$\mathcal{U} b_{\mathcal{K}}(Jg)^* = b^U_{\mathcal{K}}(Jg)^* \mathcal{U}\,, \tag{4}$$

$$\mathcal{U} b_{\mathcal{K}}(Jg) = b^U_{\mathcal{K}}(Jg) \mathcal{U}\,, \quad \text{for } g \in \mathcal{K}_-\,, \tag{5}$$

where for $f \in \mathcal{K}_+$ and $g \in \mathcal{K}_-$,

$$a^U_{\mathcal{K}}(f) := \pi_{P\oplus R}(a(Uf)) = a_{\mathcal{K}}(U_{++}f) + b_{\mathcal{K}}(JU_{-+}f)^*\,,$$

$$b^U_{\mathcal{K}}(Jg) := \pi_{P\oplus R}(a(Ug)) = a_{\mathcal{K}}(U_{+-}g) + b_{\mathcal{K}}(JU_{--}g)^*\,.$$

Proposition 3.1. *The contraction* $\mathcal{F}(T) := F(W)^*\mathcal{U}F(W)$, *with* $W : \mathcal{H} \hookrightarrow \mathcal{K}, h \mapsto h \oplus 0$, *is an intertwiner from* π_P *to* $\pi_P \circ \rho_T$.

Proof. From (2), we have that

$$F(W)^*\mathcal{U}a_{\mathcal{K}}(f\oplus f')^*F(W) = F(W)^*a^U_{\mathcal{K}}(f\oplus f')^*\mathcal{U}F(W)\,, \quad \text{for all } f\oplus f'\in\mathcal{K}_+$$

which obviously implies that

$$F(W)^*\mathcal{U}a_{\mathcal{K}}(f\oplus 0)^*F(W) = F(W)^*a^U_{\mathcal{K}}(f\oplus 0)^*\mathcal{U}F(W)\,, \quad \text{for all } f\in\mathcal{H}_+\,. \tag{6}$$

However,

$$U_{++} = \begin{pmatrix} T_{++} & (1-P)(TT^*-1)(1-R) \\ 0 & (1-R)T^* \end{pmatrix}$$

and

$$U_{-+} = \begin{pmatrix} T_{-+} & P(TT^*-1)(1-R) \\ 0 & 0 \end{pmatrix},$$

so that for $f\in\mathcal{H}_+$,

$$a^U_{\mathcal{K}}(f\oplus 0) = a_{\mathcal{K}}(T_{++}f\oplus 0) + b_{\mathcal{K}}(J_1T_{-+}f\oplus 0)^*\,.$$

Thus, since

$$F(W)^*a_{\mathcal{K}}(v\oplus v')^* = a_{\mathcal{H}}(v)^*F(W)^*\,, \quad \text{for all } v\oplus v'\in\mathcal{K}_+\,, \quad \text{and} \tag{7}$$

$$F(W)^*b_{\mathcal{K}}(w\oplus 0) = b_{\mathcal{H}}(w)F(W)^*\,, \quad \text{for all } w\in\mathcal{H}_-\,, \tag{8}$$

it follows from (6) that

$$\mathcal{F}(T)a_{\mathcal{H}}(f)^* = (a_{\mathcal{H}}(T_{++}f)^* + b_{\mathcal{H}}(J_1T_{-+}f))\mathcal{F}(T) = a^T_{\mathcal{H}}(f)^*\mathcal{F}(T)\,, \tag{9}$$

where $a^T_H(f) := \pi_P(a(Tf))$ for $f\in\mathcal{H}_+$. Note that if, in (7) and (8), we interchange the a's and the b's, then these equations obviously hold for all $v\oplus v'\in\mathcal{K}_-$ and for all $w\in\mathcal{H}_+$. Hence, from (3), it is clear that we have

$$\mathcal{F}(T)a_{\mathcal{H}}(f) = a^T_{\mathcal{H}}(f)\mathcal{F}(T)\,, \quad \text{for all } f\in\mathcal{H}_+\,. \tag{10}$$

Similarly, for $g\in\mathcal{H}_-$, we have that

$$b^U_{\mathcal{K}}(J(g\oplus 0)) = a_{\mathcal{K}}(T_{+-}g\oplus 0)^* + b_{\mathcal{K}}(J_1T_{--}g\oplus 0)\,.$$

Then again via (7) and (8) and their counterparts obtained by interchanging the a's and b's, it follows from (4), respectively (5), that

$$\mathcal{F}(T)b_{\mathcal{H}}(J_1g)^* = b^T_{\mathcal{H}}(J_1g)^*\mathcal{F}(T)\,, \tag{11}$$

and

$$\mathcal{F}(T)b_{\mathcal{H}}(J_1g) = b^T_{\mathcal{H}}(J_1g)\mathcal{F}(T)\,, \tag{12}$$

where $b^T_{\mathcal{H}}(J_1g)^* := a_{\mathcal{H}}(T_{+-}g) + b_{\mathcal{H}}(J_1T_{--}g)^* = \pi_P(a(Tg))$, for $g\in\mathcal{H}_-$. Thus, since $\mathcal{F}(T)$ is a contraction on $F_a(\mathcal{H})$, it follows by continuity from (9)–(12) that

$$\mathcal{F}(T)\pi_P(x) = \pi_P(\rho_T(x))\mathcal{F}(T)\,, \quad \text{for all } x\in A^{\mathrm{CAR}}(\mathcal{H})\,. \qquad \square$$

By the above proposition, the following is easily verified:

Proposition 3.2. $\mathcal{F}(T)^*\mathcal{F}(T)\in\mathbf{C}1$.

