MAT 3153 Assignment 4.

(1) Let $\alpha \in \Omega(X, x_0)$. Since α and the constant path at x_0 are paths from x_0 to x_0, they are homotopic (taking $x_1 = x_0$). Thus $\pi_1(X, x_0)$ is trivial.

(2) Suppose $F: c \simeq f$, where $c: S^1 \to X$ is the constant map, $c(z) = x_0 \; \forall z \in S^1$. Thus $F: S^1 \times I \to X$, and $F(z, 0) = x_0$ $\forall z \in S^1$. Define an equivalence relation on $S^1 \times I$ by

$$(w, s) \sim (z, t) \iff (w, s) = (z, t) \text{ or } s = t = 0.$$

Let Y be the set of equivalence classes.

$B^2 \cong S^1 \times I/\sim$ 2 $\quad g|_{S^1} = f$ 2

$S^1 \times I/\sim \; \to X$ 2

so $Y = \{ \{(z,t)\} \mid t \neq 1 \} \cup \{ \{(z,0) \mid z \in S^1\} \}$.

<u>Claim</u> Y, with the quotient topology from $S^1 \times I$, is homeomorphic to B^2.

<u>Proof</u> Define $p: S^1 \times I \to B^2$ by
$$p(z,t) = tz.$$
Then p is continuous (scalar mult.), and $p(w,0) = 0 \ \forall w$, so p defines a continuous map $g: Y \to B^2$. Since p is surjective, ($x = p\left(\frac{x}{\|x\|}, \|x\|\right)$ if $x \neq 0$, $0 = p(z,0)$) g is a continuous bijection.

Suppose $V \subset B^2$, $p^{-1}(V)$ open in $S^1 \times I$. Let $x \in V$, $(z,t) \in p^{-1}(\{x\})$ (so $x = tz$).

~~Let~~ Suppose $x \neq 0$. Note that p restricts to a homeomorphism

$$p|_{S^1 \times (0,1]} : S^1 \times (0,1] \to B^2 - \{0\}$$

with inverse $x \mapsto \frac{x}{\|x\|} \times \|x\|$.

So $\exists!\ z \times t \in S^1 \times (0,1]$ s.t. $p(z \times t) = x$.
Let $U = (S^1 \times (0,1]) \cap p^{-1}(V)$. Then U is open in $S^1 \times (0,1]$, so $p(U)$ is open in $B^2 - \{0\}$, whence in B^2. So

$$x = p(z \times t) \in p(U) \subset pp^{-1}(V) \subset V.$$

If $x = 0$, then $p^{-1}(\{x\}) = S^1 \times \{0\}$.

$\forall z \in S^1$, $\exists U_z$ open in S^1, $\varepsilon_z > 0$, s.t. $z \times \{0\} \in U_z \times [0, \varepsilon_z) \subset p^{-1}(V)$ (since $p^{-1}(V)$ is open). In particular, $\{U_z\}$ is an open cover of the compact space S^1.

Thus $\exists z_1, \dots, z_n \in S^1$ s.t. $\{U_{z_1}, \dots, U_{z_n}\}$ cover S^1. Let $\varepsilon = \min\{\varepsilon_{z_1}, \dots, \varepsilon_{z_n}\}$.

<u>CLAIM</u> $S^1 \times [0, \varepsilon) \subset p^{-1}(V)$.

Indeed, if $z \times t \in S^1 \times [0,\varepsilon)$, then $z \in U_{z_i}$ for some i, so $t < \varepsilon \leq \varepsilon_{z_i}$, so $z \times t \in U_{z_i} \times [0, \varepsilon_{z_i}) \subset p^{-1}(V)$.

<u>CLAIM</u> $p(S^1 \times [0,\varepsilon)) = B_0(\varepsilon) = \{x \mid \|x\| < \varepsilon\}$.

