FINAL 2019S

#3 SOCCER PENALTY KICK

a)

		STRIKER L	STRIKER C	STRIKER R
GOALIE	L	$\underline{\alpha}, 1-\alpha$	$0 \quad \underline{1}$	$0 \quad \underline{1}$
	C	$0 \quad \underline{1}$	$\underline{1} \quad 0$	$0 \quad \underline{1}$
	R	$0 \quad \underline{1}$	$0 \quad \underline{1}$	$\underline{\beta}, 1-\beta$

From the above underlined gains, we see that, at every combination of pure strategies at least one player wants to deviate.

b) If the striker assigns a strictly positive probability to L, C and R, then she must be indifferent between all three. In other words, the expected gains must be the same, i.e.,

$$E[\Pi_s^L] = E[\Pi_s^C] = E[\Pi_s^R]$$

where $E[\pi_i^k]$ is the expected gain for player i with strategy k. We have:

$$E[\Pi_s^L] = p_L \cdot (1-\alpha) + p_C \cdot 1 + p_R \cdot 1 = 1 - \alpha p_L$$

$$E[\Pi_s^C] = p_L \cdot 1 + p_C \cdot 0 + p_R \cdot 1 = p_L + p_R$$

$$E[\Pi_S^R] = p_L \cdot 1 + p_C \cdot 1 + p_R \cdot (1-\beta) = 1 - \beta p_R$$

$$E[\Pi_S^R] = E[\Pi_S^C] \Rightarrow 1 - \beta p_R = p_L + p_R$$

$$\Rightarrow p_L = 1 - (1+\beta) p_R \qquad (1)$$

$$E[\Pi_S^R] = E[\Pi_S^L] \Rightarrow 1 - \beta p_R = 1 - \alpha p_L$$

$$\Rightarrow p_L = \frac{\beta}{\alpha} p_R \qquad (2)$$

$$(1) \,\&\, (2) \Rightarrow \left(\frac{\beta}{\alpha}\right) \cdot p_R = 1 - (1+\beta) p_R$$

$$\Rightarrow \left(\frac{\beta}{\alpha} + 1 + \beta\right) p_R = 1$$

$$\Rightarrow (\beta + \alpha\beta + \alpha) p_R = \alpha$$

$$\Rightarrow \boxed{p_R = \frac{\alpha}{\alpha + \alpha\beta + \beta}} \qquad (3)$$

$$\Rightarrow (2) \Rightarrow \boxed{p_L = \frac{\beta}{\alpha + \alpha\beta + \beta}} \qquad (4)$$

Since $p_C = 1 - p_L - p_R$, we have

$$\boxed{p_C = \frac{\alpha\beta}{\alpha + \alpha\beta + \beta}}. \qquad (5)$$

(3), (4) and (5) is the mixed strategy for the goalie for which the striker will play a mixed strategy with positive probabilities on all three L, C and R.

Using the same procedure for the goalie, we must have:

$$E[\pi_G^L] = E[\pi_G^C] = E[\pi_G^R].$$

where

$$E[\pi_G^L] = q_L \cdot \alpha + q_C \cdot 0 + q_R \cdot 0 = \alpha q_L$$

$$E[\pi_G^C] = q_L \cdot 0 + q_C \cdot 1 + q_R \cdot 0 = q_C$$

$$E[\pi_G^R] = q_L \cdot 0 + q_C \cdot 0 + q_R \cdot \beta = \beta q_R$$

$$E[\pi_G^L] = E[\pi_G^C] \Rightarrow \alpha q_L = q_C$$

$$E[\pi_G^L] = E[\pi_G^R] \Rightarrow \alpha q_L = \beta q_R$$

Moreover, $q_L + q_C + q_R = 1$.

$$\Rightarrow q_L + \alpha q_L + \frac{\alpha}{\beta} q_L = 1$$

$$\Rightarrow \left(1 + \alpha + \frac{\alpha}{\beta}\right) q_L = 1 \Rightarrow \boxed{q_L = \frac{\beta}{\alpha + \alpha\beta + \beta}} \qquad (6)$$

$$\Rightarrow \boxed{q_C = \frac{\alpha\beta}{\alpha + \alpha\beta + \beta}} \qquad (7)$$

$$\Rightarrow \boxed{p_R = \frac{\alpha}{\alpha + \alpha\beta + \beta}} \qquad (8)$$

(6), (7) and (8) denote the striker's mixed strategy for which the goalie will be willing to assign a positive probability of playing all three L, C and R.

Consequently, expressions (3) to (8) define a MSNE for this game.

c) Since $\alpha\beta < \alpha < \beta$, both players will play C with lowest probability.

One way to interpret this is that since the goalie can stop the center shot with certainty, the striker will be inclined to shoot left or right more often.

d) With $\alpha = 0.4$ and $\beta = 0.6$, we have:

$$q_L = p_L = \frac{.6}{.4 + (.4 \cdot .6) + .6} = \frac{.6}{1.24} = .4839$$

$$q_C = p_C = \frac{.4 \cdot .6}{1.24} = \frac{.24}{1.24} = .1935$$

$$q_R = p_R = \frac{.4}{1.24} = .3226$$

It is faster to calculate the proba. of no goal occurring since this can only happen under three scenarios: (L,L), (C,C) and (R,R). Accounting for joint probas, we have:

$$p_L \cdot q_L \cdot \alpha + p_C \cdot q_C \cdot 1 + p_R \cdot q_R \cdot \beta$$

$$= (.4829)^2 (0.4) + (.1935)^2 + (.3226)^2 \cdot 0.6$$

$$= .0937 + .0374 + .0624$$

$$= .1935$$

Hence, the equilibrium proba. of a goal is

$$1 - .1935 = 80.65\%.$$

THE LONG WAY

d) With $\alpha = 0.4$ and $\beta = 0.6$, we have

$$q_L = p_L = \frac{0.6}{0.4 + (0.4 \cdot 0.6) + 0.6} = \frac{0.6}{1.24} = .4839$$

$$q_C = p_C = \frac{0.4 \cdot 0.6}{1.24} = \frac{0.24}{1.24} = 0.1935$$

$$q_R = p_R = \frac{.4}{1.24} = 0.3226$$

We take the joint probability for each of the 9 possible outcomes and multiply by the probability that a goal is scored at each. And then sum up.

For instance, at (L, L), we have

$$\underbrace{p_L \cdot q_L}_{\text{joint prob. that both play L.}} \cdot \underbrace{1-\alpha}_{\text{prob. of a goal when both play L.}} = .4839 \cdot .4839 \cdot 0.6 = 0.1405$$

Overall, the prob. of a goal occuring is thus:

$$p_L(q_L \cdot (1-\alpha) + q_C + q_R) + p_C(q_L + q_C \cdot 0 + q_R)\ p_R(q_L + q_C + q_R(1-\beta))$$

$$= p_L(1 - \alpha q_L) + p_C(1 - q_C) + p_R(1 - \beta q_R)$$

3.6)

$$= \underbrace{p_L + p_C + p_R}_{=1} - \alpha p_L q_L - p_C q_C - \beta p_R q_R$$

$$= 1 - 0.4(.4839)^2 - (.1835)^2 - 0.6(.3226)^2$$

$$= 80.65\%$$