Proof. For all $x \in A^{\mathrm{CAR}}(\mathcal{H})$, we have

$$\mathcal{F}(T)^*\mathcal{F}(T)\pi_P(x) = \mathcal{F}(T)^*\pi_P(\rho_T(x))\mathcal{F}(T) = \pi_P(x)\mathcal{F}(T)^*\mathcal{F}(T)\,,$$

so that the claim follows from the irreducibility of π_P. □

4. A Complete Set of Implementers

We now proceed to produce a complete set of implementing isometries.

Let T be an isometry and P a projection on $\mathcal{H}$, with

$$[T, P] \in \mathrm{HS}(\mathcal{H}) \text{ and } d := \dim\ker T^* < \infty\,. \tag{13}$$

Then let $\{h_1, \ldots, h_d\}$ be an orthonormal basis for $\ker T^*$ and $\mathcal{K}$, J_1, U, P_+ and P_- be as described at the beginning of Sec. 3.

Definition 4.1.

$$\mathcal{A} := \{S = \{h_{s_1}, \ldots, h_{s_r}\} \subseteq \{h_1, \ldots, h_d\}$$
$$: 1 \leq s_1 < s_2 < \cdots < s_r \leq d\,,\ r = 0, 1, \ldots, d\}\,,$$

where $r = 0$ if $S = \emptyset$.

Definition 4.2. Then for each $S \in \mathcal{A}$, we define the following projections:

(i) $p(S) :=$ the projection of $\mathcal{H}$ onto lin $\{h_{s_1}, \ldots, h_{s_r}\} \subseteq \ker T^*$,
(ii) $P(S) := \sum_{i=0}^{\infty} T^i p(S) T^{*i}$, and
(iii) $Q(S)_+ := P_+ \oplus P(S)$ and $Q(S)_- := P_- \oplus P(S^c)$ on $\mathcal{K}$,

where $S^c := \{h_1, \ldots, h_d\} \setminus \{h_{s_1}, \ldots, h_{s_r}\} \in \mathcal{A}$. Furthermore, with $A \in B(\mathcal{K})$, $A^S_{\alpha,\beta} := Q(S)_\alpha A Q(S)_\beta$, for $\alpha, \beta \in \{+, -\}$.

Note that $T^i p(S) T^{*i} \perp T^j p(S) T^{*j}$, for $i \neq j$. Then we have,

$$P(S)T = TP(S)\,, \quad \text{for all } S \in \mathcal{A}\,, \tag{14}$$

as of course $p(S)T = 0$, and

$$P(S) + P(S^c) = \sum_{i=0}^{\infty} T^i (p(S) + p(S^c)) T^{*i}$$
$$= \sum_{i=0}^{\infty} T^i (1 - TT^*) T^{*i} = 1\,.$$

Then from (13) and (14), $T \in C(P, P(S))$, and so for each $S \in \mathcal{A}$, let $\mathcal{U}(S)$ be the unitary implementer of α_U in $\pi_{Q(S)_-}$, and $\mathcal{F}(S)$ be the contraction $F(W)^*\mathcal{U}(S)F(W)$.

Suppose for simplicity, thought the result still holds when this assumption is dropped, that we can choose an orthonormal basis, $\{h_1, \ldots h_d\}$ for $\ker T^*$ such that

$$h_1, \ldots, h_n \in \mathcal{H}_- \quad \text{and} \quad h_{n+1}, \ldots, h_d \in \mathcal{H}_+\,, \tag{15}$$

with $0 \leq n \leq d$, where $n = 0$ if $\ker T^* \subset \mathcal{H}_+$, and $n = d$ if $\ker T^* \subset \mathcal{H}_-$.

Normalizing each $\mathcal{F}(S)$ so that $\mathcal{F}(S)^*\mathcal{F}(S) = 1$, and denoting the resultant operator by the same symbol, we have the following:

Theorem 4.3. *The* 2^d *isometries* $\{\mathcal{F}(S)\}_{S\in\mathcal{A}}$ *can be identified with the isometric implementers of* ϱ_V *in* π_{P_1} *constructed in* [3], *where* V *is the Bogoliubov operator* $T \oplus J_1TJ_1$ *and* P_1 *is the basis projection* $P_+ \oplus P_-$ *on* $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$, *with respect to the distinguished conjugation*

$$\Gamma = \begin{pmatrix} 0 & J_1 \\ J_1 & 0 \end{pmatrix}.$$

This, then, of course means the following:

Corollary 4.4. *The* 2^d *isometries* $\{\mathcal{F}(S)\}_{S\in\mathcal{A}}$ *form a complete set of implementers for* ρ_T *in* π_P.

The proof of this theorem will comprise the rest of Sec. 4. We begin by considering in detail the isometries $\mathcal{F}(S)$, for $S \in \mathcal{A}$.

Let $S = \{h_{s_1}, \dots h_{s_u}, h_{s_{u+1}}, \dots, h_{s_r}\} \in \mathcal{A}$, with $h_{s_1}, \dots, h_{s_u} \in \mathcal{H}_-$ and $h_{s_{u+1}}, \dots, h_{s_r} \in \mathcal{H}_+$, where $0 \le r \le d$ and $0 \le u \le r$, so that $u = 0$ if lin $S \subset \mathcal{H}_+$ and $u = r$ if lin $S \subset \mathcal{H}_-$.

