The University of Ottawa First Year Math Survival Guide

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Chapter 1

The social aspect

1.1 Introduction

Do you still remember the first time you were able to ride a bicycle on your own? your first day in grade 7 or grade 9?

Throughout our lives we encounter a series of transitional experiences which shape our lives and make us who we are. Although each transition has its own experiences and can have a great impact on an individual’s life, the cross over from high school to University is definitely one of the most remarkable changes in a youth life.

You should have no illusion, the transition to University is a period filled with stress, anxiety and hard work. But if it is well planned, it certainly leads to a lot of joy and pride of accomplishment. More importantly, how you prepare for and deal with this transition can make or break your hopes and dreams of a rewarding and successful career for the rest of your life.

A lot of literature on the subject of transition to University was written and is accessible to public and a lot more will probably appear in the future as many institutions, government agencies and individual educators strive to make this experience as successful and smooth as possible. After all, this is one of the biggest investment one can make in the future of a nation.

This probably makes most of the aspects of the transition discussed in this chapter applicable to any Canadian University and to almost any program you choose to enroll in. The goal of this manual is however, to help you succeed in your first year math courses from the University of Ottawa perspective. It is a reflection of years of experience with many waves of high schools students coming into the world of
higher learning from many Canadian and oversees backgrounds. The daily contact with students in classrooms and in the Help Center helped me to put together a list of common misconceptions, mistakes and misjudgments I noticed over the years. The hope is that the list will shorten and eventually disappear with time.

As you look through the table of content of this manual or as you quickly scan its pages, you will probably freak out and ask yourself: "Am I suppose to read all this?". Beside Chapter 1, the answer is No. The intention is to read a section if you feel not comfortable with the topic or there are some gaps in your knowledge you need to fill.

1.2 So what is different?

O.K, you made it through high school and here you are probably away from home on this big campus, where your entire high school population can probably fit in one or two auditoriums.

Before coming here, You have definitely read and heard a lot of articles and stories about this new life and you have certainly built your own image and expectations about this new adventure. The fact is, no matter how much you think you are well prepared, many facts about University life will still surprise you. The quicker you are aware of them, the easier and faster you will adapt and succeed. So how different University life is from your high school experience?

• **Different School, Different mandate.** Generally speaking, high school education is designed to focus on the "teaching" where students are taught "how to learn" and are evaluated on material covered repeatedly and entirely in the classroom. University education however, is much more focused on the "learning" where students are expected to demonstrate an ability to learn on their own and be able to meet their courses requirements. In many occasions, your University professor will quickly explains a topic without a lot of details and expects you to dig deeper and do examples.

• **Personal attention to your progress is dramatically reduced.** If you ask any student finishing the last year in high school what is he or she looking for the most going to University, the most likely answer you will get is "more freedom". It is absolutely true, you will definitely have more freedom and more control in choosing and managing your courses and work habits at the University. The question is are you sure how to do deal with this freedom? It is your responsibility to stay on track and make necessary adjustments if needed. No one will be on your back, looking over your shoulder to see how your are managing your time or progressing in your studies. No one will pressure you to finish a homework, or take you on the side during lunch hour to tell you that you should work harder. In fact, if your professor or a University consultant asks to see you to go over your progress, then more likely you are already off track and things are not going so well.
• **Considerably less time in the classroom, considerably more time with the books.** After making your weekly timetable for your first semester, you could be shocked how little you "physically" have to be on campus compared with your long days at high school. In most cases, your classes and labs (or DGDs) combined will require that you are on campus for as little as twelve to fifteen hours a week. When you make your personal term schedule, make sure to remember that the time outside class hours is not all free time.

• **Yes, first year courses are huge.** Large classes of 150 to 200 (and sometimes more) are very common in First-year Calculus and Linear Algebra courses. You no longer have your "seat" in the classroom and like in the movie theater, the best seats are quickly taken. You probably know a handful of students in the auditorium but the rest of the audience are new faces. This could create an atmosphere of intimidation that stops you from raising your hand and ask a clarification or a question. Rest assured, the same question is probably crossing the mind of many other students in the auditorium who will be grateful you asked the question. If you still feel the moment is not right to ask the question, make a note so you look at it after class or put it on the "to ask" list when you go see your professor during his or her office hours.

• **Textbook.** In high school, you probably rarely looked into your Math textbook for a reason other than doing some assigned homework your teacher assigned from it. This is about to change big time at the University level, in most cases. In Large multi-sectioonal classes like your Calculus and Linear Algebra courses, instructors would carefully choose the course Textbook and will usually closely follow the content of its sections. In many cases, the course outline will even contain detailed scheduled of what sections of the book are covered each week. Let us face it, University professors can talk very fast and write even faster and before you know it, the blackboard is erased or he is on another slide. Also, for some sections of the Textbook, your professor will simply explain the main idea of the subject at hand without diving into the details or doing examples. This is where your Textbook becomes your best resource to fill these gaps.

• **Teachers versus Professors** As part of their formation, high school teachers are trained to assist in imparting knowledge to students in various ways and forms. Most of University Professors, on the other hand, have little or no pedagogical training at all. Most Universities offer a training programs but on a voluntary basis. University Professors are, however, trained as experts in their particular areas of research. That is not to say at all that University professors are less qualified to deliver a remarkable pedagogical task but it is important to understand that Professors’ approaches to teaching are in most cases influenced by their training as researchers. As a consequence, their conceptions of teaching is facilitating the understanding and bringing about some conceptual change.

• **Different lecturing styles.** Unlike the "almost" uniform teaching style during high school years, University professors have more flexibility on choosing how to give their lectures, more so for large
audience. Some would use computer based presentations, some would prefer overheads or templates and others, like myself, still use the same old chalkboard. In many cases, a professor would use a combination of the above methods. Regardless of the method of delivery, professors may lecture nonstop for 80 or so minutes, expecting you to take good notes and to identify the important points in the lecture.

- **The grading philosophy.** Let us face it, the daily contact in a much smaller group with your high school teacher gives him or her a very good idea about your personality, your work athletics and especially the effort you made during the term in the course. This will definitely have an impact (in most cases positive) on your overall mark of the course. With 150 or more students in a class and very limited personal contact, don’t expect any credit for your efforts or class participation from your University professor. In a typical first year math course, the final grade is based on Assignments notes (10-15 % for up to 8 assignments), midterm(s) evaluations (typically two midterms worth 10-15 % each) and a final exam (55-60 % of your final grade). In general, High School courses are usually designed to reward students “good faith effort.” Although “good faith effort” is a first step required in achieving good results at University, it is “factored in” in the grading process.

- **Diversity.** It is probably fair to say that in high school, students in the same class are all of the same age group and have a certain homogeneity in terms of their socioeconomic background. Nothing can be farther from the truth at the University. A quick look around you in the auditorium is enough to see how diverse the room is. Students are from different age, backgrounds and cultural groups and many are foreign students with first language neither English nor French. This diversity is in fact a source of enrichment for your University experience. It gives you new perspectives on many aspects of your life through interactions with fellow students from different parts of the country and the world.

- **Tests.** In high school, tests are frequent and structured to cover small amounts of material. Your teacher may even prepare you for the test by conducting one or more review sessions or preparing a review sheet pointing out the most important concepts you need to know. Also, teachers will usually schedule tests around school activities to avoid conflicts and they are ready to write makeup test in case conflicts cannot be avoided. At the University level, Tests are usually less frequent, cumulative covering large amounts of material. Makeup tests are rarely accommodated and in most cases weights of missed tests (for well determined valid reasons) are transferred to the final exam. As for the questions on a test, the guideline in high school is to test students on the ability to reproduce solutions of the kinds of problems there were exposed to in the classroom. Although this guideline still apply for most tests in first year University courses, some questions will test students ability to apply what they have seen in class to new situations or to solve new types of problems.

- **Hands off Parents.** In high school, teachers contact parents in case of problems. The school expects parents to interfere in times of crisis. In University, students are expected to take responsibility for
their own actions. Not only parents are not informed in case of a problem, but the privacy act forbids University officials from releasing any information about the students to anyone, including parents.

- **Endless Opportunities.** You may have participated in your school politics by being on a class or school student committee. If you are interested in this kind of activities, the University life offers a much broader chance to be politically active, have issues concerning academic life and gives you an exposer to all sort of decision makers at the local, provincial and some cases the national levels. The University Students Association is a well organized body involved in campus politics and often consulted about changes that could affect student life on campus. The Association is governed by an elected student body that will represent all student population on campus. You can place your candidacy for any position and like in any other election, you go through an election campaign based on your vision of things should be run.

The Following table gives a quick summary of the above discussed topics presented in a slightly different way and in a different order.

<table>
<thead>
<tr>
<th>In high School</th>
<th>In University</th>
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</thead>
<tbody>
<tr>
<td>Students progress is closely monitored by teachers with constant comments and feedback</td>
<td>Rarely any comments will given during the term but students are expected to be more self-directed and able to make adjustments if needed</td>
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<tr>
<td>Attendance is taken on a daily basis</td>
<td>Although most universities have rules for attendance, but in general it is almost never taken</td>
</tr>
<tr>
<td>Class participation and &quot;good faith effort&quot; are factors in awarding grades</td>
<td>Grades are based on submitted works and monitored exams only</td>
</tr>
<tr>
<td>Students are in daily contact with teachers</td>
<td>In most cases, students meet their instructor for a total of three hours a week</td>
</tr>
<tr>
<td>Make-up tests offered for struggling students and missed tests</td>
<td>Make-up tests are almost never given</td>
</tr>
<tr>
<td>Tests material are 100% covered in class</td>
<td>Students are sometimes tested on material that they were told to prepare on their own</td>
</tr>
<tr>
<td>There are between 20 to 30 students in a class</td>
<td>First year courses classes could have 150-200 students</td>
</tr>
<tr>
<td>Study schedule is done in consultation with teachers and counselors</td>
<td>Students are expected to make their own study schedules and to follow the schedule conscientiously</td>
</tr>
<tr>
<td>Students are all the same age, mostly from the same socioeconomic background</td>
<td>Students are from different backgrounds, age and cultural groups</td>
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<tr>
<td>High school is mandatory and available to anyone for free</td>
<td>University is voluntary and expensive.</td>
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<tr>
<td>Each week, you spend about 30 hours in classes</td>
<td>12 to 15 hours between classes and labs combined</td>
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1.3 How is learning Mathematics different from learning other subjects?

Let us be frank and honest, there is no "one set" of rules that makes students absorb a subject universally. Different students have different approaches and reactions when it comes to learning a new subject. In fact, even for the same student the approach and reaction are different from one subject to another. Understanding what distinguish one subject from others is a first step toward developing the right attitude to learn the subject.

1. **Mathematics are a foreign language.** At many occasions, I have heard students comparing listening to certain mathematical proofs to hearing a foreign language. Well guess what? Mathematics to a large extend are a foreign language. Not convinced? think about the definition of a language; it is mainly a set of words (vocabulary) constructed from finite set of symbols (alphabet) and subject to set of rules (grammar). Similarly, mathematics have a vocabulary of their own constructed from a set of symbols, like ∀, ∃, →, ←, ∨, ∧, +, −, 1, ×, Σ and a grammar represented mainly by the mathematical logic rules.

This is of course a very naive way of viewing mathematics as an natural language, but treating mathematics as such can certainly help you process the information more easily and efficiently and it can certainly justify some of the techniques used to learn the subject. Think about the following: Can you imagine yourself learning a new language just by watching someone speaking it day after day? The same thing applies to your math courses. Regardless of the percentage of the material you understand during the lecture, you will not fully understand the notes unless you read them again, do the examples yourself and practice as many suggested problems as possible. This way, the next time you go to class the language spoken by your professor will sound more and more like the one you speak.

2. **Mathematics can easily fool you.** I cannot remember how many times a student comes to see me in my office hours and say: *When I am in class, I understand 100% but when it comes to doing my homework I don't seem to know how to start.* My answer in this case is usually simple: *Are you sure you understand 100%?* A quick look in the student notes is usually enough to find an example, a theorem or a certain fact that allows to start the assignment problem. In many cases, the students "think" they get it simply because they the steps used by the professor are fairly natural and the end solutions make sense for them. **You only get it when you can do it on your own with a complete understanding of each step in the process.** It is exactly like watching somebody swimming and thinking: *yes, I can do that.* But you only can know for sure when you jump in the water and you don't get drowned.

3. **Mathematics is a progressive subject.** Admittedly, the progressive nature is not proper to the area
of mathematics as it is common in many other areas, but it is probably most visible in mathematics. In areas like history or geography, there is a big chance you skip Chapter 1 or 2 and still do well in chapter 3. This almost never happens in mathematics. A student once told me: "Every time my professor starts in new chapter, I breathe a sigh of relief thinking I can start fresh and recover from the troubles I had with the previous chapter. It usually takes 10 or 15 minutes into the new chapter to discover that I am stuck again on those little details. Yes, you math background matters and you can only build on what we already know. The department of Mathematics and Statistics has recently created a self-diagnostic test in mathematics to help incoming students know exactly what we expect from them in terms of their math background and where they stand. Remedies are also proposed for struggling students.

