## 0.1 Basic concepts in capital theory

Elementary concepts of *capital theory* are introduced in this section. These concepts are fundamental to our understanding of social and economic problems such as the valuation of assets, the consideration of future generations by today's decision-makers, or sustainable development. Despite its simplicity, the theory is extremely powerful when it comes to interpret the intertemporal optimality conditions in resource management problems. The role of future discounting and its mechanics is introduced.[1] Those with knowledge of basic financial economic concepts may skip this section.

### 0.1.1 The pricing of bonds

A *bond* is a financial instrument in which the *bond issuer* promises to make some future payments to the *bond holder* in return for a present payment by the holder. This way, the bond issuer borrows money from the bond holder, who respectively become the *debtor* (borrower) and the *creditor* (lender). The schedule of future payments make take many forms and usually include periodic interest payments with a final payment equal to the initial amount lent, called the *principal*. The *maturity date* is the date at which the final payment is being made.

Bonds are liquid assets, meaning that they can be quickly and cheaply bought and sold on financial markets at any time before their maturity date. Let us now see how they are valued.

#### A one-year bond

Consider a bond that promises to make one single payment of $100 one year from now. There is no other intermediate payment. Note that we assume away the presence of risk and inflation, i.e., the promise will be respected with certainty and one dollar next year buys the same amount of goods and services as one dollar today. But be careful, this does not imply that one dollar to be received tomorrow has the same value as one dollar today! It will all depend on asset returns. Let's see how.

Let $p_{s,t}$ denote the price at the beginning of year $t$ for a bond that matures at year $t + s$. In what follows, we typically refer to period $t$ as the current

[1] SEE SECTION ??? FOR A DISCUSSION OF THE CONTROVERSIES SURROUNDING THE USE USE OF A DISCOUNT RATE.

period. Subscript $s, t$ refers to a $s$-year maturity bond at period $t$. The price at $t$ of a one-year maturity bond is thus denoted $p_{t,1}$. Since the buyer is effectively investing the amount $p_{1t}$ into the bond, its current *return* is expressed as:

$$r_t = \frac{\$100 - p_{1t}}{p_{1t}}. \tag{1}$$

In our analysis, $r_t$ also corresponds to the (current) one-year *interest rate*. This is because whoever buys a bond is effectively lending money to the bond seller, who in turn borrows money at interest rate $r_t$. There are thus two angles from which to view $r_t$. From the perspective of the saver who buys a bond, $r_t$ is the *return* on an investment, also called the *yield*. From the perspective of the bond seller, $r_t$ is the *cost of (financial) capital*, or the cost of borrowing money.[2]

Another, common way to look at equation (1) is to say that a bond has a yield of $r_t$ and, consequently, its price is given by

$$p_{1,t} = \frac{\$100}{1 + r_t}. \tag{2}$$

So as long as bonds provide a positive return, we have $p_{1t} < 100$. Consequently, one dollar to be received tomorrow must be worth less than one dollar received today. In fact, given that bonds provide a return of $r_t > 0$, one is indifferent between receiving \$100 next year or receiving a smaller amount of $\$100/(1 + r_t)$ today. There are two ways to convince yourself of this, depending on whether you want to spend money now or next year. Let's see why.

Suppose first that somebody promises to give you \$100 next year, but that you want to spend money now. You can then borrow amount $\$100/(1 + r_t)$ today against that promise, at interest rate $r_t$, spend the amount now, and use the \$100 next year to clear your debt. You might as well have received the amount $\$100/(1 + r_t)$ today!

Suppose instead that someone gives you the amount $\$100/(1 + r_t)$ today, but that you want to spend the money next year. You can invest the amount received today at interest rate $r_t$, and collect \$100 next year. You might as well have received a \$100 payment next year!

[2]The fact that borrowing costs and bond returns are the same suggests that there are no intermediaries in the financial markets. This is a simplification that does not affect the qualitative nature of our analysis for our present purpose.

Based on the foregoing argument, the value $p_{1,t}$ as defined in (2) is called the *present value*, or discounted value, of \$100 to be received next year.