Lemma 4.5.

$$U^S_{++} = \begin{pmatrix} T_{++} & -p(\{h_{s_{u+1}}, \dots, h_r\}) \\ 0 & T^*P(S) \end{pmatrix},$$

$$U^S_{+-} = \begin{pmatrix} T_{+-} & -p(\{h_{n+1}, \dots, h_d\} \cap S^c) \\ 0 & 0 \end{pmatrix},$$

$$U^S_{-+} = \begin{pmatrix} T_{-+} & -p(\{h_{s_1}, \dots, h_{s_u}\}) \\ 0 & 0 \end{pmatrix},$$

$$U^S_{--} = \begin{pmatrix} T_{--} & -p(\{h_1, \dots, h_n\} \cap S^c) \\ 0 & T^*P(S^c) \end{pmatrix}.$$

Proof. By computation, using the fact that $P_{\ker T^*}P(X) = p(X)$ for $X \in \mathcal{A}$, where for a subspace, $\mathcal{R} \subset \mathcal{H}$, we denote by $P_{\mathcal{R}}$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}$. □

Assuming now that $\{k_i\}_{i=1}^I$, respectively $\{l_j\}_{j=1}^J$, is an orthonormal basis for $\ker T_{++}$, respectively $\ker T_{--}$, where of course $I, J < \infty$, we compute $\ker U^S_{\varepsilon\varepsilon}$ and $\ker {U^S_{\varepsilon\varepsilon}}^*$, $\varepsilon = +, -$.

First we make the observation that for $f \in \mathcal{H}$,

$$P(S)f \in \ker T^* \text{ if and only if } P(S)f = p(S)f\,. \tag{16}$$

Then for $h, k \in \mathcal{H}$:

$$\begin{aligned} P_+h \oplus P(S)k \;\in\; \ker U^S_{++} & \\ & \Leftrightarrow T_{++}h = p(\{h_{s_{u+1}}, \dots, h_{s_r}\})k \quad \text{and} \quad T^*P(S)k = 0 \\ & \Leftrightarrow P_+h \in \ker T_{++} \quad \text{and} \quad P(S)k \in \text{lin}\,\{h_{s_1}, \dots, h_{s_u}\} \end{aligned}$$

where the second equivalence follows by (16) and since $T_{++}h \in \ker T^*$ implies $T_{++}h = 0$. Thus $g'_i = (k_i \oplus 0)$, $i = 1, \dots, I$, $g'_{I+i} = (0 \oplus h_{s_i})$, $i = 1, \dots, u$ is an orthonormal for $\ker U^S_{++}$.

Similarly, for $h, k \in \mathcal{H}$:

$$\begin{aligned} P_-h \oplus P(S^c)k \;\in\; \ker U^S_{--} & \\ & \Leftrightarrow T_{--}h = p(\{h_1, \dots, h_n\} \cap S^c)k \quad \text{and} \quad T^*P(S^c)k = 0 \\ & \Leftrightarrow P_-h \in \ker T_{--} \quad \text{and} \quad P(S^c)k \in \text{lin}\,\{h_{n+1}, \dots, h_d\} \cap S^c \end{aligned}$$

where again the second equivalence follows by (16) and since $T_{--}h \in \ker T^*$ implies $T_{--}h = 0$. So $f'_j = (l_j \oplus 0)$, $j = 1, \dots, J$, $f'_{J+j} = (0 \oplus h_{x_j})$, $j = 1, \dots, d-n-r+u$ is an orthonormal basis for $\ker U^S_{--}$, where $\{h_{x_j}\}_{j=1}^{d-n-r+u} = \{h_{n+1}, \dots, h_d\} \cap S^c$ and $n + 1 \leq x_1 < x_2 < \cdots < x_{d-n-r+u} \leq d$.

Then for $j = 1, \dots, J$, $f_j = (T_{+-}l_j \oplus 0)$, and for $j = 1, \dots, d-n-r+u$, $f_{J+j} = (-h_{x_j} \oplus 0)$, $\{f_j\}_{j=1}^{J+d-n-r+u}$ is an orthonormal basis for $\ker {U^S_{++}}^*$ [15]. Similarly, for $i = 1, \dots, I$, $g_i = (T_{-+}k_i \oplus 0)$, and for $i = 1, \dots, u$, $g_{I+i} = (-h_{s_i} \oplus 0)$, $\{g_i\}_{i=1}^{I+u}$ is an orthonormal basis for $\ker {U^S_{--}}^*$.

For each $S \in \mathcal{A}$ we now turn our attention to the operator which we shall call $\Lambda_S(U)$, i.e. the associate of U defined in [15], here with respect to the projections $Q(S)_+$ and $Q(S)_-$ on $\mathcal{K}$, and compute its components.

First we have:

Lemma 4.6.

$$ {U^S_{--}}^{-1} = \begin{pmatrix} T^{-1}_{--} & 0 \\ -p(\{h_1, \dots, h_n\} \cap S^c) & T \end{pmatrix}, \tag{17}$$

where T^{-1}_{--} *is defined as* ${V_{11}}^{-1}$ *is defined in* [3].