4. **The Anxiety factor.** Do you get emotional, upset or angry when doing or thinking about math homework or test? do you skip your math lectures for no reason? do you freeze on your math test? did you choose your University major based on having as little math as possible? These are few of a math anxiety symptoms that you might have developed through the years. I am sure that students develop anxieties for other subjects as well, but I am yet to meet a student who expresses history, geography or even science or chemistry anxiety the same way as a math anxiety. There are many reasons why people develop math anxiety that could vary from an early childhood embarrassing math grade to this social conception of math as being the "measure for intelligence". What is more important is perhaps not the "why" but rather the "how". The following are some steps that could help coop with math anxiety.

- Have a positive attitude and always think: "I am smart enough to deal with my math course
- Be persistent and patient
- never skip lectures
- Avoid gathering where the only topic is to talk about how mathematics are impossible to understand
- keep on hammering these exercises on a daily basis
- Use all resources the University offers you
- Your student federation office will provide you with previous tests and exams. Make sure your try some of them over and over
- You have to accept the fact that there is price you have to pay for your success. With the No pain, No gain mentality, you can go a long way.

5. **The big picture is hard to see.** One of the major difficulties encountered by students in first year courses is that they do not see the relevance of the material for their chosen fields. This can seriously affect their motivation in the course, and their ultimate success. This effect seems to be most pronounced in first year mathematics. One question instructors get to hear often is: why do we
need to know that? and it usually come from serious hard working students. Because mathematics is widely applied in every branch of sciences, engineering, economics and even in some social sciences it is very hard to explore many applications of mathematics at the first year level. You have to be patient and trust the system. Think of your first courses as the seed time, the harvest will come later.

1.4 How to succeed in your Math courses?

Now that you have a better understanding of the main differences between old and new environments and between learning mathematics and other subjects, how can you better prepare yourself to do well in your first year mathematics courses? the purpose of this section is to give you some tips to help you achieve exactly that. Most of these tips apply whether you freshly come from high school or returning to University years after finishing high school.

1. **Attend every class.** Let me be clear, attend every class, not almost every class. I agree that it can be sometimes tempting to cut a class, join a couple of friends to do something else. It is hard to say "no" to fun and social activities, but it is much harder to make up what you miss. **Do not fool yourself into thinking that missing a class won't make a difference** as long as you get copies of the notes. Getting copies of someone's notes is an extreme measure you take only when you really have to skip class for an emergency and it should not become a habit or a substitute of going to class. The notes you get from someone else might be neat and clean, but they will more likely miss important remarks and tips the professor might have made (not in writing) during class.

2. **Before class.** The best preparation for your class time starts actually before the class. We all have always something big or small on our minds, it is in our nature. Before your Math class, take five minutes to clear your mind in whatever way that works best for you. Get to class early, get a good seat maybe near the front so you can clearly see and hear with as little distraction as possible. Always remember, the professor you are about to listen to is a person actively engaged in applying, refining, and inventing mathematics, in addition to "hopefully" being dedicated to teaching. His or her research work might be applied in topnotch engineering, technology or even medical sciences. Taking a math lesson with a university professor is like taking a painting course from a successful painter, or a music course from a professional musician. This is an opportunity that was rarely available to you in high school and you shouldn't miss on it. Yes, sometimes what your music teacher tells you does not make a lot of sense, but when you are alone practicing your lesson, the words will come to your mind and you will appreciate the lesson more.

3. **During class.** Try to isolate yourself from your surrounding, avoid any unnecessary conversations with your friends. Some of them might not like at the beginning, but they will get used to it. Remember that the less you write, the more you can listen. This is why it is crucial that you develop a
shorthand note-taking technique so you are not way behind the professor explanations. If you find yourself always struggling to write everything down, there is nothing wrong with bringing a recorder to class and ask the professor to record the lectures. I am yet to know any colleague who says No. No matter what, don't stop taking notes even if you are completely lost and steps don't make any sense anymore.

4. **After class.** Be careful, attending all lectures and taking notes is not a sufficient condition for passing the course. For every classroom hours, you should spend between two to four hours on your own reading, working on practice problems and doing assignments. Do the math: if you are spending 12 hours on average in class during a week, then you would effectively spend an average of 36 hours a week on University work. From that perspective, studying at University is equivalent to a full time job, and more.

5. **Edit your notes.** As soon as you have the chance after class, edit your class notes to fill in any gaps and fix all mistakes. It might be a good idea to start by reading the corresponding section(s) in the textbook and then ensure it is integrated with the lecture notes in a way you will understand. Rewrite the lecture notes if necessary, only you can be the judge of that. Make sure you review your notes before the next class. Editing and reviewing your notes soon after your class is critical and the reason for that is simple: how many times have you said to yourself: “Why this Theorem (or example, or paragraph) does not make any sense any more, I thought I did understand it in class?” well, there are two possible answers. The first one is that you probably only understood the main idea in class, but you are now stuck on the mechanics of it. A more likely answer is that you left it long after class, and your brain had the chance to forget it.

6. **Understand so you don't forget.** You certainly have to memorize the statements of Definitions, Theorems, formulas and major facts from your lecture notes, but you **only** do that after you have a solid understanding of them. **Your Notes must be understood, not just read and memorized.** Add some enthusiasm in reading your notes by thinking of them as one chapter in a novel you are reading. Try to think of what is expected to happen in the next chapter. If you have time, read a bit ahead in your textbook. I know that made many of my lectures look much easier in my undergraduate years.

7. **Attend Labs.** As explained earlier, learning Mathematics is like learning a new language. In any language course, the actual learning is done in the labs where students can listen and practice taking the language. Much like any other language, **mathematics is a subject that must be struggled through by reading and practicing problems.** In most cases, lecture time is mainly devoted to explaining and prove the main definitions, theorems and facts and probably go through few examples. Your professor counts on you to attend the lab, usually conducted by a graduate student, to practice the theory through a series of carefully chosen exercises. The labs are conducted in smaller groups to give students more chance to interact with their TA, to ask questions and to try problems on their own rather than just copy what the TA is witting on the board.
8. **Do your Assignments.** A typical first year math course would have between 4 and 8 Assignments worth 10 to 15% of the total course work. The most important rule when it comes to doing an assignment is **Not to wait until the day before submission date to begin your work on it.** By leaving things till the last minute, you are adding a killer factor into the mix, the stress factor. Start by surveying all the questions in your Assignment, identifying the sections in the textbook (or your notes) they belong to. Such a survey will also allow you to identify questions that you should start with, namely those that you are most comfortable with and you think you can finish. Before you start on a question, take five minutes to quickly review the corresponding section. If a question seems to be very hard and you are stuck in it, look in your textbook or notes for a similar example. If difficulty persists, it might be a good idea to leave for some time, work on something else and then give it another try later on. If you have exhausted all your ideas trying to solve the problem, record whatever work you have done on it and make a note that you should seek help.

9. **Take advantage of various resources.** Unlike what you might think, you are not left alone in the "jungles" of Calculus and Linear Algebra. Help could be just few minutes away from where you are on Campus. The following is a list of some of the resources available to you.

   (a) **The Math Help Center.** this is the place where you need to be if you are stuck while reading your notes, textbook, doing your assignment or practice problems. The Center operates on a first come first served basis but it is open for long hours during the week. Don’t wait until the last minute to seek help, the center is usually much busier before midterms and assignments submission dates and there could a long waiting period. Make time in your weekly schedule, between your courses and labs, to show up and get your questions answered. Most importantly, don’t expect the staff at the Help Center to give you a full solution for your Assignment question. Their mandate is to help put you on the right track to be able to solve it yourself. Always remember, a trip to the Help Center is successful only if you can the problem you came for help with on your own. The Center has a study area where you can sit and do your work.

   (b) **Get a Private Tutor.** If you think you are really lacking behind and spending countless hours reading textbooks to get up to speed with the course, consider hiring a private tutor for limited time. maybe one or two hours a week. You can access a list of private tutors on campus at the Peer Help Center or drop by the math department, the main office usually has a list of tutors or can tell you where you might be able to find one. Be careful, private tutors are not cheep and even if you can afford it, you will not be able to find one who is willing to spend 10 hours with you every week. The ideal is to combine the help of a private tutor with the (free) service of Math Help Center. In most cases, hiring a tutor for maybe one or two hours a week to get you through the basics and seek more help at the Math Help Center should be enough to get you up to speed in the course.

   (c) **The libraries.** There is a network of libraries on campus for all programs of studies. Most math
related textbook and articles are located in Morrisset library. Each library is equipped with a user friendly search catalogue, many study areas where you can work on your notes, your assignment or prepare for your test with access to thousands of books. Using your student card, books can be borrowed and taken outside the library.

(d) **Your professor office hours.** Office Hours are the posted days and times a professor can be expected to be in his or her office available to meet students. These office hours are usually clearly specified on the course outline distributed during the first lecture. Some students think it is not appropriate to bother the professor outside the classroom, trust me nothing can be further from the truth. I can assure you that professors are committed to help you succeed in their classes, fully understand the class notes, be able to finish the assignment and providing office hours is one way to show that commitment. However, office hours should not be treated as a "happy hour" where students can drop by with little or no preparation at all and expect the professor to walk you through every single line of your notes or show you every single step to solve an assignment question. Some students think they can "pretend" that they have attempted the question, but trust me an unprepared student is easy to detect. Coming unprepared to professor office hours is exactly like declaring to him or her "Do the thinking for me and I will take care of the writing". To get the full benefit from meeting your professor and to clear the time for other students to ask their questions, you must come to the office hours fully prepared with very specific questions and ready to show what you have done to understand. Equally important is not to use office hours as a kind of a "therapy session" where you explain the social and psychological events of your life that led to you current situation in order to justify lack of understanding. This is neither the right place to do that nor the right person to talk this issues with and this will certainly leads nowhere. If for some reason you cannot see your professor during the specified office hours (due to schedule conflict or an emergency), there is usually a “by appointment” option for you by which you can contact your professor to ask for an appointment.

10. **Always keep a positive Attitude.** How many times, during a intense Gym workout session, you stop and say to yourself *I am board to death?* Probably Never. The reason is simple, during a workout session your body is busy training its muscles so its does not send a signal of boredom to your brain. In the same way, lecture time should be treated as an intense brain workout, think about it as the gym time to your brain. Nothing could more disturbing for both the professor and the fellow students than seeing a student taking a nap or yawning constantly or indulging in social conversations with other fellow students during lecture time. If you get board in math lecture, you are definitely taking the wrong approach to learning. The quality of teaching, the content of lecture are no excuse for a negative attitude. You don't choose the instructor or the content of your course, but you definitely choose the approach on learning the course.
1.5 Most common math misconceptions and errors

When I first decided to add this chapter to the guide, the goal was to give a somehow complete list of most common math errors, academic errors that is, that I have encountered through my daily contact with students for many years here at the University of Ottawa. After I have started to write it however I realized that some of these errors originate from gaps not in the student academic background but rather from wrong attitude toward math or bad study habits.

1.5.1 Most common wrong attitudes toward math

Throughout its development, Math has build a reputation, unfortunately a bad one. With that bad reputation, people developed a certain "attitude" toward math. Some people even talk about a "math anxiety" and they go as far as describing every single detail of its symptoms. If you find Mathematics a hard subject to learn, having a bad attitude and keeping the preconceptions will only make the subject harder to learn.

• I was always bad at Math. Yes, and I was always bad at Volleyball until I realized that I was trying to play it the same way I played Football. Students who do really poor in their first year University math courses are in majority students who simply do not have good math study skills. This is not because they are not capable of studying math, but because they have never developed a skill to study math. Don't get me wrong, I am not saying that the subject is all that easy and anyone should be able to get through it with a minimum effort, but one has to accept the fact that it takes a different approach than most of other subjects to get satisfactory results in mathematics. From my experience, the first step to remedy this "I was always bad at Math" disease is to take a big dose from the "get serious, study smart, be patient and ready to make sacrifices" medicine. Always remember that at your first year mathematics courses, your professors are not testing your ability to conduct research in the subject and pursue a PhD degree (as much as they love to see you do that), they are only after your ability to reproduce what they taught you in class and in some cases apply your knowledge to certain situations.

• I am not good with numbers. I am not good with numbers either and I have a PhD in Mathematics. Let me put it this way, a carpenter can do amazing cuts on a plywood sheet to fit a subfloor, perfect circular and multi sided dining tables and other geometrical figures but if you ask him to do his tax return by hand, in most cases he will get audited. Would you say that he is good at math? If you measure "being good in math" with the ability of identifying similarities and patterns, working with some constraints and able to draw conclusions, then the answer is definitely yes. Not being able to add or multiply big numbers in your head is not at all a sign that you are not good in Math. This myth about math and numbers has solid roots in our society so much so that mathematicians often get the question: "what is the last number you invented?"