Indeed, if $z \times t \in S^1 \times [0,\varepsilon)$, then $p(z \times t) = tz$, so $\|p(z \times t)\| = |t|\,\|z\| = t < \varepsilon$.

Thus
$$0 \in B_0(\varepsilon) = p(S^1 \times [0,\varepsilon)) \subset pp^{-1}(V) \subset V.$$

$\Rightarrow V$ open

$\Rightarrow p$ quotient map

$\Rightarrow g$ homeomorphism

The homotopy

$$F : S^1 \times I \longrightarrow X$$

defines

$$\tilde{F} : Y \to X$$

so defining $\tilde{f}$ to be the composite

$$B^2 \xrightarrow{g^{-1}} Y \xrightarrow{\tilde{F}} X$$

we obtain the desired map (noting

that if $\|x\| = 1$, then

$$\tilde{f}(x) = \tilde{F}(g^{-1}(x))$$

$$= \tilde{F}(x \times 1)$$

$$= f(x)$$

so $\tilde{f}|_{S^1} = f$.) whew!

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3 $f: S^1 \to S^1$ null htpic

$\Rightarrow f$ extends to $g: B^2 \to S^1$.

Let $j: S^1 \to B^2$ be the inclusion.

Then $j \circ g: B^2 \to B^2$ is continuous, and so has a fixed point: $j \circ g(x) = x$.

But $j(y) \in S^1$ $\forall y \in S^1$, so $x \in S^1$ i.e. $g(x) = f(x) = x$.

Next, let $h(x) = -f(x)$. Then $f \simeq$ const $\Rightarrow h \simeq -$const and any two constant maps are homotopic because S^1 is path connected. So $h \simeq$ const. Thus $\exists y \in S^1$ s.t. $h(y) = y$ i.e. $-f(y) = y$ i.e. $f(y) = -y$.

4 Let v be a continuous, nowhere vanishing vector field on B^2.

Define
$$f: B^2 \to B^2$$
by
$$f(x) = \frac{v(x)}{\|v(x)\|}.$$

Then $\|v(x)\|$ is continuous & non-vanishing, so $f(x)$ is continuous.

$\Rightarrow \exists\, x \in B^2$ s.t. $f(x) = x$.

~~But $f(x) \in S^1$ $\forall x$~~.

$$\Rightarrow v(x) = \|v(x)\| \cancel{\cdot} f(x) = \|v(x)\| x.$$

(5) If $\mathbb{R} \approx \mathbb{R}^n$, then

$$\mathbb{R} - \{0\} \approx \mathbb{R}^n - \{\vec{0}\}$$

But $\mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$ is not connected whereas $\mathbb{R}^n - \{0\}$ is connected (since path connected)

$\mathbb{R}^{n+1} - \{0\}$ contains S^n as a deformation retract. As a result,

$$\pi_1(\mathbb{R}^{n+1} - \{0\}, e_1) \cong \pi_1(S^n, e_1)$$

$$\cong \begin{cases} \mathbb{Z} & n=1 \\ \text{trivial} & \text{otherwise} \end{cases}$$

$$\Rightarrow \pi_1(\mathbb{R}^{n+1} - \{0\}, e_1) \not\cong \pi_1(\mathbb{R}^2 - \{0\}, e_1)$$

if $n \neq 1$

$$\Rightarrow \mathbb{R}^{n+1} - \{0\} \not\approx \mathbb{R}^2 - \{0\}$$

$$\Rightarrow \mathbb{R}^{n+1} \not\approx \mathbb{R}^2.$$

(6) Let $g \in G$, $h \in H$ be non-neutral elements. Recall that $gh \in G*H$ is the equivalence class that contains g, h in S, the set of words $g_1 h_1 g_2 h_2 \cdots g_n h_n$. hg is the class containing $e_G h g e_H$. It suffices to show that this second class does not contain a word of length ≤ 2. Suppose

$$g'h' \sim e_G h g e_H.$$

Then $h = e_H$ and $g = e_G$, $\lightning$.