Proof. First let v denote the above matrix in (17), and recall that the ranges of U^S_{--} and ${U^S_{--}}^*$ are closed. Thus an element of $\text{ran}U^S_{--}$ can be written as

$$U^S_{--}{U^S_{--}}^* \begin{pmatrix} P_-h \\ P(S^c)k \end{pmatrix} = \begin{pmatrix} T_{--}T^*_{--}h + P(\{h_1, \dots, h_n\} \cap S^c)h \\ P(S^c)k \end{pmatrix} \tag{18}$$

for $P_-h \oplus P(S^c)k \in \mathcal{K}_-$. Then v applied to the right hand side of (18) gives

$$\begin{pmatrix} T^*_{--}h \\ -P(\{h_1, \dots, h_n\} \cap S^c)h + TP(S^c)k \end{pmatrix},$$

as $T_{--}^{-1}T_{--} = P_{\mathrm{ran}T^*_{--}}$ and

$$\ker T_{--}^{-1} = \ker T^*_{--} \,. \tag{19}$$

This is ${U^S_{--}}^*(P_-h \oplus P(S^c)k)$, so that v and ${U^S_{--}}^{-1}$ coincide on $\mathrm{ran}U^S_{--}$. Furthermore, again by (19), $vg_i = 0$ for $i = 1, \dots, I+u$. Thus $v = {U^S_{--}}^{-1}$. $\square$

Then by (19), by direct computation we have:

Lemma 4.7.

$$\Lambda_S(U)_{++} = \begin{pmatrix} T_{++} - P_+ - T_{+-}T_{--}^{-1}T_{-+} & -p(\{h_{s_{u+1}}, \dots, h_{s_r}\}) \\ 0 & T^*P(S) - P(S) \end{pmatrix},$$

$$\Lambda_S(U)_{+-} = \begin{pmatrix} T_{+-}T_{--}^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Lambda_S(U)_{-+} = \begin{pmatrix} T_{--}^{-1}T_{-+} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Lambda_S(U)_{--} = \begin{pmatrix} P_- - T_{--}^{-1} & 0 \\ p(\{h_1, \dots, h_n\} \cap S^c) & P(S^c) - T \end{pmatrix}.$$

Further, again by direct computation:

Proposition 4.8. *With* $\Lambda_S((\overset{+}{-})T) := W^*\Lambda_S((\overset{+}{-})U)W$, *we have that as* S *varies in* $\mathcal{A}$, $\Lambda_S((\overset{+}{-})T)$ *remains constant and is equal to* $\Lambda((\overset{+}{-})T)$, *where*

$$\Lambda((\overset{+}{-})T) := = \begin{pmatrix} (\overset{+}{-})T_{++} - P_+(\overset{-}{+})T_{+-}T_{--}^{-1}T_{-+} & T_{+-}T_{--}^{-1} \\ T_{--}^{-1}T_{-+} & P_-(\overset{-}{+})T_{--}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} -P_+(\overset{+}{-}){T_{++}^{-1}}^* & -{T_{++}^{-1}}^*{T_{-+}}^* \\ -{T_{+-}}^*{T_{++}^{-1}}^* & P_-(\overset{-}{+})T^*_{--}(\overset{+}{-}){T_{+-}}^*{T_{++}^{-1}}^*{T_{-+}}^* \end{pmatrix}.$$

For $S \in \mathcal{A}$, let $M_S := \dim\ker U^S_{++}$ and $L_S := \dim\ker U^S_{--}$. Thus $M_S = I + u$ and $L_S = J + d - n - r + u$, and $\mathcal{F}(S)$ is given on $D_{\mathcal{H}}$ by,

$$\det[Q(S)_- + \Lambda_S((-1)^{L_S+M_S}U)^*_{+-}\Lambda_S((-1)^{L_S+M_S}U)_{+-}]^{1/2}$$

$$\cdot \sum_{(\rho,\tau)\in\mathcal{P}(S)} \mathrm{sgn}(\rho,\tau) \prod_{i=1}^{l} a_{\mathcal{H}}(W^*f_{\rho_i})^* \prod_{j=1}^{m} b_{\mathcal{H}}(J_1W^*g_{\tau_j})^*$$

$$\cdot : \exp(b((-1)^{L_S+M_S}\Lambda(T))) : \prod_{i=1+1}^{L_S} b_{\mathcal{H}}(J_1 l_{\rho_i}) \prod_{j=m+1}^{M_S} a_{\mathcal{H}}(k_{\tau_j})\,, \tag{20}$$

where an element (ρ, τ) of $\mathcal{P}(S)$ is defined to be a partition of $\{1, \dots, L_S\} \cup \{1, \dots, M_S\}$ into two subsets,

$$\{\rho_1, \dots, \rho_l\} \cup \{\tau_1, \dots, \tau_m\} \quad \text{and} \quad \{\rho_{l+1}, \dots, \rho_{L_S}\} \cup \{\tau_{m+1}, \dots, \tau_{M_S}\}\,,$$

with the indices in the natural order, such that

(i) $\{\rho_{l+1}, \dots, \rho_{L_S}\} \subseteq \{1, \dots, J\}$,
(ii) $\{\tau_{m+1}, \dots, \tau_{M_S}\} \subseteq \{1, \dots, I\}$,

and as in [15], $\operatorname{sgn}(\rho, \tau)$ is defined to be the sign of the permutation

$$\begin{pmatrix} \rho_{l+1} \cdots \rho_{L_S}, \tau_{m+1} + L_S \cdots \tau_{M_S} + L_S, \rho_1 \cdots \rho_l, \tau_1 + L_S \cdots \tau_m + L_S \\ 1 \cdots\cdots\cdots\cdots L_S + M_S \end{pmatrix}.$$