• Mathematics do not matter. Pressed by time constraints and the need to finish the curriculum,
most high school teachers and first year (and even beyond) University math instructors find little or no chance in their courses to present real life applications of the elegant theories they develop in their classes. Students constantly asked me the question “why do we need to know that?” or even a stranger one ”Research in Math? haven't we discovered everything in Math?”. The fact is most of these daunting theories started as an attempt to solve a real life problem and ongoing research in these areas goes hand in hand with development of new technologies and ways to improve our lives. Many of the studies on the issue of learning mathematics conducted by experts in the field show a strong correlation between learning the subject and the awareness of its relevance in practice. This is the way we are built, our brain becomes more interested in absorbing the information when it realizes it has a chance to use it later on. Of course, this not the place to start talking about the use of Mathematics in our daily life, nor this is the intent of these notes but it could be reassuring to you to know that efforts are made at the department of Mathematics and Statistics at the University of Ottawa to make the math learning experience more hands-on. The following are examples of efforts made to show the practical side of what you learn in your class.

- Applications of Linear Algebra: [aix1.uottawa.ca/~jkhoury/linearmain.htm](http://aix1.uottawa.ca/~jkhoury/linearmain.htm)
- Some Applications in modern Technology: [aix1.uottawa.ca/~jkhoury/](http://aix1.uottawa.ca/~jkhoury/)

Another point that could be of comfort to you is to know that many organizations in the industrial world exist with the main goal of connecting industry with academic, in particular mathematics. Research on improving certain aspects of an industry often results in mathematical problems whose solutions are beyond the training skills of staff members. Here in Canada, MITACS is a federally- and provincially-funded research network bringing together academia, industry and the public sector to develop cutting edge tools.

**I will put off my math courses for now.** If you find Math hard on you now, imagine how hard it is going to be one or two semesters from now. As mentioned in many place above, mathematics is a cumulative subject, what you learn now is founded on what you have already learnt in previous courses and in high school. By putting off a required Math course, your are probably putting off other courses for which the math course is a prerequisite but more dangerously you are putting yourself at a much greater risk of failing the course by allowing more time for whatever math knowledge you have now to fade away.

**I know the stuff, I can pass with no effort** When I teach Calculus I, many students often tell me that they have seen most the material in high school. I agree, many topics in first semester University math courses are just review, with bit more details, proofs and applications, of the same topics seen in high school math courses. But here is the catch, "bit more details, proofs and applications" is the part where most questions on a test will come from. We want you to have a solid high school background to be able to build on it. I cannot remember how many times I invited some students
to be office after a midterm and said: "This should be a wake up call for you". It does have to be that way, it is a grave mistake to do the bare minimum in your course and even graver if you stop going to classes thinking that you master the stuff and you only have to write the tests.

• **I did very good on the first midterm, I think I worried too much.** True story. Before writing this section, I was finishing the marking of the final exam on a second Semester Calculus class. The class had about 60 students, 7 of which did very well on the midterm (25 or higher on 30), but still managed to either fail the course or barely pass it. You might think that the reason is simply because the final was very hard. Wrong, because many students did (on average) better on the final than the midterm. Most logical explanation is that some or all of these 7 students took it easy after the midterm thinking that they probably can have similar results on the final with much less effort. Why do the work if you are getting paid anyway, right? What is noticeable is that these students were regular visitors to the Math Help Center and to my office hours at the beginning of the term (before the midterm) and almost disappeared and became occasional visitors after the midterm. If you found the test easy, it is more likely because you studied well for it.

1.5.2 **Most common math errors**

This section is probably the reason why I started this project. My main goal was to make a list of most common Math errors I have encountered during the many years of contact with students coming fresh from high school or returning to university studies after years in the work force. If you think that there is no way you could make any of these errors, then I sincerely congratulate you. If you made one or more of these errors before or you think that you could have done them in certain cases, rest assured that you are not alone. Be aware though that being just "embarrassed" or "feeling ashamed" is not enough to make them go away or stop you from committing them over and over, but understanding these errors will. These common errors are not listed in any significant order, what could be "clear and obvious" for you is not necessarily the same for others.

• **The square root confusion.** One of the biggest math misconceptions is about the use of square root. A large percentage of students seem to be under the misconception that the square root of a positive number could be negative, so they would write $\sqrt{4} = \pm 2$. While it is true that $2^2 = 4$ and $(-2)^2 = 4$, "the square root of 4" is defined to be only the positive option, $+2$. Square roots are always positive or zero.

• **Everything is additive (or should I say linear).** A whole range of common mistakes originate from the incorrect assumption that all functions are linear. That is the assumption that for any function $f$, one has $f(a+b) = f(a) + f(b)$ and $f(ca) = cf(a)$ are always true for any choices of real numbers $a$, $b$ and $c$. The most common (wrong) consequences of this assumption are the following: $(a+b)^2 = a^2 + b^2$ and $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$. Just think of the following example: $\sqrt{1+4} = \sqrt{5}$ but $\sqrt{1} + \sqrt{4} = 1 + 2 = 3 \neq \sqrt{5}$. 
In general, \((a + b)^n \neq a^n + b^n\) and \(\sqrt[n]{a + b} \neq \sqrt[n]{a} + \sqrt[n]{b}\)

Remember, in your Calculus courses, only functions with straight lines as graphs are linear. In other words, these are functions of the form \(f(x) = mx + b\). This notion of "linearization of everything" extends well beyond the scope of the exponents and radicals to things like fractions, logarithms and trigonometric functions. It is very often that students would write something like 
\[a \cdot b + c = a \cdot b + a \cdot c,\]
\[\ln(a + b) = \ln a + \ln b\]
\[\cos(a + b) = \cos a + \cos b\]
which are of course all wrong.

In general:

- \(\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}\),
- \(\log_{a}(x+y) \neq \log_{a} x + \log_{a} y\),
- \(\cos(a + b) \neq \cos a + \cos b\),
- \(\sin(a + b) \neq \sin a + \sin b\)

**Dividing by zero.** While most students agree that \(\frac{0}{1} = 0\), a surprisingly high percentage of students still think that the value of \(\frac{1}{0}\) is zero. Always remember, you cannot divide with zero and consequently you cannot simplify with an expression that could take the zero value. I cannot remember how many times I have seen students attempting to solve an equation reduced to the form 
\[x(x+1)(2x-1) = x(x+1)(-x+2)\]
simply by canceling the two common factors \(x\) and \(x+1\) on both sides of the equation and end up with the simple equation \(2x-1 = -x+2\) with the unique solution \(x = 1\). While this is a valid solution, it is certainly not the only one. By "canceling" \(x\) and \(x+1\) on both sides of the equation, you have eliminated two other valid solutions, namely \(x = 0\) and \(x = -1\).

**Improper Distribution.** While most students are comfortable using distribution of the multiplication over addition (that is \(a(b+c) = ab + ac\)), some will distribute multiplication over multiplication. Here is an actual line written by a student on a test:

\[2(3 + 4x) = 2(3) + 2(4)2(x) = 6 + 16x.\]

What the student did is actually multiply the outside "2" with both the 4 and the \(x\) in the term \(4x\).

**Wrong simplification.** This error can take many forms, each of which has its own reason. Let me start with an actual error I came across recently while marking a final exam. At one point in his solution of an exam problem, the student (a good one by the way) made the following simplification:

\[
\frac{(2x + 3)(x - 1) + 3x^2}{(2x + 3)(x^2 + 1)} = \frac{(2x+3)(x-1) + 3x^2}{(2x+3)(x^2 + 1)} = \frac{(x-1) + 3x^2}{(x^2 + 1)} = \frac{3x^2 + x - 1}{(x^2 + 1)}.
\]

If you made this kind of an error in the past, it is most likely because you let your mind play tricks on you by inserting invisible parentheses. What I am trying to say is that the two expressions \(ab + c\) and \(a(b+c)\) are completely different. In the first one, \(a\) multiplies \(b\) only, while in the second one, \(a\)
multiplies $b + c$. Another form of wrong simplification is illustrated in the following simplification a student actually made in the process of evaluating a certain integral:

$$\left. \cos \frac{2\pi}{x} \right|_{2\pi} = \frac{\cos 2\pi}{2\pi} - \frac{\cos \pi}{\pi} = \cos - \cos = 0.$$  

My only explanation for such an error is that students tend to treat every "juxtaposition" as a multiplication, so they see an expression like $\cos 2\pi$ as being $(\cos) \times (2\pi)$ without realizing that "cos" alone has no value whatsoever.

• **Confusing the hypothesis with the conclusion.** The statement "If $A$, then $B$" is called an implication and it means "Condition $A$ implies condition $B$" or "If condition $A$ is true, then condition $B$ is also true." In the Mathematical language, the implication "If $A$, then $B$" is written as "$A \implies B$" where $A$ is called the hypothesis of the implication and $B$ is called its conclusion. The two implications "$A \implies B$" and "$B \implies A$" are not equivalent in the sense that they state two different things, one is called the inverse of the other. It is customary that theorem and results in Mathematics are stated as implications. On many occasions in your first year Math courses (and of course more beyond first year) you will be asked to prove some statements written in implication form. In this case, read the statement more than once, make sure you clearly understand what is given and what is it that you are trying to prove as very often students confuse an implication with its inverse. Here is an example to illustrate this common error. In a first year discrete Math course, students were asked to prove the following simple statement: "For any integer $n$, if $3n + 1$ is odd, then $n$ must be even." Some students started their proof by assuming that $n$ is even and then proving that $3n + 1$ is odd. While the proof itself is valid, those students got no credit at all since they did not actually answer the question but rather provided a proof for the inverse statement.

• **It is true for some, it must be true for all.** A probably more appropriate name for that common error is **Proof by Example.** To prove a certain result, some students would verify that the result is true in some particular cases, and they would conclude that it must be always true. For example, in a first year calculus course students are often asked to prove a statement like: $e^x > x$ for all real number $x$. Every time such a question appears on a test, a handful of students would actually attempt it by considering values of $x$ like $x = -2, -1, 0, 1, \ldots$, plug them in $e^x$ and verify in each case that the inequality $e^x > x$ is satisfied.

• **Working Backward.** This is closely related to the confusion between the hypothesis and the conclusion we mentioned above. To the question "prove that $\sqrt[3]{3} > \sqrt[2]{2}$", many students would unfortunately attempt a solution using the following reasoning:

$$\sqrt[3]{3} > \sqrt[2]{2} \implies 3^{\frac{1}{3}} > 2^{\frac{1}{2}} \implies \left(3^{\frac{1}{3}}\right)^6 > \left(2^{\frac{1}{2}}\right)^6 \implies 3^2 > 2^3 \implies 9 > 8.$$  

Since the last statement ($9 > 8$) is correct, the students would conclude that the original statement $\sqrt[3]{3} > \sqrt[2]{2}$ must be correct. Nothing could be further from the truth. **You should never begin a proof by assuming the very thing that you are trying to prove is actually true. I hope that you see the point.**
Chapter 2

Basic Set Theory

2.1 Basic Definitions and notations

A set is a collection of objects called the elements of the set. If \( A \) is a set, and \( x \) is an element of \( A \), we say that \( x \) belongs to \( A \) and we write \( x \in A \) for short. Given a set \( A \), we say that a set \( B \) is a subset of \( A \), and we write \( B \subseteq A \), if every element of \( B \) is at the same time an element of \( A \). The notation \( B \subset A \) is often used if we want to stress the fact that \( B \) is a subset of \( A \) which is not equal to \( A \). Curly brackets "{ }" are used to indicate the elements of a set either by listing them or by specifying a common properties satisfied by all the element of the set.

Example 2.1.1. (1) The set \( A = \{-1, 1, 2, \delta, \epsilon\} \) contains 5 elements: \(-1, 1, 2, \delta \) and \( \epsilon \).

(2) The set \( B = \{ 2m \text{ such that } m \text{ is an integer } \} \) is the set of all even integers.

Two sets \( A \) and \( B \) are said to be equal, and we write \( A = B \), if they have precisely the same elements. Thus \( A = B \) if every element of \( A \) is an element of \( B \) and every element of \( B \) is an element of \( A \) or equivalently:

\[
A = B \iff A \subseteq B \text{ and } B \subseteq A.
\]

The order in which we list the elements of a set is not important. For instance the two sets \{1,2,3\} and \{3,2,1\} are equal since they have the same list of elements.

There is a unique set containing no elements, called the empty set which we and denote by \( \emptyset \). The empty set is a subset of any other set.

Example 2.1.2. Let \( A = \{2,4,6,8,10\} \), \( B = \{2,6,9,13\} \) and \( C = \{0,9,13\} \). Then \( 2 \in A, 2 \in B \) but \( 2 \notin C \). Also \( C \subset B \) but \( C \nsubseteq A \).
2.1.1 Operations on sets

Basic operations on sets include (among other operations) intersection and union. If \( A \) and \( B \) are two subsets of a set \( \mathbb{U} \), then the intersection of \( A \) and \( B \), denoted by \( A \cap B \), is the subset of \( \mathbb{U} \) of all elements common to both \( A \) and \( B \).

\[
\begin{align*}
A \cap B & \\
A & \\
B & 
\end{align*}
\]

The union of \( A \) and \( B \) is the set of elements, which are either in \( A \) or in \( B \).

\[
\begin{align*}
A \cup B & \\
A & \\
B & 
\end{align*}
\]

In Example 2.1.2 above, \( A \cap B = \{2, 6\} \), \( A \cap C = \emptyset \), \( B \cap C = \{9, 13\} \), \( A \cup B = \{2, 4, 6, 8, 9, 10, 13\} \), \( A \cup C = \{0, 2, 4, 6, 8, 9, 10, 13\} \) and \( B \cup C = \{0, 2, 6, 9, 13\} \).