### Arbitrage

Note that (2) can be expressed as follows:

$$p_{1,t}(1 + r_t) = \$100. \tag{3}$$

This equality can be interpreted as a *no-arbitrage condition*.[3] To see why, suppose that you can choose between two types of bonds: A-bonds offers return $r_t$, while B-bonds offer to pay \$100 one year from now and cost $p_{1,t}$ today. If $p_{1,t}(1 + r_t) > \$100$, the issuer of B-bonds can borrow amount $p_{1t}$ in order to purchase A-bonds, and produce an easy surplus. Repeating this operation a million times, the issuer of B-bonds will become immensely rich. In real life, given that bond A offers a higher return, no-one will show interest in bond B at price $p_{1t}$. The B-bond issuer will be forced to reduce her price if she wants to borrow anything, thus recovering equality (3). The reader is encouraged to verify that the same type of reasoning will apply if the inequality is flipped around. We now continue with multi-period bond pricing.

### A two-year bond

Suppose that a two-year maturity bond promises to pay \$100 two years from now at the beginning of year $t + 2$. There is no other payment. Its current price is denoted $p^b_{2t}$. Suppose further that the yearly returns on bonds are constant over time and given by $r$. Following the argument made above with a one-year bond, you know that this bond will be worth $\$100/(1 + r)$ at the beginning of period $t+1$. By the same logic, it is worth $(1/(1+r))(\$100/(1+r))$ at the beginning of period $t$. We therefore have that given a constant yearly interest rate of $r$, a promise to pay \$100 two years from now has a present value of

$$p^b_{2t} = \frac{\$100}{(1 + r)^2}. \tag{4}$$

[3]Arbitrage corresponds to a situation where an investor can make "easy" gains from asset purchases, i.e., gains that are not justified by additional efforts or risks. Such a situation cannot persist if information is public, which we assume in our analysis. Insider trading corresponds to arbitrage and is generally considered illegal.

Extending the above argument to a promise of a \$100 payment to be received $s$ years from now, its present value is given by:

$$p^b_{st} = \frac{\$100}{(1+r)^s}. \tag{5}$$

**The discount factor**

Based on the above discussions, if bonds provide a positive one-year return $r$, then future payments are discounted to a lower value today. More precisely, a dollar to be received next year is worth $\beta \equiv 1/(1+r)$ dollar today. This lower value accorded to future payments is the reason why $r$ is called the *discount rate*. Similarly, $\beta$ is called the *discount factor*. While $r$ and $\beta$ convey the same basic information, make sure not to confuse them in your analysis. Moreover, because these rates plays such an important role in comparing future payments with present ones, they play a crucial role in natural resource management because today's decisions affect future possibilities through their impact on the state of the resource's stock.

### 0.1.2 An application to a specific investment decisions

Suppose that you have a project that only you can implement. The project may be unique because it requires the use of a natural resource, say an underground oil pool on which you have exclusive rights. We shall refer to such a unique project as a *specific project*.

Assume that in order to implement the project, a *drilling rig* must be built at initial investment cost $I$. $I$ is called a *sunk cost* because once built, the rig has no resale value. The project is expected to bring the following stream of net profits over the next three periods: $\pi_1$, $\pi_2$ and $\pi_3$. After period 3, the rig stops producing and is scrapped at no cost. To simplify, we assume that there is no risk associated with this project; all of the profits and costs are known with certainty. Should you build that rig?

The answer to the question hinges on the *opportunity cost* of the amount $I$ being set aside for that project; that is, it must be compared with the best alternative use that you have for that money. To simplify things, suppose that the best you can do with that money is to buy government bonds. Applying the method learned in section 0.1.1, we have that the present value sum of the future benefits of building the rig can be expressed as follows:

$$V_0 = \frac{\pi_1}{1+r} + \frac{\pi_2}{(1+r)^2} + \frac{\pi_3}{(1+r)^3}. \tag{6}$$

Note that we assume that the rig is built at period 0 and that the first profit payment $\pi_1$ is received one period later. If $V_0 > I$, then the project brings a higher return than the bonds and the rig should be built. A good way to see this is to bring forward the gains from both investment decision and compare their values at period 3. If amount $I$ is invested into bonds, then the value at period 3 is $B_3 = (1+r)^3 I$. If amount $I$ is spent in building the oil rig, then the value of the future net profits at period 3 is $V_3 = \pi_1(1+r)^2 + \pi_2(1+r) + \pi_3$. (To make the comparison valid, we must assume that profits received at periods 1 and 2 are turned into bonds). Obviously, the project is preferable to the bonds if $V_3 > B_3$; the reader is encouraged to verify that this is indeed equivalent to $V_0 > I$.