For each $S \in \mathcal{A}$, let $\mathrm{d}_S(U)$ denote the term,

$$\det[Q(S)_- + \Lambda_S((-1)^{L_S+M_S}U)_{+-}^{\;*}\Lambda_S((-1)^{L_S+M_S}U)_{+-}]\,.$$

Then abbreviating $:\exp(b((-1)^{L_S+M_S}\Lambda(T))):\Omega$ by Ω', we have that

$$\begin{aligned}
&\langle \mathcal{F}(S)^* \mathcal{F}(S)\Omega, \Omega\rangle \\
&\quad = \langle \mathcal{F}(S)\Omega, \mathcal{F}(S)\Omega\rangle = \mathrm{d}_S(U)^{-1} \\
&\qquad \times \left\langle \prod_{i=1}^{L_S} a_{\mathcal{H}}(W^* f_i)^* \prod_{j=1}^{M_S} b_{\mathcal{H}}(J_1 W^* g_j)^* \Omega',\ \prod_{i=1}^{L_S} a_{\mathcal{H}}(W^* f_i)^* \prod_{j=1}^{M_S} b_{\mathcal{H}}(J_1 W^* g_j)^* \Omega' \right\rangle \\
&\quad = \mathrm{d}_S(U)^{-1} \\
&\qquad \times \left\langle \Omega',\ \prod_{j=M_S}^{1} b_{\mathcal{H}}(J_1 W^* g_j) \prod_{i=L_S}^{1} a_{\mathcal{H}}(W^* f_i) \cdot \prod_{i=1}^{L_S} a_{\mathcal{H}}(W^* f_i)^* \prod_{j=1}^{M_S} b_{\mathcal{H}}(J_1 W^* g_j)^* \Omega' \right\rangle \\
&\quad = \mathrm{d}_S(U)^{-1}\langle \Omega', \Omega'\rangle = \mathrm{d}_S(U)^{-1}\mathrm{d}_S(T)\,,
\end{aligned}$$

with $\mathrm{d}_S(T) := \det[P_- + \Lambda((-1)^{L_S+M_S}T)_{+-}^{\;*}\Lambda((-1)^{L_S+M_S}T)_{+-}]$, where the fourth equality follows from the CAR's and since, by [15, (5.12)],

$$a_{\mathcal{H}}(W^* f)\Omega' = 0 \quad \text{and} \quad b_{\mathcal{H}}(W^* J g)\Omega' = 0\,,$$

for $f \in \ker {U^S_{++}}^*$ and $g \in \ker {U^S_{--}}^*$, and the fifth since $\langle \Omega', \Omega'\rangle = \mathrm{d}_S(T)$, again a result of [15].

Thus for each $S \in \mathcal{A}$, if we replace $\mathcal{F}(S)$ with $\mathrm{d}_S(U)^{1/2}\mathrm{d}_S(T)^{-1/2}\mathcal{F}(S)$, which we shall still denote by $\mathcal{F}(S)$, we then have that each of these 2^d normalized operators is an intertwining isometry.

Observe that for $S_0 := \{h_{n+1}, \dots, h_d\} \in \mathcal{A}$, $\mathcal{F}(S_0)$ is given on $D_{\mathcal{H}}$ by

$$\begin{aligned}
d_{S_0}(T)^{-1/2} &\sum_{(\rho,\tau)\in\mathcal{P}(S_0)} \operatorname{sgn}(\rho,\tau) \prod_{i=1}^{l} a(T_{+-} l_{\rho_i})^* \prod_{j=1}^{m} b(J_1 T_{-+} k_{\tau_j})^* \\
&\cdot :\exp(b((-1)^{I+J}\Lambda(T))): \prod_{i=l+1}^{L_{S_0}=J} b(J_1 l_{\rho_i}) \prod_{j=m+1}^{M_{S_0}=I} a(k_{\tau_j})\,.
\end{aligned}$$

Now let $S \in \mathcal{A}$ and $S_1 := S \cap S_0 \in \mathcal{A}$. Then with $d' := d - n - r + u$, for an arbitrary element $(\mu, \nu) \in \mathcal{P}(S)$ given by:

$$\{\mu_1, \dots, \mu_{p-d'}, J+1, \dots, J+d'\} \cup \{\nu_1, \dots, \nu_{q-u}, I+1, \dots, I+u\} \quad \text{and}$$
$$\{\mu_{p+1}, \dots, \mu_{J+d'}\} \cup \{\nu_{q+1}, \dots, \nu_{I+u}\}\,,$$

let (μ^1, ν^1) be the element of $\mathcal{P}(S_1)$ given by,

$$\{\mu_1, \dots, \mu_{p-d'}, J+1, \dots, J+d'\} \cup \{\nu_1, \dots, \nu_{q-u}\} \quad \text{and}$$
$$\{\mu_{p+1}, \dots, \mu_{J+d'}\} \cup \{\nu_{q+1}, \dots, \nu_{I+u}\}\,.$$

Then it is clear from the definitions that $\operatorname{sgn}(\mu, \nu) = \operatorname{sgn}(\mu^1, \nu^1)$, where on the right hand side, the function sgn is that defined on $\mathcal{P}(S_1)$.