2.2 Integers, rationals, reals and complexes

The set of all non-negative integers (or natural numbers) 0, 1, 2, 3, … is denoted by \( \mathbb{N} \). This is clearly a subset of the set \( \mathbb{Z} \) of all integers: \( \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \). A rational number is a fraction of the form \( \frac{p}{q} \) where \( p \) and \( q \) are both integers, with \( q \neq 0 \). In particular, every integer is a rational number since if \( n \in \mathbb{Z} \), then \( n = \frac{n}{1} \) is also rational. The set of all rational numbers is denoted by \( \mathbb{Q} \). A class of numbers that includes all rational number (and much more) is of course the set \( \mathbb{R} \) of all real numbers. Real numbers
that are not elements of $\mathbb{Q}$ are called **irrationals**. These are numbers like the famous $\pi$, $e$, $\ln 2$ (that we will see later) and $\sqrt{n}$ where $n \in \mathbb{N}$ is not a perfect square.

Unlike rational numbers, irrationals don't have a periodic part (repeating part) when written in decimal form. For example, the decimal form of the rational number $\frac{27}{111}$ is $0.285714285714285714$ with $285714$ as the periodic part. The decimal form of $\sqrt{2}$ is $1.6931471806...$ with no repeating part.

Although we don't assume any knowledge of it, but depending on your background, you may have also seen the set of **complex numbers** denoted by $\mathbb{C}$. Complex numbers are needed to solve equations like $x^2 + 1 = 0$ which fails to have any real roots. Using the number $i$ satisfying $i^2 = -1$, a complex number (in standard form) is an expression of the form $a + bi$ where $a, b \in \mathbb{R}$. Since any real number can be written under this form (with $b = 0$), real numbers are in particular complex numbers. One then has the following list of (strict) inclusions of sets:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}. $$

## 2.3 Intervals

Large sets are often labeled by the properties that characterize their elements. The notation

$$A = \{x \in \mathbb{R}; \ p(x)\}$$

means that $A$ is the set of all real numbers satisfying a certain property $p(x)$. The semi-column ";" used after $\mathbb{R}$ stands for "such that" (in some book, this is replaced with a vertical bar $|$. For example $\{x \in \mathbb{R}; \ x \geq 0\}$ is the set of all non-negative real numbers and $\{x \in \mathbb{R}; \ x^2 \geq 1\}$ is the union of the two sets $\{x \in \mathbb{R}; \ x \geq 1\}$ and $\{x \in \mathbb{R}; \ x \leq -1\}$ as we will see later. In most of your University math courses, these sets are written using the "interval" notation.

An important property of real numbers resides in the fact that we can order them (in the natural way): if $x, y \in \mathbb{R}$, the notation $x < y$ means $x$ is strictly less than $y$. Equivalently, $y > x$ means $y$ is strictly greater than $x$. The notation $x \leq y$ means that $x$ is less than or equal to $y$ and $x \geq y$ means that $x$ is greater than or equal to $y$. For example, $1 < \frac{7}{4}$ and $\frac{7}{4} \geq 0.8$. This ordering on the reals allows us to think of $\mathbb{R}$ as a set of points on a straight line, called the **real line**. We normally choose 0 as a reference point on the line, so any number to the right of 0 is positive and any number to the left of 0 is negative. If $x < y$, then that would translate to $x$ being to the left of $y$ on the real line (or that $y$ is to the right of $x$). We also make the
distinction between non-negative and positive numbers. A real number is called \textbf{positive} if it is strictly greater than 0 and is called \textbf{non-negative} if it is positive or zero.

\[ -\pi \approx -3.141592654 \quad e \approx 2.718281828 \]

Mathematical notions like domain, range, continuity, differentiability, convergence and others are often defined on a certain interval of real numbers. Let \( a, b \) be two real numbers with \( a < b \). The set of all real number strictly between \( a \) and \( b \) (that is excluding \( a \) and \( b \)) is referred to as the \textbf{open interval} \( (a, b) \) or \( (a, b] \) depending on the book used and the taste of your instructor. In these notes, we will use the notation \( [a, b] \) for an open interval. If we want to include one (or both) of the points \( a, b \), we replace the brackets \( ] \) and \( [ \) with \( [ \) and \( ] \) respectively. A standard way of graphically representing intervals on the real line is to use a thick line segment with filled or/and empty circles at its ends. An empty circle represents a point not belonging to the interval, while a filled circle represents a point in the interval. This is illustrated in the following diagrams.

- \( [a, b] = \{ x \in \mathbb{R}; a < x < b \} \):
  \[ \begin{array}{c}
    a \\
    \hline
    b
  \end{array} \]

- \( [a, b] = \{ x \in \mathbb{R}; a \leq x \leq b \} \):
  \[ \begin{array}{c}
    a \\
    \hline
    b
  \end{array} \]

- \( [a, b] = \{ x \in \mathbb{R}; a \leq x < b \} \):
  \[ \begin{array}{c}
    a \\
    \hline
    b
  \end{array} \]

- \( ]a, b] = \{ x \in \mathbb{R}; a < x \leq b \} \):
  \[ \begin{array}{c}
    a \\
    \hline
    b
  \end{array} \]

the following are some special (infinite) Intervals:
• \( [a, \infty) = \{ x \in \mathbb{R}; x > a \} \):

\[ a \]

• \( ]a, \infty) = \{ x \in \mathbb{R}; x \geq a \} \)

\[ a \]

• \( ]-\infty, b] = \{ x \in \mathbb{R}; x < b \} \)

\[ b \]

• \( ]-\infty, b) = \{ x \in \mathbb{R}; x \leq b \} \)

\[ b \]

Clearly, \( ]-\infty, +\infty) = \mathbb{R} \).

**Warning 2.3.1.** Never use a square bracket before or after \( \infty \) or \(-\infty \) since the symbol \( \infty \) is not a "real" number and therefore we cannot include it in any of these intervals.

**Example 2.3.1.** Let \( I = ]-1, 3] \) and \( J = ]1, 5] \). Then \( I \cap J = ]1, 3] \) and \( I \cup J = ]-1, 5] \). This is illustrated in the following diagram where the interval \( I \) is drawn in blue and the interval \( J \) in red. The two colors overlap in \( ]1, 3] \) giving the intersection of the two intervals.

**Try it yourself 2.3.1.**

1. Represent each of the following sets as intervals on the real line.
   (a) \( A = \{ x \in \mathbb{R}; -2 \leq x < 4 \} \).
   (b) \( B \) is the set of all real numbers greater than -2.

2. Find \( I \cap J \) and \( I \cup J \) in each of the following cases.
   (a) \( I = ]-\infty, -2] \) and \( J = ]-3, \infty] \).
   (b) \( I = [0, 4] \) and \( J = [-1, 3] \)
Chapter 3

Brushing up on Basic Algebra

This chapter deals with basic algebraic manipulations on real numbers. Very often, the lack of these basic tools is the main reason for poor performance in first year Math courses. Your instructors assume that you have seen these basics over and over and expect you to be fully comfortable working with them. Students should not expect that instructors would dedicate any time to review them.

The content of this chapter is the bone that will hold the meat of your math learning, so roll up these sleeves and invest some effort to review them.

3.1 Integer Exponent

If \( a \in \mathbb{R} \) and \( n \in \mathbb{N} \), \( a^n \) is the product of \( a \) by itself \( n \) times, with the convention that \( a^0 = 1 \) if \( a \neq 0 \):

\[
a^n = \begin{cases} 
1, & \text{if } n = 0 \text{ and } a \neq 0 \\
\underbrace{a \times a \times \ldots \times a}_{n \text{ times}}, & \text{if } n > 0
\end{cases}
\]

In \( a^n \), \( a \) is called the base and \( n \) is called the exponent. If \( n \) is a negative integer, we define \( a^n = \frac{1}{a^{-n}} \) (note that in this case \(-n > 0\)). The following table summarizes the basic properties of integer exponents. You definitely need to memorize these basic properties.
**Integer Exponents Properties.**

If $a$ and $b$ are nonzero real numbers, $m$ and $n$ are integers, then:

- $a^m \cdot a^n = a^{m+n}$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $\frac{a^n}{a^m} = a^{n-m}$
- $(a^m)^n = a^{mn}$
- $(\frac{a}{b})^{-n} = (\frac{b}{a})^n$

**Examples 3.1.1.**

1. $2^3 \cdot 2^4 = 2^{3+4} = 2^7 = 128$
2. $\left(\frac{-1}{2}\right)^4 = \frac{(-1)^4}{2^4} = \frac{1}{16}$
3. $\frac{2^5}{2^2} = 2^{5-2} = 2^3 = 8$
4. $\frac{1}{2^{-2}} = 2^2 = 4$
5. $\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2 = \frac{4^2}{3^2} = \frac{16}{9}$

We usually don’t like to include a negative exponent when writing the final answer of a problem. For example, we would write $\frac{2}{x}^4$ rather than $2x^{-4}$ (although the two expressions are equal).

**Example 3.1.1.** Often you would need to write an expression like

$$\frac{3^{2n-3}}{5^{n+2}}$$

under the form $ar^n$ where $a$ and $r$ are real numbers. This could be done using the above properties:

$$\frac{3^{2n-3}}{5^{n+2}} = \frac{3^2}{5^2} \cdot \frac{3^{-3}}{5^{-2}} = \frac{(3^2)\cdot 3^{-3}}{5^2 \cdot 5^{-2}} = \frac{9 \cdot 3^{-3}}{5^2 \cdot 5^{-2}} = \frac{9}{675} \cdot \frac{5}{5} = \frac{9}{675} \cdot 5^n$$

**Example 3.1.2.** Let us rewrite the sum $21(7^3) + 2(7^4)$ in one term $ar^n$:

$$21(7^3) + 2(7^4) = (3 \times 7)(7^3) + 2(7^4) = 3(7^4) + 2(7^4) = 5(7^4)$$

**Try it yourself 3.1.1.**

1. Find the value of each of the following expressions.

   (a) $(-2)^4$  (b) $4^{-2}$  (c) $2^{-2} \cdot 3^3$  (d) $(\frac{2}{3})^2$

2. Write each of the following expressions in the form $a \cdot r^n$:

   (a) $\frac{2^{2n+3}}{3^{n+1}}$  (b) $10(5)^3 + 3(5)^4$
3.2 Rational Exponent.

The integer exponents are just members of a larger family, the rational (or radical) exponents.

We start by reviewing the notion of square root. If \( a \) is a non-negative real number, then the square root of \( a \) is defined to be the non-negative number \( x \) whose square is equal to \( a \). In other words: if \( x = \sqrt{a} \), then \( x \geq 0 \) and \( x^2 = a \). Be careful, if \( x^2 = a \), then also \((-x)^2 = a\) since \((-x)^2 = x^2\) which means that in theory there are two square roots of a positive real number, one positive and another negative. But in order to avoid confusion, the convention is that \( \sqrt{a} \) stands for the non-negative square root of \( a \). A common mistake is to write \( \sqrt{x^2} = x \) (unless \( x \) is clearly non-negative in the context of the problem you are working with) instead of \( \sqrt{x^2} = |x| \) (the absolute value of \( x \) that we will review below). For example, the square root of 9 is 3 (since \( 3^2 = 9 \)), not \(-3\). Note also that \( \sqrt{9} \) can be written in rational exponent as \( 9^{\frac{1}{2}} \). Similarly, the fourth root of the non-negative number \( a \) is the non-negative number whose fourth power is \( a \): \( x = \sqrt[4]{a} \iff x \geq 0 \) and \( x^4 = a \).

With rational exponent notation, the fourth root of the non-negative number \( a \) is written as \( a^{\frac{1}{4}} \). Thus, \( \sqrt[4]{16} = 16^{\frac{1}{4}} = 2 \), since \( 2^4 = 16 \). We can similarly define sixth roots, eighth root, etc... In general, if \( a \) is a non-negative real number and \( n \in \mathbb{N} \) is even, the \( n \)-th root of \( a \), written as \( \sqrt[n]{a} \) (in radical form) or \( a^{\frac{1}{n}} \) (in rational exponent form), is the non-negative real number \( x \) satisfying: \( x^n = a \):

\[
\sqrt[n]{a} = x \iff x \geq 0 \text{ and } x^n = a.
\]

What about odd \( n \)-th roots?

If \( n \) is odd, we can talk about the \( n \)-th root of any real number \( a \) (not necessarily non-negative): this is the unique number whose \( n \)-th power is equal to \( a \):

\[
\sqrt[n]{a} = x \iff x^n = a.
\]

Thus, the cubic root of 8 is 2 (since \( 2^3 = 8 \)) and the cubic root of -8 is -2, (since \((-2)^3 = -8\)). Note that if \( n \) is odd, then \( \sqrt[n]{a} \) has the same sign as the number \( a \).

