#2 Show that $u(\vec{x})$ is quasi-concave iff $\succsim$ is convex.

(see solution to assigned problem 1.23.)

#3 $u(x_1, x_2) = (x_1 - \alpha_1)^{1-\theta} (x_2 - \alpha_2)^{\theta}, \quad \theta \in (0,1)$

a) max problem:

$$\max_{x_1, x_2} u(x_1, x_2) \text{ s.t. } p_1 x_1 + p_2 x_2 = y$$

(NB We impose a binding budget constraint since utility is strictly increasing in x_1 & x_2.)

b) $\mathcal{L} = u(x_1, x_2) + \lambda(y - p_1 x_1 - p_2 x_2)$

$$\mathcal{L}_{x_1} = 0 \Rightarrow \frac{\partial u}{\partial x_1} - \lambda p_1 = 0$$

$$\Rightarrow (1-\theta)\left(\frac{x_2 - \alpha_2}{x_1 - \alpha_1}\right)^{\theta} = \lambda p_1 \qquad [A]$$

$$\mathcal{L}_{x_2} = 0 \Rightarrow \frac{\partial u}{\partial x_2} - \lambda p_2 = 0 \Rightarrow \theta\left(\frac{x_1 - \alpha_1}{x_2 - \alpha_2}\right)^{1-\theta} = \lambda p_2 \qquad [B]$$

$$\mathcal{L}_{\lambda} = 0 \Rightarrow y = p_1 x_1 + p_2 x_2$$

c) In order to derive the indirect utility fctn $v(\vec{p}, y)$, we begin by deriving the demand functions $x_1(\vec{p}, y)$ and $x_2(\vec{p}, y)$. These demands will then be inserted back into the direct utility fctn.

$[A] \& [B] \Rightarrow \dfrac{(1-\theta)\left(\frac{x_2-\alpha_2}{x_1-\alpha_1}\right)^{\theta}}{\theta\left(\frac{x_1-\alpha_1}{x_2-\alpha_2}\right)^{1-\theta}} = \dfrac{p_1}{p_2}$

$\Rightarrow \dfrac{p_2x_2 - p_2\alpha_2}{p_1x_1 - p_1\alpha_1} = \dfrac{\theta}{1-\theta}$

Since $p_2x_2 = y - p_1x_1$, we get, after rearranging:

$$x_1 = \frac{(1-\theta)(y - p_2\alpha_2) + \theta p_1\alpha_1}{p_1} \qquad [D1]$$

And similarly,

$$x_2 = \frac{\theta(y - p_1\alpha_1) + (1-\theta)p_2\alpha_2}{p_2} \qquad [D2]$$

Those are the (ordinary) demand fctns. Substituting back into $u(x_1, x_2)$, we get, after rearranging:

$$v(\vec{p}, y) = \frac{(1-\theta)^{1-\theta}\theta^{\theta}(y - p_2\alpha_2 - p_1\alpha_1)}{p_1^{1-\theta}p_2^{\theta}}$$

d) The ordinary dmd fctns were derived above. See [D1] & [D2].

e) In order to derive the consumer's demand for manufactured goods using the indirect utility function, the easiest is to use Roy's identity, i.e.,

$$x_2(\vec{p}, y) = -\frac{\partial v / \partial p_2}{\partial v / \partial y}. \quad \text{THE END.}$$

(THIS NEXT PART WAS NOT REQUIRED.)

$$\frac{\partial v}{\partial p_2} = \frac{(1-\theta)^{1-\theta}\theta^{\theta}(-\alpha_2)}{p_1^{1-\theta}p_2^{\theta}} - \frac{(1-\theta)^{1-\theta}\theta^{\theta}(y - p_2\alpha_2 - p_1\alpha_1)}{(p_1^{1-\theta}p_2^{\theta})^2} \cdot \theta p_1^{1-\theta} p_2^{\theta-1}$$

$$\frac{\partial v}{\partial y} = \frac{(1-\theta)^{1-\theta}\theta^{\theta}}{p_1^{1-\theta}p_2^{\theta}}$$

$$\Rightarrow x_2(\vec{p}, y) = -\left[(-\alpha_2) - \frac{(y - p_2\alpha_2 - p_1\alpha_1)\theta p_2^{\theta-1}}{p_2^{2\theta}} \cdot p_2^{\theta}\right]$$

$$= \frac{\theta(y - p_2\alpha_2 - p_1\alpha_1) + p_2\alpha_2}{p_2}$$

This effectively corresponds to the demand function [D2].