In some situations, investors may need to decide between many projects other than buying government bonds. In the next section, we look at how the above approach may be adapted to compare projects.

### 0.1.3 Ranking alternative investment projects

Suppose that you have $n$ projects to choose from and that you would like to rank them in a priority list. You might want to do this because the total amount that you need to invest exceeds the cost of the individual projects but you do not have enough to implement them all. A straightforward way to do this is to borrow the method introduced in section 0.1.2 as follows.

Suppose that project $i$, $i \in \{1, 2, ..., n\}$, have initial investment cost $I_i$ while the present value of the stream of future profits is equal to $V_{0i}$. Those costs and benefits are still assumed to be known with certainty. You can then rank them in terms of the *profitability index*, $V_{0i}/I_i$, which is a measure of the benefits per dollar invested.[4] By choosing the subset of projects with the highest profitability index, the average yield on the total amount invested is maximized.

**Exercise 1** ***On present-discounted values and investment decisions*** *You consider building a rig above an underground oil pool. The following table*

[4] For a neat application of the profitability index used to compare the various 2015 "Sustainable Development Goals" by the United Nations, read "The economics of optimism", *The Economist*, January 24th 2015.

*gives the net profits that you expect to receive at the beginning of each future year with certainty. The yearly interest rate on bonds* $r = 3\%$*. There is no inflation and the factory becomes obsolete after its fourth year of operation.*

| | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ |
|---|---|---|---|---|
| *Expected net profit* | *100* | *150* | *200* | *200* |

a) *Suppose that the initial outlay required to build the rig is* $I = 500$*. Determine whether it is a worthwhile investment.*

b) *Calculate the profitability index for this project.*

c) *Suppose that the yield on bonds is 15%. Is the project still worth undertaking?*

d) *Calculate the* cutoff *discount rate, i.e., the one that corresponds to a profitability index of 1. (You may proceed by trial and error.)*

### 0.1.4 Investment decisions and the pricing of assets

Suppose that at the *beginning* of period $t$, you own a unit of an asset and you are considering whether to sell it now or hold on to it for the whole of period $t$. If you sell it now, you receive an amount $p_t$ which can be invested in your best alternative project which produces a return $r$. If you hold on to the asset, it produces a benefit of value $d_t$ during period $t$ and the asset unit can then be sold at price $p_{t+1}$ at the beginning of the next period. All prices, benefits and returns are known with certainty.

Typical examples of assets that you may own this way include a house, a plot of agricultural land or a machine. The house can either be sold at price $p_t$ at the beginning of period $t$ or rented out during period $t$ at rental price $d_t$ (net of maintenance and other costs) and then sold at the end of that period at price $p_{t+1}$. For a hectare of agricultural land, $p_t$ and $p_{t+1}$ denote its selling prices now and next year and $d_t$ is the net proceeds from the crops produced. A machine may be a sawmill that can be sold at prices $p_t$ now or $p_{t+1}$ next year and yield net revenues $d_t$ during year $t$ as payments for converting logs into lumber. Another important example is that of a company's shares bought on the stock market, in which case $p_t$ and $p_{t+1}$ are the share prices at the beginning of the respective periods and $d_t$ is the dividend per share received during period $t$. For concreteness, we shall refer to the asset under consideration as a *machine*.

In the finance literature, the best alternative asset with return $r$ is typically referred to as a government bond or a bank deposit certificate. For our purpose, all that counts is that the proceed from selling the presently owned asset can be reinvested during period $t$ to yield a return $r$.