Furthermore, it is not difficult to verify that

$$(-1)^{d'(I+u-q)} \times \operatorname{sgn}(\mu^1, \nu^1) = \operatorname{sgn}(\mu_0, \nu_0)\,, \tag{21}$$

where $(\mu_0, \nu_0) \in \mathcal{P}(S_0)$ is given by:

$$\{\mu_1, \dots, \mu_{p-d'}\} \cup \{\nu_1, \dots, \nu_{q-u}\} \quad \text{and}$$
$$\{\mu_{p+1}, \dots, \mu_{J+d'}\} \cup \{\nu_{q+1}, \dots, \nu_{I+u}\}\,,$$

and sgn is the function defined on $\mathcal{P}(S_0)$.

Now let Ψ be the unitary in $B(F_a(\mathcal{H}))$ implementing α_{-1} in π_P, that satisfies $\Psi\Omega = \Omega$. Then note that since $\Lambda(T)_{+-} = \Lambda(-T)_{+-}$, we of course have,

$$:\exp(b(\Lambda(T))):\Omega =: \exp(b(\Lambda(-T))):\Omega\,,$$

so that

$$\Psi :\exp(b(\Lambda(T))): =: \exp(b(\Lambda(-T))):\ . \tag{22}$$

Hence for an arbitrary term of (20), with $(\mu, \nu) \in \mathcal{P}(S)$ as above, we have:

$$\begin{aligned}
\operatorname{sgn}(\mu, \nu) & \prod_{i=1}^{p-d'} a_{\mathcal{H}}(T_{+-} l_{\mu_i})^* \prod_{n=1}^{d'} a_{\mathcal{H}}(-h_{x_n})^* \\
& \cdot \prod_{j=1}^{q-u} b_{\mathcal{H}}(J_1 T_{-+} k_{\nu_j})^* \prod_{t=1}^{u} b_{\mathcal{H}}(-J_1 h_{s_t})^* \\
& \cdot :\exp(b(\Lambda((-1)^{L_S+M_S} T))): \prod_{i=p+1}^{L_S} b_{\mathcal{H}}(J_1 l_{\mu_i}) \prod_{t=g+1}^{M_S} a_{\mathcal{H}}(k_{\nu_t}) \\
= {} & (-1)^{d'(I+u-q)} (-1)^{d'(p-d')} (-1)^{u(q-u+p)} \operatorname{sgn}(\mu_0, \nu_0) \\
& \cdot \prod_{t=1}^{u} b_{\mathcal{H}}(-J_1 h_{s_t})^* \prod_{n=1}^{d'} a_{\mathcal{H}}(-h_{x_n})^* \prod_{i=1}^{p-d'} a_{\mathcal{H}}(T_{+-} l_{\mu_i})^* \prod_{j=1}^{q-u} b_{\mathcal{H}}(J_1 T_{-+} k_{\nu_j})^*
\end{aligned}$$

$$
\begin{aligned}
&\cdot \Psi^{d'+u} : \exp(b(\Lambda((-1)^{I+J}T))) : \prod_{i=p+1}^{L_S} b_{\mathcal{H}}(J_1 l_{\mu_i}) \prod_{t=q+1}^{M_S} a_{\mathcal{H}}(k_{\nu_t})\\
&= (-1)^{d'(I+u-q)}(-1)^{d'(p-d')}(-1)^{u(q-u+p)}(-1)^{(q-u+p-d')(d'+u)}(-1)^{ud'}\\
&\quad \cdot \prod_{i=1}^{u}(b_{\mathcal{H}}(-J_1 h_{s_{u+1-i}})^*\Psi)\prod_{i=1}^{d'}(a_{\mathcal{H}}(-h_{x_{d'+1-i}})^*\Psi)\\
&\quad \cdot \operatorname{sgn}(\mu_0,\nu_0)\prod_{i=1}^{p-d'} a_{\mathcal{H}}(T_{+-}l_{\mu_i})^* \prod_{j=1}^{q-u} b_{\mathcal{H}}(J_1 T_{-+}k_{\nu_j})^*\\
&\quad \cdot : \exp(b(\Lambda((-1)^{I+J}T))) : \prod_{i=p+1}^{L_S} b_{\mathcal{H}}(J_1 l_{\mu_i}) \prod_{t=q+1}^{M_S} a_{\mathcal{H}}(k_{\nu_t})\\
&= (-1)^{d'I}\prod_{i=1}^{u}(b_{\mathcal{H}}(-J_1 h_{s_{u+1-i}})^*\Psi)\prod_{i=1}^{d'}(a_{\mathcal{H}}(-h_{x_{d'+1-i}})^*\Psi)\\
&\quad \cdot \operatorname{sgn}(\mu_0,\nu_0)\prod_{i=1}^{p-d'} a_{\mathcal{H}}(T_{+-}l_{\mu_i})^* \prod_{j=1}^{q-u} b_{\mathcal{H}}(J_1 T_{-+}k_{\nu_j})^*\\
&\quad \cdot : \exp(b(\Lambda((-1)^{I+J}T))) : \prod_{i=p+1}^{L_S} b_{\mathcal{H}}(J_1 l_{\mu_i}) \prod_{t=q+1}^{M_S} a_{\mathcal{H}}(k_{\nu_t})\,,
\end{aligned}
$$

where the first equality follows from (21), the CAR's, and (22) since $L_S + M_S = d' + u + I + J$.

Furthermore, the second equality follows from the fact that for $n \in \mathbf{N}$ and $f_1, \ldots, f_n \in \mathcal{H}_+$ we have,

$$a_{\mathcal{H}}(f_1)^* \cdots a_{\mathcal{H}}(f_n)^* \Psi^n = a_{\mathcal{H}}(f_n)^*\Psi \cdots a_{\mathcal{H}}(f_1)^*\Psi\,,$$

and the corresponding equality for the $b_{\mathcal{H}}(\cdot)^*$'s.