With the understanding that \( \sqrt[n]{a} \) does not exist if \( a \) is a negative number and \( n \) is an even integer, the following table gives basic rules for rational (or radical) exponents.
 Rational Exponents Properties.  
If $a$ and $b$ are real numbers, $m$ and $n$ are integers, then:

$$\sqrt[n]{a}, \sqrt[n]{b} = \sqrt[n]{ab}, \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \quad (\text{for } b \neq 0), \quad \sqrt[n]{a^m} = a^{\frac{m}{n}}$$

**Warning 3.2.1.** Students often have tendency to treat the $n$-th root as a "linear" operator, so they would write $\sqrt[n]{a+b} = \sqrt[n]{a} + \sqrt[n]{b}$, which is of course wrong in general. Just think of the following example: $\sqrt[4]{1+3} = \sqrt[4]{4} = 2$, but $\sqrt[4]{1} + \sqrt[4]{3} = 1 + \sqrt[4]{3} \neq 2$.

**Example 3.2.1.** Without the help of a calculator, evaluate (when possible) each of the following expressions.

1. $(-64)^{\frac{3}{4}}$
2. $(-64)^{\frac{2}{3}}$
3. $(\frac{25}{64})^{\frac{3}{2}}$
4. $\left(\sqrt[3]{-\frac{8}{27}}\right)^2$

**Solution.** We use the properties of rational exponents.

1. $(-64)^{\frac{3}{4}} = (-4)^3 = (-4)^{(\frac{3}{4})} = (-4)^{2} = 16$. This could also be done a bit differently:

$$(-64)^{\frac{3}{4}} = \sqrt[4]{(-64)^3} = \sqrt[4]{(64)(-64)} = \sqrt[4]{(4^3)(-4^3)} = \sqrt[4]{4^3} \sqrt[4]{(-4)^3} = (4)(4) = 16.$$  

2. This is not a real number. Note that $(-64)^{\frac{3}{4}} = \sqrt[4]{(-64)^3}$ by the above properties. Since $(-64)^3$ is a negative number, its fourth root $\sqrt[4]{(-64)^3}$ is not a real number and therefore it cannot be computed (it is a complex number, but we don’t deal with this class of numbers in this manual).

3. $\left(\frac{25}{64}\right)^{\frac{3}{2}} = \left(\frac{5}{8}\right)^3 = \left(\frac{125}{512}\right) = \frac{125}{512} = \frac{125}{512}$.

4. $\left(\sqrt[3]{-\frac{8}{27}}\right)^2 = \left(\sqrt[3]{-\frac{8}{27}}\right)^2 = \left(-\frac{2}{3}\right)^2 = \frac{4}{9}$.

**Examples 3.2.1.** Simplify each of the following expressions.

(1) $\sqrt[4]{16}$,  (2) $\sqrt[3]{x^2}$,  (3) $\left(\frac{2}{3}\right)^{-\frac{1}{3}} \left(\frac{2}{3}\right)^{\frac{3}{2}}$,  (4) $\frac{3^{-1}x^2y^{-1}}{(2xy^{-2})^3}$,  (5) $\sqrt[3]{80}$
Again, we use the properties of rational exponents.

1. \( \sqrt{\sqrt{x}} = \sqrt{x^{\frac{1}{2}}} = x^{\frac{1}{4}} \)

2. \( xx^{\frac{5}{2}}x^2 = x^{1+\frac{5}{2}+2} = x^{\frac{17}{4}} \)

3. \( \left( \frac{3}{2} \right)^{-\frac{1}{3}} \left( \frac{2}{3} \right)^{\frac{1}{2}} = \left( \frac{2}{3} \right)^{\frac{1}{2}} \left( \frac{3}{2} \right)^{\frac{1}{3}} = \left( \frac{3}{2} \right)^{\frac{1}{3}+\frac{1}{2}} = \left( \frac{3}{2} \right)^{\frac{5}{6}} \)

4. \( \frac{(3^{-1}x^2 y^2 z^{-1})^{-3}}{(2xy^{-1}z^2)^2} = \frac{1}{(3^{-1}x^2 y^2 z^{-1})(2xy^{-1}z^2)^2} = \frac{1}{3^{-3}2x^3y^4z^2} = \frac{3^3 y}{4x^8z} = \frac{27y}{4x^8z} \)

5. \( \frac{\sqrt{90}}{\sqrt{80}} = \frac{\sqrt{2}}{\sqrt{5}} = \frac{\sqrt{2}}{\sqrt{5}} \frac{\sqrt{2}}{\sqrt{5}} = \frac{2}{4} = \frac{1}{2} \sqrt{\frac{2}{5}} \)

Try it yourself 3.2.1.

1. Without the use of your calculator, find the value of each of the following expressions.

   \( (a) \ (−27)^{\frac{1}{3}}, \quad (b) \ \left( \frac{−9}{27} \right)^{\frac{2}{3}}, \quad (c) \ \left( −\frac{\sqrt{27}}{3} \right)^{−\frac{1}{2}}. \)

2. Simplify each of the following expressions.

   \( (a) \ \frac{(2^{-1}x^2 y^2 z^{-1})^{-2}}{(4x^2 y^{-1} z^2)^2}, \quad (b) \ \frac{(a^4 b)^{-4}}{a^{-4} b^4}, \quad (c) \ \frac{(m^3 n)^5 (-mn^2)^3}{(3m^2 n)^4}. \)

3.3 Basic properties of operations on real numbers

Next, we look at the basic properties of operations on real numbers. These properties include, among other things, the distributivity of the multiplication with respect to addition, cross multiplication (in the case of equality between two fractions) and writing the sum or difference of two or more fractions as a single fraction (common denominator).
Basic properties of operations on real numbers.

If \(a, b, c\) and \(d\) are real numbers then:

\[(a + b)(c + d) = ac + ad + bc + bd\] (Distributivity)

\[-\frac{a}{b} = \frac{-a}{b}, \quad (b \neq 0)\]

If \(\frac{a}{b} = \frac{c}{d}\) (for \(b, d \neq 0\), then \(ad = bc\) (Cross multiplication)

\[c \frac{a}{b} = \frac{ca}{db}, \quad (b \neq 0)\]

\[\frac{a}{b} \frac{c}{d} = \frac{ad}{bd}, \quad (b, d \neq 0)\]

\[\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}\] (for \(b, d \neq 0\)) (Common denominator)

The last property in the above list shows how to write two fractions in common denominator. Surprisingly, some students find this hard to achieve, especially when fractions involve one or more variables in them. So make sure you understand the following examples and that you try the suggested Exercises.

**Examples 3.3.1.**

1. \[(x^2 - 2xy)(x + 2y^3) = x^2(x) + x^2(2y^3) - 2xy(x) - 2xy(2y^3) = x^3 + 2x^2y^3 - 2x^2y - 4xy^4\]
2. \[\frac{1}{2} + \frac{3}{7} = \frac{(1)(7) + (3)(2)}{(2)(7)} = \frac{13}{14}\]
3. \[2 - \frac{2}{3} = \frac{2}{1} - \frac{2}{3} = \frac{(2)(3) - (1)(2)}{(1)(3)} = \frac{4}{3}\]
4. \[\frac{1}{x^2} + \frac{x}{x+2} = \frac{1(x+2) + x(x-2)}{(x-2)(x+2)} = \frac{x^2 + x^2}{x^2 - 4}\]
5. \[\frac{2x+3}{x-4} + \frac{1}{x+1} = \frac{(2x+3)(x+1) + (x^2-4)(1)}{(x^2-4)(x+1)} = \frac{3x^2 + 5x - 1}{(x^2-4)(x+1)}\]

**Example 3.3.1.** In this example, we solve for \(x\) in the following equation:

\[\frac{2}{5} = \frac{3}{x+1}\]

Note first that \(x\) cannot equal to \(-1\) since this is the value that makes \(x + 1\) equal 0. Cross multiplication gives \(2(x + 1) = 5(3) \Rightarrow 2x + 2 = 15 \Rightarrow 2x = 13 \Rightarrow x = \frac{13}{2}\).

**Remark 3.3.1.** Multiplying the denominators in \(\frac{a}{b} + \frac{c}{d}\) to get a common denominator is not always the most efficient way. If \(b\) and \(d\) have a common factor, then one would look at the least common multiple (lcm for short). Let us explain this with a numerical example.

**Example 3.3.2.** Suppose you want to write \(\frac{5}{60} + \frac{13}{90}\) in common denominator. Note first that \(60 = 2^2 \cdot 3 \cdot 5\) and \(90 = 2 \cdot 3^2 \cdot 5\). So to get the lcm of 60 and 90, we take the “largest power” in each factor of 60 and 90:
the factor 2 appears with an exponent of 2 in the decomposition of 60, and with an exponent of 1 in the decomposition of 90. We take the largest power $2^2$ as a factor in the lcm of 60 and 90. Similarly, we take $3^2$, and 5 as the other factors in the lcm of 60 and 90. Thus, the lcm of 60 and 90 is $2^2 \cdot 3^2 \cdot 5 = 180$ and we take this as a common denominator (as opposed to $60 \cdot 90 = 540$):

$$\frac{7}{60} + \frac{13}{90} = \frac{7 \times 3}{180} + \frac{13 \times 2}{180} = \frac{47}{180}.$$  

Here is another example involving some variables.

**Example 3.3.3.**

$$\frac{z}{x(y + z)} + \frac{y}{z(y + z)} = \frac{z^2}{xz(y + z)} + \frac{xy}{xz(y + z)} = \frac{z^2 + xy}{xz(y + z)}.$$ 

**Try it yourself 3.3.1.** Write each of the following as a single fraction.

1. $\frac{1}{3} + \frac{2}{9}$
2. $\frac{5}{47} + \frac{1}{48}$
3. $\frac{1}{x-3} + \frac{2x-1}{x+4}$
4. $\frac{1}{x(x-1)} + \frac{1}{x(x+1)}$
5. $\frac{1}{1 + \frac{1}{x+2}}$
6. $\frac{1}{x+1} + \frac{1}{x^2-1}$

### 3.3.1 Absolute Value.

When someone asks me about the mathematical notions that cause most trouble to students taking a first year University Math course, my list always includes the notion of absolute value. For some reason, students find this notion very "unnatural". You probably won’t see any "interesting" use of this notion at the beginning, but you can rest assured that it is at the heart of many uses.

**Definition 3.3.1.** Given a real number $x$, we define the absolute value of $x$, denoted by $|x|$, as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This is rather a technical and formal definition. It could be helpful to think of the absolute value as a measure of distances on the real line: if $x$ is any real number, then $|x|$ can be thought of as the "distance"
between \( x \) and the origin \( O \) on the line. This explains why \( |x| \) is always a nonnegative. If \( x \) and \( y \) are two real numbers, then \( |x - y| \) can be thought of as the "distance" between the two numbers \( x \) and \( y \).

### Properties of the Absolute Value:

Let \( x, y \) be two real numbers.

\[
\begin{align*}
|x - y| &= |y - x| \\
\left| \frac{x}{y} \right| &= \frac{|x|}{|y|}, \text{ (for } y \neq 0) \\
|xy| &= |x||y| \\
|x^n| &= |x|^n \text{ for any integer } n \\
\sqrt[n]{x} &= \begin{cases} 
|\sqrt[n]{x}| & \text{if } n \text{ is even} \\
x & \text{if } n \text{ is odd}
\end{cases}
\end{align*}
\]

\( |x + y| \leq |x| + |y| \)

The last property is called the **Triangular inequality** which is widely used in many mathematical proofs.

Before we look at some examples, a warning.

**Warning 3.3.1.** Do not treat absolute as parentheses. The two do not behave the same way. For instance, \(-|-1| = -1 \) but \(-(-1) = 1\).

**Examples 3.3.2.**

1. \(|-1| = -(-1) \) (since \(-1 < 0\) = 1.
2. \(|2| = 2 \) (since \(2 > 0\)).
3. \(-|-1| = -1\).
4. \(-|2| = -2\).
5. \(|-| - 2| = -(-| - 2|) \) (since \(|-2| < 0\) = \(-2\) = \(-(-2) \) (since \(-2 < 0\) = 2.
6. \(|-2 - 2| = 2 - 2| = 0| = 0\).
7. The distance between \(-2\) and \(-5\) is \(| -2 - (-5)| = |3| = 3.

**Try it yourself 3.3.2.**

1. Decide if each of the following statements is true or false.
   
   \[(a) \quad | - | - 3| > 1 \quad (b) \quad -| - 3| - -2| = -1 \quad (c) \quad -(| - 2| - |3|) = -1 \quad (d) \quad (-| - 2|)^3 = -8.\]

2. If \( x = -2\), what is the value of \(|x^2 - x| - | - x^3 - x|?\)
Chapter 4

Plane Analytic Geometry

Analytic geometry (also called the coordinate or cartesian geometry) is the study of plane geometry using the properties of Algebra. This allows the use of the algebraic tools in our disposition to analyze and represent well defined geometrical.