The upshot is that if you hold on to the machine during period $t$ and sell it at the end, you begin period $t+1$ with $p_{t+1} + d_t$ dollars in your pocket. If, alternatively, you sell the asset at the beginning of period $t$ and invest it in bonds, then you end up with $p_t(1+r)$ dollars at the beginning of period $t+1$. Obviously, you will choose the alternative that gives you the best outcome to begin period $t+1$. But if both types of assets are being held simultaneously during period $t$, either by you or between you and other people who have the same information about prices and returns, then an argument can be made that all must be indifferent between holding either type of asset during period $t$, that is, the following *no-arbitrage condition* must hold:

$$p_{t+1} + d_t = p_t(1+r). \tag{7}$$

The best way to understand why condition (7) must hold in equilibrium is to see what happens when it does not hold. Take $p_{t+1} + d_t > p_t(1+r)$. This means that the machine brings a strictly higher return than bonds. But if that is the case, then no investor wants to sell a machine at price $p_t$. (This argument rests on the assumption that the values $p_{t+1}$, $d_t$ and $r$ are pre-determined by other factors.) Similarly, all bond holders want to part with their bonds and buy machines. We thus have a situation where, given the preceding strict inequality, the demand for machines strictly exceeds the supply. This will lead to an increase in the price $p_t$ until the no-arbitrage condition is re-established.

More precisely, a bond holder's willingness to pay for a machine at the beginning of period $t$ is $(p_{t+1} + d_t)/(1+r)$, no more, no less. If $p_t$ is lower than that value, then the machine yields a strictly higher return than bonds. If $p_t$ is higher, then the bond yields a strictly higher return and all machine holders want to sell their machines at the beginning of period $t$. The only possible equilibrium in which investors hold both assets during period $t$ is given by the no-arbitrage condition above.

**Problem 2** *Suppose that a machine can only be owned for one period. At any period* $t$*, therefore, an investor may buy a machine at price* $p_t$ *but she is* *forced to sell it at price $p_{t+1}$ at the beginning of period $t+1$. In order to buy a machine, an investor must borrow the amount $p_t$ at interest rate $r$ and pay back principal and interest at the beginning of period $t+1$. (Treasury bonds are not part of the problem here.) A machine produces a dividend $d_t$ during period $t$. Assume throughout that $p_{t+1}$, $d_t$ and $r$ are considered fixed at the beginning of any period $t$.*

*a) Argue that the no-arbitrage condition (7) must hold in equilibrium.*

*b) Whether it denotes the return from a treasury bond or the interest rate on loans, discuss what $r$ represents generally.*

*c) Analyze separately the effect of an increase in $p_{t+1}$, $d_t$ and $r$ on $p_t$. Explain.*

Since the machine could be sold at price $p_t$ at the beginning of period $t$, it may be said that holding on to the machine during the period is equivalent to investing the amount $p_t$ into the machine. The return from the machine is consequently given by the following expression:

$$\frac{(p_{t+1} - p_t) + d_t}{p_t}. \tag{8}$$

In the financial literature, expression (8) is called the *internal rate of return* from the project. The denominator represents the amount invested, or *financial capital*. The numerator represents the net gain to be had from the investment. This net gain is composed of two parts: A *capital gain*, $p_{t+1} - p_t$, and a dividend, or fruit, $d_t$. According to the no-arbitrage condition, asset prices must be such that the internal rate of return is equal to the opportunity cost of financial capital, that is,

$$\frac{(p_{t+1} - p_t) + d_t}{p_t} = r. \tag{9}$$

When considering inter-temporal natural resource use problems, we will often refer back to the breakdown of the no-arbitrage condition into the following four components: the capital gain, the fruit, the invested financial capital, and the opportunity cost of financial capital. Though simple, the formulation is extremely powerful for our understanding of decision making in a dynamic setting. Make sure to fully understand its workings.

**Exercise 3** *Discuss the implication on the evolution of asset prices of a dividend that is smaller than the gain from bond yields, $d_t < rp_t$?*

The above discussion was based on a discrete time analytical framework. In continuous time, if $p(t)$, $d(t)$ and $r(t)$ respectively denote the instantaneous asset price, dividend flow and interest rate, then one can show that the continuous-time analog to the no-arbitrage condition is

$$\frac{\dot{p}(t) + d(t)}{p(t)} = r(t), \tag{10}$$

where $\dot{p}(t) \equiv dp(t)/dt$ and represents the instantaneous rate of change of the asset's price, or capital gain.