Then from the above, it of course follows that we have

$$\mathcal{F}(S) = (-1)^{d'I}\prod_{i=1}^{u}(b_{\mathcal{H}}(-J_1 h_{s_{u+1-i}})^*\Psi)\prod_{i=1}^{d'}(a_{\mathcal{H}}(-h_{x_{d'+1-i}})^*\Psi)\mathcal{F}(S_0)\,.$$

Now with $\mathcal{K}, P_1, V$ and Γ as in Theorem 4.3, and $P_2 = 1 - P_1$, consider the associate $\Lambda(V)$, of V, defined in [3]. More precisely, first look at the term, ${V_{11}^{-1}}^*$ V_{21}^* $P_{\ker V_{22}^*}$, which appears in each component of $\Lambda(V)$. Let $h \oplus J_1 k \in \ker V_{22}^* = \ker T_{--}^* \oplus J \ker T_{++}^*$. Then $V_{11}V_{21}^*(h \oplus J_1 k) = T_{++}T_{-+}^* h \oplus J_1 T_{--}T_{+-}^* k$, and

$$
\begin{aligned}
T_{++}T_{-+}^* h &= P_+ T P_+ T^* P_- h = P_+ T T^* P_- h\\
&= P_+ T T^* P_{\mathcal{H}_- \cap \mathrm{ran} T} P_- h + P_+ T T^* P_{\mathcal{H}_- \cap \ker T^*} P_- h = 0\,.
\end{aligned}
$$

Similarly, $J_1 T_{--}T_{+-}^* k = 0$, so that $V_{21}^*(\ker V_{22}^*) \subset \ker V_{11} = \ker {V_{11}^{-1}}^*$.

Thus we have that,

$$\begin{aligned}
\Lambda(V)_{11} &= \Lambda(T)_{++} \oplus -J_1\Lambda(T)^*_{--}J_1\,,\\
\Lambda(V)_{12} &= \Lambda(T)_{+-} \oplus -J_1\Lambda(T)^*_{+-}J_1\,,\\
\Lambda(V)_{21} &= \Lambda(T)_{-+} \oplus -J_1\Lambda(T)^*_{-+}J_1\,,\\
\Lambda(V)_{22} &= \Lambda(T)_{--} \oplus -J_1\Lambda(T)^*_{++}J_1\,. \qquad (23)
\end{aligned}$$

As an orthonormal basis for $\ker V_{11}$, choose $f'_i := k_i \oplus 0$ and $f'_{I+j} := 0 \oplus J_1 l_j$, for $i = 1, \ldots, I$ and $j = 1, \ldots, J$.

Now since $\dim\ker V_{11} = I + J$, let $\mathcal{Q}$ denote the set of partitions of the index set $\{1, \ldots, I + J\}$ into two subsets so that an element ρ of $\mathcal{Q}$ is given by two disjoint subsets, $\{\rho_1, \ldots, \rho_l\}$ and $\{\rho_{l+1}, \ldots, \rho_{I+J}\}$, with indices in the natural order. Then define a function s on $\mathcal{Q}$ by

$$s(\rho) := \operatorname{sign}\begin{pmatrix} \rho_{l+1}\cdots\rho_{I+J}, \rho_1\cdots\rho_l \\ 1\cdots\cdots\cdots\cdots I+J \end{pmatrix}.$$

By [16, Sec. 7], $\mathcal{F}(S_0)$ can then be expressed in the self-dual context and language as

$$\begin{aligned}
&[\det(P_1 + \Lambda(V)_{12}\Lambda(V)_{12}{}^*)]^{-1/4}\\
&\quad\cdot\sum_{\rho\in\mathcal{Q}} s(\rho)\prod_{i=1}^{l} a(V_{12}\Gamma f'_{\rho_i})^* : \exp(b(\Lambda((-1)^{I+J}V))/2) : \prod_{i=1+1}^{I+J} a(f'_{\rho_i})\,, \qquad (24)
\end{aligned}$$

on the dense subspace $\mathcal{D} = \pi_{P_1}(A^{\mathrm{SDC}}(\mathcal{K},\Gamma))\Omega_{P_1}$. Moreover, this is clearly the operator $\Psi_0(V)$ of [3, Sec. 4.2] for appropriate choice of orthonormal basis $\{e_i\}_{i=1}^{L_V=I+J}$ for $\ker V_{11}$.

Consequently, consider next the ordered orthonormal set of vectors,

$$\{-J_1h_n, \ldots, -J_1h_1, (-1)^{I+1}h_d, \ldots, (-1)^{I+1}h_{n+1}\}\,.$$

For $i = 1, \ldots, n$ and $j = n + 1, \ldots, d$,

$$\Psi_i := b_{\mathcal{H}}(-J_1h_i)^*\Psi \quad\text{and}\quad \Psi_i := a_{\mathcal{H}}((-1)^{I+1}h_i)^*\Psi\,.$$

Then for each $S \in \mathcal{A}$, let $\bar{S}$ be defined as,

$$\{-J_1h_{s_u}, \ldots, -J_1h_{s_1}, (-1)^{I+1}h_{x_{d'}}, \ldots, (-1)^{I+1}h_{x_1}\}\,,$$

with order inherited from above, and define Ψ_S to be

$$\Psi_{s_u}\cdots\Psi_{s_1}\Psi_{x_{d'}}\cdots\Psi_{x_1}\,,$$

so that we have,

$$\mathcal{F}(S) = \Psi_S\mathcal{F}(S_0)\,.$$

Of course for $i = 1, \ldots, n$, $\Psi_i = \pi_{P_1}(B(0 \oplus -J_1 h_i))\Psi$, and for $i = n+1, \ldots, d$, $\Psi_i = \pi_{P_1}(B((-1)^{I+1} h_i \oplus 0))\Psi$.