The basic idea behind analytic geometry is to identify points in the plane using a coordinate system system formed by two perpendicular axes (the $x$-axis and the $y$-axis), meeting at a point $O$ called the origin. Every point in the plane is represented with two coordinates $(a, b)$ in a unique way. Thus, the $x$-axis can be thought of as being the set of all points with the second coordinate equals to 0 and the $y$-axis as being the set of all points with first coordinate equals to 0. The coordinate system divides the plane into four infinite regions called Quadrants and labeled using the Roman numbers I, II, III and IV:

Quadrant I is the set of all points $(x, y)$ with both coordinates positive. Quadrant II consists of all points
(x, y) with x < 0 and y > 0. Quadrant III consists of all points (x, y) with both coordinates negative.

**Example 4.0.4.** In the diagram below A is the point (−1,2), B is the point (1,3), C is the point (2,−3) and D is the point (−3,0).

4.0.2 Distance between two points

If the coordinates of the points A and B in the plane are given, then (using the Pythagorean Theorem) we can compute their distance apart.

**Example 4.0.5.** In example 4.0.4 above, the distance between the points A and B is \(\sqrt{(1−(-1))^2 + (3-2)^2} = \sqrt{5}\) and the distance between C and D is \(\sqrt{(-3-2)^2 + (0-(-3))^2} = \sqrt{34}\).

**Try it yourself 4.0.3.** (1) In each case, find the distance between the points A and B in the plane.

\(a\) A(−1,−1), B(1,2)

\(b\) A(−2,3), B(1,−2)

\(c\) A(0,−3), B(5,0).

(2) Given the two points A(−1,1) and B(0,1) in the plane, find a relation between the coordinates of the point M(x, y) in the plane which has the same distance to A and B (we say that M is equidistant to A and B).
4.1 Algebraic representation of some geometric plane figures

In this section, we look at analytic representations of plane geometric figures: lines, circles, ellipses etc... This basically means looking for an equation satisfied by the two coordinates $x$ and $y$ of an arbitrary point $M(x, y)$ on one of these figures.

4.1.1 Lines

In the plane, two things are needed to write the equation of a line: the slope $m$ and one point on the line. For a non vertical line, the slope of the line is the ratio $\frac{y_2 - y_1}{x_2 - x_1}$ where $(x_1, y_1)$ and $(x_2, y_2)$ are two points on the line with $x_1 \neq x_2$:

If the line is vertical, then all the points on the line have the same $x$-value and its equation in this case is of the form $x = a$. The slope of a vertical line is defined to be $\infty$. If, on the other hand, all the points on the line have the same $y$-value, then the line is horizontal and its slope $m = \frac{y_2 - y_1}{x_2 - x_1}$ is equal to $0$ since $y_1 = y_2$. The equation of a horizontal line is of the form $y = b$.

Knowing the slope and one point on the line, we can write the equation of the line.
Point-Slope equation of a line
If \( m \) is the slope of a line \( L \) going through the point \( A(x_0, y_0) \), then the equation of \( L \) is
\[
y - y_0 = m(x - x_0).
\]
Notice that the Point-Slope equation can be written under the form \( y = mx + b \) with \( m \) being the slope and \( b \) a constant that can be determined by the coordinates of one point on the line. If \( m > 0 \), the line is *increasing* and if \( m > 0 \), the line is *decreasing*.

In some cases, the line equation is written under the form \( ax + by + c = 0 \), called the *standard form*.

**Example 4.1.1.** The standard equation of a line \( L \) is \( 4x + 2y - 6 = 0 \). Isolating \( y \) in this equation, we get
\[
y = -2x + 3.
\]
The slope of the line is then \( m = -2 \) and setting \( x = 0 \) in the equation gives the point \((0,3)\) on the line. Another point on the line is the point \( x = 1, y = 1 \). To draw the line, all what we need is to joint the two points \((0,3)\) and \((1,1)\):

---

**Parallel lines, Perpendicular lines**
The two lines given by the equations \( y = mx + b \) and \( y = m'x + b' \) are:

- parallel if and only if \( m = m' \) (they have the same slope)
- perpendicular if and only if \( mm' = -1 \) (the slope of one is minus the reciprocal of the other)
Example 4.1.2. Consider the line $L$ going through the points $A(-1, 2)$ and $B(3, -1)$.

(1) Find the equation of the line $L_1$ going through the point $P(1, -2)$ and parallel to $L$.

(2) Find the equation of the line $L_2$ going through the point $Q(1, 0)$ and perpendicular to $L$.

Solution.

(1) Since $L_1$ is parallel to $L$, its slope is equal to that of $L$. Namely, $\frac{3-(-1)}{3-(-2)} = -\frac{4}{3}$. Knowing the slope and the point $P$ on $L_1$, we can write the Point-Slope equation of the line:

$$y - (-2) = -\frac{4}{3}(x - 1) \iff y = -\frac{4}{3}x - \frac{2}{3}.$$

(2) If $m$ is the slope of $L_2$, then we must have $-\frac{4}{3}m = -1$ since $L_2$ and $L$ are perpendicular. Solving for $m$ gives $m = \frac{3}{4}$. Now we use the Point-Slope equation:

$$y - 0 = \frac{3}{4}(x - 1) \iff y = \frac{3}{4}x - \frac{3}{4}.$$

4.1.2 Circles

Formally, A circle can be described as the set of all points in the plane at equal distant $r$, called the radius of the circle, from a fixed point $C$, called the center of the circle. Suppose that the center $C$ is the point $(a, b)$ in the plane $xOy$. If $M(x, y)$ is an arbitrary point on the circle, the Pythagorean Theorem gives that $(x - a)^2 + (y - b)^2 = r^2$. 

![Circle Diagram](image)
The equation of a circle does not always come in the clear form \((x - a)^2 + (y - b)^2 = r^2\) described above and one often has to work a bit to bring it to that form and be able to extract the coordinates of the center and the radius. The following example explains how.

**Example 4.1.3.** The equation \(x^2 + y^2 + 2x - 4y = 4\) represents a circle in the plane. To see this, we start by grouping the terms in \(x\) and the terms in \(y\) and completing the squares (see section 5.2.1.2):

\[
x^2 + y^2 + 2x - 4y = 4 \iff (x^2 + 2x + 1) - 1 + (y^2 - 4y + 4) - 4 = 4 \iff (x + 1)^2 + (y - 2)^2 = 9 = 3^2.
\]

This is the equation of a circle centered at the point \((-1, 2)\) and of radius 3.

### 4.1.3 Ellipses

If \(a, b\) are two positive real numbers, the equation

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]  

represents the standard equation of an ellipse centered at the origin in the plane. The constants \(a\) and \(b\) in equation (1) are called the ellipse's semi-axes:
The ellipse given in equation (1) intersects the x-axis at the points ±a and the y-axis at the points ±b. Note that in the case where the two semi-axes are equal, the ellipse is simply a circle.

If the center of the ellipse is the point \((x_0, y_0)\) in the plane rather than the origin, then the equation of the ellipse becomes:

\[
\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1
\]  

(2)

and the ellipse looks as follows:

Like in the case of a circle, the equation of an ellipse is not always given neatly as in (2) and one has to "twist" it a bit to bring it to that form.

**Example 4.1.4.** (1) The equation \(x^2 + 4y^2 = 4\) can be rearranged as \(\frac{x^2}{2} + \frac{y^2}{1} = 1\) by dividing the whole equation with 4. The second form shows that this is the equation of an ellipse centered at the origin with semi-axes 2 and 1. The ellipse intersects the x-axis at the points ±2 and the y-axis at the points ±1:

(2) At the first glance, the equation \(9x^2 + 4y^2 - 18x + 8y = 23\) does not look at all like that of an ellipse. As we did in Example 4.1.3 above, we start by grouping terms in \(x\) and terms in \(y\) and then we complete the
squares:

\[9x^2 + 4y^2 - 18x + 8y = 23 \iff 9 \left(x^2 - 2x + 1\right) - 1 + 4 \left(y^2 + 2y + 1\right) - 1 = 23\]

\[9(x - 1)^2 + 4(y + 1)^2 = 36\]

\[\frac{9}{36}(x - 1)^2 + \frac{4}{36}(y + 1)^2 = 1\]

\[\frac{(x - 1)^2}{4} + \frac{(y + 1)^2}{9} = 1.\]

It is clear now that this is the equation of an ellipse centered at the point (1, -1) and of semi-axes 2 and 3:

\[\text{oval graph} \]

**Remarks 4.1.1.** (1) It could very well happen that an equation of the form \(ax^2 + by^2 + cx + dy = f\) does not define any curve in the plane. For example, there is no point \((x, y)\) in the plane that satisfies the equation \(x^2 + y^2 + 2x - 2y = -4\) since after regrouping terms in \(x\) and \(y\) and completing the squares, we end up with the equation \((x + 1)^2 + (y - 1)^2 = -4\) which is not possible (positive = negative).

(2) If the equation \(ax^2 + by^2 + cx + dy = f\) is well defined in the plane for real numbers \(a, b, c\) and \(f\) with \(ab > 0\) (so \(a\) and \(b\) are both positive or both negative), then examples 4.1.3 and 4.1.4 above suggest that the equation represents either a circle or an ellipse in the plane. If \(a = b\), the equation represents a circle and if \(a \neq b\), the equation is that of an ellipse.

**Try it yourself 4.1.1.** In each case, determine if the given equation is that of a circle or an ellipse. If it is a circle, give the center and the radius and if it is an ellipse, give the center and the semi-axes. Represent the equation graphically.

1. \(2x^2 - 4x + 2y^2 - 6y = 3\)
2. \(-2x^2 + 2x - y^2 + 5y = 0\)
3. \(x^2 + y^2 + 6x - 4y = 17\)
4.1.4 Hyperbolas

If \(a, b\) are two positive real numbers, a hyperbola is a curve in the plane satisfying one or the other of the two equations

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]  

The hyperbola \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\) intersects the \(x\)-axis at the points \(x = \pm a\) (by setting \(y = 0\) in the equation, we get \(x^2 = a^2\) so \(x = \pm a\)) but there is no \(y\)-intercepts since if we put \(x = 0\) in the equation \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\), we get \(-\frac{y^2}{b^2} = 1\) which has no real roots. Similarly, the hyperbola \(-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) intersects the \(y\)-axis at the points \(y = \pm b\) and it has no \(x\)-intercepts. A hyperbola differs from a circle or an ellipse by the fact that it is not a "closed" curve in the sense that it does not inclose a bounded region. A hyperbola consists of two disconnected branches limited between the two lines \(y = -\frac{b}{a}x\) and \(y = \frac{b}{a}x\) called asymptotes.

A special type of hyperbolas is what is known as the rectangular hyperbolas. These are curves in the plane satisfying an equation of type: \((x - h)(y - k) = m\) where \(h, k\) and \(m\) are real numbers. A rectangular hyperbola has one horizontal and one vertical asymptote given by the equations \(y = k\) and \(x = h\) respectively. A particular case of a rectangular hyperbola is when \(h = k = 0\). The hyperbola in this case has an equation of the form \(xy = m\) with the two axes as asymptotes.
Example 4.1.5. The graphs of the rectangular hyperbolas \((x - 1)(y - 2) = 3\) and \(xy = -1\) are given below:

![Graphs of rectangular hyperbolas](image)

Example 4.1.6. Let us look at the equation \(9x^2 - 4y^2 - 18x - 16y = 43\) in the plane. Note first that the coefficients of \(x^2\) and \(y^2\) are of opposite signs (one is positive and the other is negative). By remark [4.1.1] above, we know that this equation cannot represent a circle nor an ellipse. Regrouping terms in \(x\) and terms in \(y\) and completing the squares like in the above examples would give the equation

\[
\frac{(x - 1)^2}{4} - \frac{(y + 2)^2}{9} = 1.
\]

To represent the equation graphically, we start by making new system of perpendicular axes at the point \((1, -2)\). With respect to the new system (drawn in blue in the below figure), the above equation is of the form \(\frac{x^2}{4} - \frac{y^2}{9} = 1\) which a hyperbola with the lines \(y = -\frac{3}{2}x\) and \(y = \frac{3}{2}x\) as asymptotes. The graph looks as follows.

![Graph of hyperbola](image)

Try it yourself 4.1.2. In each case, represent the equation graphically.

\((1)\) \(-4x^2 + y^2 + 8x - 4y = 16\) \((2)\) \(4x^2 - 9y^2 - 36y = 72\) \((3)\) \((x - 2)(y + 3) = 1\)
4.1.5 Parabolas

A parabola in the plane is the curve with equation \( y = ax^2 + bx + c \) for real numbers \( a, b \) and \( c \), called the standard equation. Like the hyperbola, a parabola is not closed curve but it differs from the hyperbola in the fact that it has no asymptotes. A more useful form of the equation of a parabola can be obtained by completing the square (see section 5.2.1.2) in the equation:

\[
y = ax^2 + bx + c = a \left( x^2 + \frac{b}{a} x \right) + c
\]

\[
= a \left( x^2 + \frac{b}{a} x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} \right) + c
\]

\[
= a \left( x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a} + c
\]

This suggests the following equation of a parabola, called the vertex equation.

