ECO 6122 MICRO IV
MID-TERM II March 11, 2015
SOLUTIONS.

#1 The profit function:

a) Definition:
The profit function is a function that gives the <u>maximum profit level</u> as a function of <u>output and input prices only</u>. Formally:

$$\pi(p,\vec{w}) = \max_{\vec{x},y} py - \vec{w}\vec{x} \quad \text{s.t.} \quad y = f(\vec{x}).$$

b) For convexity, we must show that the profit function has the following property:

$$\pi(p^t,\vec{w}^t) \leq t\pi(p^0,\vec{w}^0) + (1-t)\pi(p^1,\vec{w}^1)$$

$\forall t \in [0,1]$, where $(p^t,\vec{w}^t) = t(p^0,\vec{w}^0) + (1-t)(p^1,\vec{w}^1)$.

PROOF:
Let $\vec{x}(p,\vec{w})$ and $y(p,\vec{w})$ be the profit max. choices of inputs and output. Then

$$\pi(p,\vec{w}) = py(p,\vec{w}) - \vec{w}\vec{x}(p,\vec{w})$$

By the definition of profit maximization, we have:

$$p^0 y(p^0, \vec{w}^0) - \vec{w}^0 \vec{x}(p^0, \vec{w}^0) \geqslant p^0 y(p^t, \vec{w}^t) - \vec{w}^0 \vec{x}(p^t, \vec{w}^t)$$

and $p' y(p', \vec{w}') - \vec{w}' \vec{x}(p', \vec{w}') \geqslant p' y(p^t, \vec{w}^t) - \vec{w}' \vec{x}(p^t, \vec{w}^t)$

since the producer could have chosen $y(p^t, \vec{w}^t)$ and $\vec{x}(p^t, \vec{w}^t)$ at prices $(p^0, \vec{w}^0)$ and $(p', \vec{w}')$.

Hence,

$$t\Pi(p^0, \vec{w}^0) + (1-t)\Pi(p', \vec{w}')$$

$$\geqslant t\left[p^0 y(p^t, \vec{w}^t) - \vec{w}^0 \vec{x}(p^t, \vec{w}^t)\right] + (1-t)\left[p' y(p^t, \vec{w}^t) - \vec{w}' \vec{x}(p^t, \vec{w}^t)\right]$$

$$\geqslant (tp^0 + (1-t)p')\, y(p^t, \vec{w}^t) - (t\vec{w}^0 + (1-t)\vec{w}')\, \vec{x}(p^t, \vec{w}^t)$$

$$\geqslant p^t y(p^t, \vec{w}^t) - \vec{w}^t \vec{x}(p^t, \vec{w}^t) = \Pi(p^t, \vec{w}^t)$$

QED

c) In the above definition of a convex function, choose t, p^0, p', $\vec{w}^0$ and $\vec{w}'$ as follows:

$$t = 0.5, \quad p^0 = 0.5P, \quad p' = 1.5P, \quad \vec{w}^0 = \vec{w}' = \vec{w}$$

Then $(p^t, \vec{w}^t) = t(p^0, \vec{w}^0) + (1-t)(p', \vec{w}')$

$$= 0.5(0.5P, \vec{w}^0) + 0.5(1.5P, \vec{w}')$$

$$= (P, \vec{w})$$

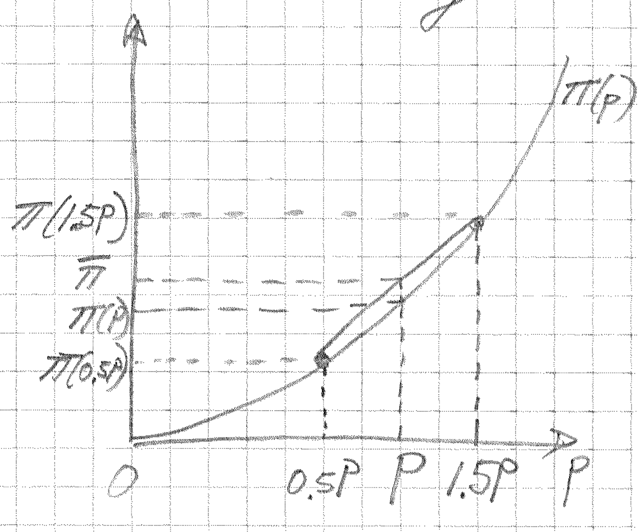
Since the profit function is convex, we must have:

$$0.5\,\pi(0.5P,\vec{w}) + 0.5\,\pi(1.5P,\vec{w}) \geq \pi(P,\vec{w})$$

But since price fluctuation are predictable, the left-hand side represents the average profit. Hence, the producer's profit will be at least as large with the fluctuating prices on average as with the constant price at the mean. QED

ALTERNATIVE GRAPHICAL PROOF:

The following graph illustrates a profit function that is strictly convex.



$\bar{\pi}$ denotes the profit that the producer makes on average when prices fluctuate.

$\pi(P)$ is the profit level with a constant price equal to the average of fluctuating prices.

We see that strict convexity gives $\bar{\pi} > \pi(P)$.

#2 Integrability theorem

a) (This was taken from example 2.3 in the textbook.)
In order for $\vec{x}(\vec{p}, y)$ to be utility generated, it must satisfy the following three conditions:

① budget balancedness
② symmetry of Slutsky matrix $s(\vec{p}, y)$
③ negative semi-definiteness of $s(\vec{p}, y)$

① Budget balancedness simply requires that $\sum_i p_i x_i(\vec{p}, y) = y$

② Symmetry of $s(\vec{p}, y)$:
This requires the following:

$$\frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial y} = \frac{\partial x_j}{\partial p_i} + x_i \frac{\partial x_j}{\partial y} \quad , \forall i, j$$

③ For n.s.d., one must follow the same procedure that one uses on the Hessian matrix to verify that a function is concave.

b) "Utility generated" means that the demand function is derived from the behavior of a consumer that wishes to maximize her utility level at given prices and income level.

3) Income and substitution effects

a) (Replicate figure 1.20 of textbook.)

b) The Slutsky matrix is defined as follows: $S(\vec{p}, y) =$

$$\begin{pmatrix} \frac{\partial x_1}{\partial p_1} + x_1 \frac{\partial x_1}{\partial y} & \cdots & \frac{\partial x_1}{\partial p_n} + x_n \frac{\partial x_1}{\partial y} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial p_1} + x_1 \frac{\partial x_n}{\partial y} & \cdots & \frac{\partial x_n}{\partial p_n} + x_n \frac{\partial x_n}{\partial y} \end{pmatrix}$$

where $x_i \equiv x_i(\vec{p}, y)$ denotes the Marshallian demand.

c) In the appendix we have the following "Slutsky equation":

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial x_i^h}{\partial p_j} - x_j \frac{\partial x_i}{\partial y}, \quad \forall i, j$$

where $x_i^h \equiv x_i^h(\vec{p}, u)$ denotes the compensated demand.

~~This implies~~ that the components of the Slutsky matrix are equal to $\frac{\partial x_i^h}{\partial p_j}$. But we also know that

$$x_i^h = \frac{\partial e(\vec{p}, u)}{\partial p_i}$$ (THIS IS SHEPHARD'S LEMMA)

where $e(\vec{p}, u)$ is the expenditure function.

Hence, $\frac{\partial x_i^h}{\partial p_j} = \frac{\partial^2 e}{\partial p_j \partial p_i}$.

The elements of the Slutsky matrix are therefore equal to $\partial^2 e / \partial p_i \partial p_j$, i.e. the elements of the Hessian of $e(\vec{p}, u)$.

Since $e(\vec{p}, u)$ is concave in prices, its Hessian matrix is n.s.d. and so is the Slutsky matrix. QED

By Young's theorem, we have

$$\frac{\partial^2 e}{\partial p_i \partial p_j} = \frac{\partial^2 e}{\partial p_j \partial p_i},$$

which means that the Slutsky matrix is symmetric.

QED

4) Fixed factors

(See solution to problem 3.50 in my web page.)