If by $\mathcal{F}(S_0)$ we now mean the expression given in (24), so that we are working in the self-dual setting, we then have:

Proposition 4.9. *Each partial isometry* Ψ_i *satisfies*:

$$\operatorname{ran} \mathcal{F}(S_0) \subset (\ker \Psi_i)^{\perp} (= \ker \Psi_i^*) \,.$$

Proof. To begin with, it is easily seen that for $i = 1, \ldots, n$ and $j = n+1, \ldots, d$ both $a(0 \oplus J_1 h_i)$ and $a(h_j \oplus 0)$ anticommute with $a(V_{12}\Gamma f)^*$ for all $f \in \mathcal{K}_1$.

Hence for $1 \le i \le n$, we have from [3, Lemma 4.1]

$$\begin{aligned} & a(0 \oplus J_1 h_i) : \exp(b(\Lambda((^{+}_{-})V))/2) : \\ & \quad = : \exp(b(\Lambda((^{+}_{-})V))/2) : a(0 \oplus J_1 h_i) \\ & \qquad - a(\Lambda((^{+}_{-})V)_{12}\Gamma(0 \oplus J_1 h_i))^* : \exp(b(\Lambda((^{+}_{-})V))/2) : \\ & \qquad - : \exp(b(\Lambda((^{+}_{-})V))/2) : a(\Gamma\Lambda((^{+}_{-})V)_{22}\Gamma(0 \oplus J_1 h_i)) = 0 \,, \end{aligned}$$

since $\Lambda((^{+}_{-})V)_{12}\Gamma(0 \oplus J_1 h_i) = 0$ and $\Gamma\Lambda((^{+}_{-})V)_{22}\Gamma(0 \oplus J_1 h_i) = 0 \oplus J_1 h_i$.

Similarly, we have that

$$a(h_j \oplus 0) : \exp(b(\Lambda((^{+}_{-})V))/2) := 0 \,, \quad \text{for } n+1 \le j \le d \,,$$

since $\Lambda((^{+}_{-})V)_{12}\Gamma(h_j \oplus 0) = 0$ and $\Gamma\Lambda((^{+}_{-})V)_{22}\Gamma(h_j \oplus 0) = h_j \oplus 0$.

Thus the claim is proved. □

Proposition 4.10.

$$\ker V^* \cap \operatorname{ran}(P_1 - \Lambda(V)_{12}^*) = \ker V^* \cap \mathcal{K}_1 \,.$$

Proof. From Proposition 4.8 and (23), we need to show that:

$$\ker T^* \cap \operatorname{ran}(P_+ + T_{-+}{T_{++}}^{-1}) = \ker T^* \cap \mathcal{H}_+ \,, \tag{25}$$

and

$$J_1 \ker T^* \cap \operatorname{ran}(P_- + J_1 T_{+-}{T_{--}}^{-1} J_1) = J_1 \ker T^* \cap \mathcal{H}_- \,. \tag{26}$$

Recalling assumption (15), these clearly follow since $\ker {T_{\varepsilon\varepsilon}}^{-1} = \ker T_{\varepsilon\varepsilon}^*$ for $\varepsilon = +, -$. □

Corollary 4.11. *The set*,

$$\{0 \oplus -J_1 h_n, \ldots, 0 \oplus -J_1 h_1, (-1)^{I+1} h_d \oplus 0, \ldots, (-1)^{I+1} h_{n+1} \oplus 0\} \,,$$

is an orthonormal basis for $\ker V^* \cap \operatorname{ran}(P_1 - \Lambda(V)_{12}^*)$.

Proof. Obvious from Proposition 4.10. ☐

Thus we have shown that the set of isometries $\{\mathcal{F}(S) : S \in \mathcal{A}\}$ is identifiable with the set of isometric implementers of ϱ_V in π_{P_1} constructed in [3], and so the proof of Theorem 4.3 is complete. This set then forms a complete set of isometric implementers for ρ_T in π_P — a fact which we remark may also be observed here by explicitly computing that

$$\sum_{S\in\mathcal{A}} \mathcal{F}(S)\mathcal{F}(S)^* a(f)^* \prod_{i=1}^{n} a(f_i)^*\Omega = a(f)^* \sum_{S\in\mathcal{A}} \mathcal{F}(S)\mathcal{F}(S)^* \prod_{i=1}^{n} a(f_i)^*\Omega\,,$$

for all $f, f_1, \ldots, f_n \in \mathcal{H}$ and $n \in \mathbf{N}$. Consequently we have provided a method for producing isometric implementers, in the *complex* formalism, which is far more natural than that of [3], and furthermore unified this latter approach with the apparently unrelated ideas of [8].

Acknowledgements

This work formed part of the author's doctoral thesis at the University of Wales, Swansea, under the supervision of Professor David E. Evans, and was prepared for submission whilst visiting the University of Rome, "La Sapienza", under contract with the University of Ancona, Italy, funded by the European Union Research and Training Network, "Geometric Analysis".

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