The vertex equation of a parabola.

\[
y = a(x - h)^2 + k.
\]

The vertex equation gives the following information about the parabola:

1. \((h, k)\) is the vertex (maximum or minimum point) of the parabola;
2. If \(a > 0\), the parabola is open upward;
3. If \(a < 0\), the parabola is open downward;
4. The parabola is symmetric with respect to the vertical line \(x = h\);
5. Working with the standard equation \( y = ax^2 + bx + c \) of a parabola, the vertex is the point \( \left( -\frac{b}{2a}, c - \frac{b^2}{4a} \right) \) and the line of symmetry is \( x = -\frac{b}{2a} \);
6. If the parabola has \( x \)-intercepts \( x_1 \) and \( x_2 \), then the \( x \)-coordinate of the vertex point is always half way between \( x_1 \) and \( x_2 \).
Example 4.1.7. Consider the equation $y = -2x^2 - 2x + 4$. We know it represents an open downward parabola in the plane (since the coefficient of $x^2$ is negative). We start by writing the vertex equation of the parabola using the completion of the square:

$$y = -2\left(x^2 + x + \frac{1}{4}\right) + 4$$

$$= -2\left(x + \frac{1}{2}\right)^2 + \frac{9}{2}$$

The vertex of the parabola is then the point $\left(-\frac{1}{2}, \frac{9}{2}\right)$ in the plane. On the other hand, note that the $x$-intercepts of the parabola are the roots of the quadratic equation $-2x^2 - 2x + 4 = 0$:

$$x_1 = \frac{-(-2) + \sqrt{2^2 - 4(-2)(4)}}{2(-2)} = -2, \quad x_2 = \frac{-(-2) - \sqrt{2^2 - 4(-2)(4)}}{2(-2)} = 1$$

which means that the $x$-intercepts are $x = -2$ and $x = 1$. The $y$-intercept is the point $(0, 4)$. The graph of the parabola looks as follows.
**Remark 4.1.1.** Parabolas defined by the equation $y = ax^2 + bx + c$ are open "along the $y$-axis" (that is open upward or downward). But a parabola could also be open along the $x$-axis. In this case, the standard equation is of the form $x = ay^2 + by + c$ for some real numbers $a$, $b$ and $c$. If $a > 0$, the parabola is open to the right and if $a < 0$, it is open to the left.

**Example 4.1.8.** Consider the equation $y^2 = x - 6 + 5y$ in the plane. After rearranging, we get $x = y^2 - 5y + 6$ which represents a parabola open to the right on the $x$-axis. Let us look at the vertex equation of this parabola:

$$x = y^2 - 5y + 6 = \left(y^2 - 5y + \frac{25}{4} - \frac{25}{4}\right) + 6 = \left(y - \frac{5}{2}\right)^2 - \frac{1}{4}.$$ 

The vertex point of the parabola is then $\left(-\frac{1}{4}, \frac{5}{2}\right)$. On the other hand, the $y$-intercepts of the parabola are...
the roots of of \( y^2 - 5y + 6 = 0 \), namely \( y = 2 \) and \( y = 3 \). The graph of the parabola is the following.

Try it yourself 4.1.3. In each case, sketch the graph of the parabola.

(1) \( y = -3x^2 - 12x - 14 \)  (2) \( y = x^2 - x - 6 \)  (3) \( y = -x^2 + 1 \)  (4) \( x = y^2 - y - 6 \)
Chapter 5

Polynomials and Factoring

Polynomials are probably the "nicest" mathematical object you get to work with in your courses. Let us start by recalling the basic definitions and terminologies.

A polynomial in the variable \( x \) is an expression of the form

\[
P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0
\]

where \( a_n, a_{n-1}, \ldots, a_1, a_0 \) are real numbers called the coefficients of the polynomial. Each of the expressions \( a_i x^i \) \((i = 0, \ldots, n)\) is called a term of the polynomial. If \( a_n \neq 0 \), we say that the polynomial \( P(x) \) is of degree \( n \), and we write \( \deg P(x) = n \). The term \( a_n x^n \) is called the leading term of the polynomial. If each of the coefficients is zero, the polynomial is called the zero polynomial. The degree of the zero polynomial is defined to be \(-\infty\). Note that one can think of any nonzero real number as being a polynomial of degree 0: if \( a \in \mathbb{R} \), then \( a = a x^0 \) since \( x^0 = 1 \). Similarly, one can also talk about polynomials in two or more variables. For example, \( 2x^2 y^3 - 4x^4 + xy^5 \) is a polynomial in the two variables \( x \) and \( y \).

A real number \( a \) is called a root (or a zero) of a polynomial \( P(x) \) if \( P(a) = 0 \). The expression \( P(a) = 0 \) means that we get 0 when we replace \( x \) everywhere with the value \( a \) in \( P(x) \). For example, \( a = -1 \) is a root of the polynomial \( P(x) = x^4 - x^3 - 4x^2 - 3x - 1 \) since \( P(-1) = (-1)^4 - (-1)^3 - 4(-1)^2 - 3(-1) - 1 = 0 \).
5.1 Operations on polynomials

Since every real number can be seen as a polynomial, operations like addition, subtraction, multiplication and division on real numbers can be extended to operations that can be performed on polynomials. Adding or subtracting two polynomials is done by combining and adding the "like terms". In the case of polynomials in one variable $x$, the "like terms" are terms of the form $ax^n$ and $bx^n$ (same $n$) called **monomials**. In the case of polynomials in two variables $x$ and $y$, the "like terms" are terms of the form $ax^n y^m$ and $bx^n y^m$ also called monomials. The same is true for 3 or more variables. Multiplying two polynomials is done by multiplying every term of one of them by every term of the other and then combining using the distributivity laws. Dividing two polynomials (called **long division**), on the other hand, requires a bit more effort and it is important to get a good handel on it in order to successfully be able to factor a polynomial. Let us start with some basic examples, we will review long division after.

**Examples 5.1.1.** Consider the two polynomials in one variable

$$P(x) = -2x^3 + 3x^2 - 11x + 6, \quad Q(x) = -3x^2 + x - 1.$$  

Then:

- $P(x) + Q(x) = (-2x^3 + 3x^2 - 11x + 6) + (-3x^2 + x - 1) = -2x^3 + 3x^2 - 10x + 5$
- $P(x) - Q(x) = (-2x^3 + 3x^2 - 11x + 6) - (-3x^2 + x - 1) = -2x^3 + 3x^2 - 11x + 6 + 3x^2 - x + 1 = -2x^3 + 6x^2 - 12x + 7$
- $P(x)Q(x) = (-2x^3 + 3x^2 - 11x + 6)(-3x^2 + x - 1) = (-2x^3)(-3x^2) + (-2x^3)(x) + (-2x^3)(-1) + (3x^2)(-2x^3) + (3x^2)(x) + (3x^2)(-1) + (-11x)(-3x^2) + (-11x)(x) + (-11x)(-1) + 6(-3x^2) + 6(x) + 6(-1) = 6x^6 - 2x^4 + 2x^3 - 9x^4 + 3x^3 - 3x^2 + 33x^3 - 11x^2 + 11x - 18x^2 + 6x - 6 = 6x^6 - 11x^4 + 38x^3 - 32x^2 + 17x - 6$ after regrouping the "like terms".
- $2P(x) - 3Q(x) = 2(-2x^3 + 3x^2 - 11x + 6) - 3(-3x^2 + x - 1) = -4x^3 + 6x^2 - 22x + 12 + 9x^2 - 3x + 3 = -4x^3 + 15x^2 - 25x + 15.$

**Examples 5.1.2.** In this example, we consider polynomials in three variables $P(x, y, z) = -2x^3 yz^2 + y^2 z - xz^3 + 1$ and $Q(x, y, z) = -y^2 x + x + y$. Then

- $P(x, y, z) + Q(x, y, z) = (-2x^3 yz^2 + y^2 z - xz^3 + 1) + (-y^2 x + x + y) = -2x^3 yz^2 - xz^3 + x + y + 1$
- $P(x, y, z) - Q(x, y, z) = (-2x^3 yz^2 + y^2 z - xz^3 + 1) - (-y^2 x + x + y) = -2x^3 yz^2 - xz^3 + 2y^2 z - x - y + 1$
5.1.1 Long Divisions of polynomials

Given two polynomials $P(x)$ and $D(x)$ with $D(x) \neq 0$, the main goal of long division of polynomials is to be able to write an expression of the form $\frac{P(x)}{D(x)}$ under the form

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

where $R(x)$ is either the zero polynomial or $R(x) \neq 0$ and $\deg R(x) < \deg D(x)$. Multiplying (1) with $D(x)$ gives

$$P(x) = D(x)Q(x) + R(x).$$

The polynomial $Q(x)$ is called the **quotient** and $R(x)$ is called the **remainder**. If the remainder is zero, then (2) gives a factorization of the polynomial $P(x)$.

In your elementary years, you learnt how to divide two numbers using a table. Here is how you were shown to divide, say 123 by 7:

\[
\begin{array}{c|c}
7 & 123 \\
\hline
7 & 70 \\
53 & 49 \\
4 & 4 \\
\end{array}
\]

In this process, 123 is called the dividend, 7 the divisor, 17 the quotient and 4 the remainder. Note that 123 can be written as $(7 \times 17) + 4$ (so Dividend=Quotient $\times$ Divisor+Remainder). Since every real number can be thought of as a polynomial, it is only natural that this notion of long division on the reals can be extended to polynomials. Let us review this with an example.

Suppose we want to divide $3x^5 + 2x^4 + x^3$ by $x^3 - 1$. This is done in steps.

- **Step 1.** We start by writing the expression $\frac{3x^5 + 2x^4 + x^3}{x^3 - 1}$ in the long division setting with $x^3 - 1$ on the left and $3x^5 + 2x^4 + x^3$ on the right:

\[
x^3 - 1) 3x^5 + 2x^4 + x^3
\]

- **Step 2.** Divide the leading term of the dividend (numerator polynomial) by the leading term of the
divisor (denominator polynomial): \( \frac{4x^5}{x^2} = 3x^2 \). Write \( 3x^2 \) in the top line above the dividend:

\[
\begin{array}{c|cc}
3x^2 \\
\hline
x^3 - 1 \) & 3x^5 + 2x^4 + x^3 \\
\end{array}
\]

**Step 3.** Now multiply \( 3x^2 \) with the divisor: \( 3x^2(x^3 - 1) = 3x^5 - 3x^2 \). Note that the leading term in \( 3x^5 - 3x^2 \) matches the leading term in the dividend.

**Step 4.** Change the signs in \( 3x^5 - 3x^2 \) to get \( -3x^5 + 3x^2 \) and write your answer under the dividend while lining the terms of the same degree.

\[
\begin{array}{c|cc}
3x^2 \\
\hline
x^3 - 1 \) & 3x^5 + 2x^4 + x^3 \\
\end{array}
\]

\[
-3x^5 + 3x^2 \\
\]

**Step 5.** Add the last line to the line above it (this amounts to subtracting \( 3x^5 - 3x^2 \) from the line above it). This step is designed to eliminate the terms in \( x^5 \). The result is \( 2x^4 + x^3 + 3x^2 \).

\[
\begin{array}{c|cc}
3x^2 \\
\hline
x^3 - 1 \) & 3x^5 + 2x^4 + x^3 \\
\end{array}
\]

\[
-3x^5 + 3x^2 \\
\]

\[
2x^4 + x^3 + 3x^2 \\
\]

**Step 6.** Now repeat Step 2 with \( 2x^4 + x^3 + 3x^2 \) as the new dividend and with the same divisor \( x^3 - 1 \): \( \frac{2x^4}{x^2} = 2x \) and add this term to the term \( 3x^2 \) already on the top line.

\[
\begin{array}{c|cc}
3x^2 + 2x \\
\hline
x^3 - 1 \) & 3x^5 + 2x^4 + x^3 \\
\end{array}
\]

\[
-3x^5 + 3x^2 \\
\]

\[
2x^4 + x^3 + 3x^2 \\
\]

**Step 7.** Now repeat Step 3: multiply \( x - 1 \) with \( 2x \): \( 2x(x - 1) = 2x^2 - 2x \).

**Step 8.** Repeat Step 4: change the signs in \( 2x^2 - 2x \) to get \( -2x^2 + 2x \) and write this under the line of the "new dividend" \( 2x^4 + x^3 + 3x^2 \):

\[
\begin{array}{c|cc}
3x^2 + 2x \\
\hline
x^3 - 1 \) & 3x^5 + 2x^4 + x^3 \\
\end{array}
\]

\[
-3x^5 + 3x^2 \\
\]

\[
2x^4 + x^3 + 3x^2 \\
\]

\[
-2x^4 + x^3 + 3x^2 \\
\]

\[
-2x^4 + 2x \\
\]

50
• **Step 9.** Repeat Step 5: add the last line to the line above it. The result is \( x^3 + 3x^2 + 2x \).

\[
\begin{array}{c}
\text{Step 9. Repeat Step 5: add the last line to the line above it. The result is } \quad x^3 + 3x^2 + 2x.
\end{array}
\]

\[
\begin{array}{c}
x^3 - 1 \quad 3x^2 + 2x
\end{array}
\]

\[
\begin{array}{c}
\text{Step 9. Repeat Step 5: add the last line to the line above it. The result is } \quad x^3 + 3x^2 + 2x.
\end{array}
\]

\[
\begin{array}{c}
x^3 - 1 \quad 3x^5 + 2x^4 + x^3
\end{array}
\]

\[
\begin{array}{c}
-3x^5 \
+3x^2 \\
2x^4 + x^3 + 3x^2 \\
-2x^4 \
+2x \\
\hline
x^3 + 3x^2 + 2x
\end{array}
\]

• **Step 10.** Back to step 2 above with \( x^3 + 3x^2 + 2x \) as the new dividend and with the same divisor \( x^3 - 1 \): \( \frac{x^3}{x^2} = 1 \) and add this term to the term \( 3x^2 + 2x \) already on the top line.

\[
\begin{array}{c}
\text{Step 10. Back to step 2 above with } \quad x^3 + 3x^2 + 2x \text{ as the new dividend and with the same divisor } \quad x^3 - 1 \text{: } \frac{x^3}{x^2} = 1 \text{ and add this term to the term } \quad 3x^2 + 2x \text{ already on the top line.}
\end{array}
\]

\[
\begin{array}{c}
x^3 - 1 \quad 3x^2 + 2x + 1
\end{array}
\]

\[
\begin{array}{c}
\text{Step 10. Back to step 2 above with } \quad x^3 + 3x^2 + 2x \text{ as the new dividend and with the same divisor } \quad x^3 - 1 \text{: } \frac{x^3}{x^2} = 1 \text{ and add this term to the term } \quad 3x^2 + 2x \text{ already on the top line.}
\end{array}
\]

\[
\begin{array}{c}
\text{Step 10. Back to step 2 above with } \quad x^3 + 3x^2 + 2x \text{ as the new dividend and with the same divisor } \quad x^3 - 1 \text{: } \frac{x^3}{x^2} = 1 \text{ and add this term to the term } \quad 3x^2 + 2x \text{ already on the top line.}
\end{array}
\]

\[
\begin{array}{c}
x^3 - 1 \quad 3x^5 + 2x^4 + x^3
\end{array}
\]

\[
\begin{array}{c}
-3x^5 \
+3x^2 \\
2x^4 + x^3 + 3x^2 \\
-2x^4 \
+2x \\
\hline
x^3 + 3x^2 + 2x
\end{array}
\]

• **Step 11.** Multiply \( x^3 - 1 \) with 1: \( 1.(x^3 - 1) = x^3 - 1 \).

\[
\begin{array}{c}
\text{Step 11. Multiply } \quad x^3 - 1 \text{ with 1: } \quad 1.(x^3 - 1) = x^3 - 1.
\end{array}
\]

• **Step 12.** Change the signs in \( x^3 - 1 \) to get \( -x^3 + 1 \) and write this under the line of the "new dividend" \( x^3 + 3x^2 + 2x \):

\[
\begin{array}{c}
\text{Step 12. Change the signs in } \quad x^3 - 1 \text{ to get } \quad -x^3 + 1 \text{ and write this under the line of the "new dividend" } \quad x^3 + 3x^2 + 2x:
\end{array}
\]

\[
\begin{array}{c}
x^3 - 1 \quad 3x^2 + 2x + 1
\end{array}
\]

\[
\begin{array}{c}
\text{Step 12. Change the signs in } \quad x^3 - 1 \text{ to get } \quad -x^3 + 1 \text{ and write this under the line of the "new dividend" } \quad x^3 + 3x^2 + 2x:
\end{array}
\]

\[
\begin{array}{c}
\text{Step 12. Change the signs in } \quad x^3 - 1 \text{ to get } \quad -x^3 + 1 \text{ and write this under the line of the "new dividend" } \quad x^3 + 3x^2 + 2x:
\end{array}
\]

\[
\begin{array}{c}
\text{Step 12. Change the signs in } \quad x^3 - 1 \text{ to get } \quad -x^3 + 1 \text{ and write this under the line of the "new dividend" } \quad x^3 + 3x^2 + 2x:
\end{array}
\]

\[
\begin{array}{c}
\text{Step 12. Change the signs in } \quad x^3 - 1 \text{ to get } \quad -x^3 + 1 \text{ and write this under the line of the "new dividend" } \quad x^3 + 3x^2 + 2x:
\end{array}
\]

\[
\begin{array}{c}
\text{Step 12. Change the signs in } \quad x^3 - 1 \text{ to get } \quad -x^3 + 1 \text{ and write this under the line of the "new dividend" } \quad x^3 + 3x^2 + 2x:
\end{array}
\]

\[
\begin{array}{c}
\text{Step 12. Change the signs in } \quad x^3 - 1 \text{ to get } \quad -x^3 + 1 \text{ and write this under the line of the "new dividend" } \quad x^3 + 3x^2 + 2x:
\end{array}
\]

\[
\begin{array}{c}
\text{Step 12. Change the signs in } \quad x^3 - 1 \text{ to get } \quad -x^3 + 1 \text{ and write this under the line of the "new dividend" } \quad x^3 + 3x^2 + 2x:
\end{array}
\]

\[
\begin{array}{c}
\text{Step 12. Change the signs in } \quad x^3 - 1 \text{ to get } \quad -x^3 + 1 \text{ and write this under the line of the "new dividend" } \quad x^3 + 3x^2 + 2x:
\end{array}
\]

51
• **Step 13.** Add the last line to the line above it. The result is $3x^2 + 2x + 1$.

```
\[
\begin{array}{r}
\phantom{-}3x^2 + 2x + 1 \\
\hline
x^3 - 1) \ 3x^5 + 2x^4 + x^3 \\
\hphantom{x^3 - 1) } -3x^5 \hspace{1cm} +3x^2 \\
\hline
2x^4 + x^3 + 3x^2 \\
\hphantom{x^3 - 1) } -2x^4 \hspace{1cm} +2x \\
\hline
x^3 + 3x^2 + 2x \\
\hphantom{x^3 - 1) } -x^3 \hspace{1cm} + 1 \\
\hline
3x^2 + 2x + 1 \\
\end{array}
\]
```

• **Step 14. Stop.** The degree of $3x^2 + 2x + 1$ is strictly less than the degree of the divisor $x^3 - 1$. This means that $3x^2 + 2x + 1$ is the remainder of the division and $3x^2 + 2x + 1$ is the quotient.

The above division allows us to write $3x^5 + 2x^4 + x^3 = (x^3 - 1)(3x^2 + 2x + 1) + 3x^2 + 2x + 1$.

Here is another example where we omit the steps of the long division.

**Example 5.1.1.** Suppose you are asked to write the polynomial $3x^4 - 5x^2 + x$ under the form

\[(x^2 - x + 1)Q(x) + R(x)\]

for some polynomials $Q(x)$ and $R(x)$ with $\deg R(x) < 2$. You start by performing the long division of $3x^4 - 5x^2 + x$ by $x^2 - x + 1$:

```
\[
\begin{array}{r}
\phantom{-}3x^2 + 3x - 5 \\
\hline
x^2 - x + 1) \ 3x^4 - 5x^2 + x \\
\hphantom{x^2 - x + 1) } -3x^4 + 3x^3 - 3x^2 \\
\hline
3x^3 - 8x^2 + x \\
\hphantom{x^2 - x + 1) } -3x^3 + 3x^2 - 3x \\
\hline
-5x^2 - 2x \\
\hphantom{x^2 - x + 1) } -5x^2 + 5 \\
\hline
-7x + 5 \\
\end{array}
\]
```

So, the quotient of this division is $3x^2 - 3x - 5$ and the remainder is $-7x + 5$. Therefore,

\[3x^4 - 5x^2 + x = (x^2 - x + 1)(3x^2 - 3x - 5) - 7x + 5.\]

**Try it yourself 5.1.1.** In each case, perform the long division $\frac{P(x)}{D(x)}$. Write $P(x)$ in the form $P(x) = Q(x)D(x) + R(x)$ for some polynomials $Q(x)$ and $R(x)$ with $\deg R(x) < \deg D(x)$.
1. \( P(x) = x^2 - 5x + 9, D(x) = 2x + 3 \)

2. \( P(x) = x^4 - 5x^3 + 9, D(x) = -x^2 + 3x \)

3. \( P(x) = 3 = x^3 + 4x^2 + x - 1, D(x) = x^2 + 1 \)

### 5.2 Factoring

Writing the number 105 in the form 105 = 3 × 5 × 7 is called factoring this number. In general, factoring an expression (a polynomial, or a function in general) means writing the expression as a product of other terms called factors, which when multiplied together give the original expression. For example, the polynomial \( 4x^4 + 6x^3 + 12x^2 \) can be factored as \( 2x^2(2x^2 + 3x + 6) \).

In some cases (if you are lucky enough), factoring polynomials can be done by regrouping. The idea is to first rewrite the polynomial in groups where each group can be separately factored and the resulting groups having a common factor. Here are two examples.

**Examples 5.2.1.**

1. 
   \[
   ab + xb - ax - x^2 = b(a + x) - x(a + x) = (a + x)(b - x)
   \]

2. 
   \[
   8x^2 - 4xy - 6xz + 3yz + 2xy^2 - y^3 = 4x(2x - y) - 3z(2x - y) + y^2(2x - y) = (2x - y)(4x - 3z + y^2)
   \]

The following particular expansions/factoring are very useful and pop up in many situations. Although they are simple to prove, memorizing them could save you a great deal of time.
Useful Formulas.

In the following, $a$ and $b$ are two real numbers.

\[(a + b)^2 = a^2 + 2ab + b^2\]
\[(a - b)^2 = a^2 - 2ab + b^2\]
\[(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3\]
\[(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\]
\[a^2 - b^2 = (a - b)(a + b)\text{ (Difference of two squares)}\]
\[a^3 - b^3 = (a - b)(a^2 + ab + b^2)\text{ (Difference of two cubes)}\]

Remarks 5.2.1.

1. Unlike the difference of two squares, the sum of two squares $a^2 + b^2$ cannot be factored or simplified any further (well, it can if we are using complex numbers but we are working only with real numbers). The same is true for $a^4 + b^4$, $a^6 + b^6$ and $a^n + b^n$ in general if $n$ is an even integer.

2. If $n$ is an odd integer, one can factor $a^n + b^n$. A useful relation is the following:

\[a^3 + b^3 = (a + b)(a^2 - ab + b^2).\]

3. In the difference and the sum of two cubes, the expressions $(a^2 + ab + b^2)$ and $(a^2 - ab + b^2)$ cannot be simplified any further (over the real numbers).

Examples 5.2.2.

1. $25 - 4x^2 = 5^2 - (2x)^2 = (5 - 2x)(5 + 2x)$ (difference of two squares)

2. $27x^3 - 8 = (3x)^3 - 2^3 = (3x - 2)((3x)^2 + 2(3x) + 2^2) = (3x - 2)(9x^2 + 6x + 4)$ (difference of two cubes)

3. $(x^3 - y)^3 = (x^3)^3 - 3(x^3)^2y + 3x^3y^2 - y^3 = x^9 - 3x^6y + 3x^3y^2 - y^3$

4. $(3 - x^2)^2 = 3^2 - 2(3)(x^2) + (x^2)^2 = 9 - 6x^2 + x^4$

Here are some examples where the above formulas can be very handy within factoring by regrouping.

Examples 5.2.3. (1)

\[x^3 - y^2 + x^2 - y^3 = (x^3 - y^3) + (x^2 - y^2)\]
\[= (x - y)(x^2 + xy + y^2) + (x - y)(x + y)\]
\[= (x - y)(x^2 + xy + y^2 + x - y).\]
\[
x^4 - 2x^3 - x + 2 = (x^4 - 2x^3) - (x - 2)
= x^3(x - 2) - (x - 2)
= (x - 2)(x^3 - 1)
= (x - 2)(x - 1)(x^2 + x + 1).
\]

5.2.1 Factoring a quadratic polynomial and completing the square

5.2.1.1 Roots and factoring of a quadratic

A quadratic polynomial is a polynomial of degree 2 which we write under the form \( ax^2 + bx + c \) with \( a \neq 0 \). The expression \( \Delta = b^2 - 4ac \) is called the discriminant of the polynomial. It gives information about the existence and nature of the polynomial’s roots. If \( b^2 - 4ac \geq 0 \), the quadratic \( ax^2 + bx + c \) will have two real roots \( x_1 \) and \( x_2 \) given by the quadratic formula:

\[
x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.
\]

In this case, the quadratic can be factored as \( a(x - x_1)(x - x_2) \). Note that if \( b^2 - 4ac = 0 \), then \( x_1 = x_2 = -\frac{b}{2a} \) and the quadratic has one (double) real root and it can be factored in this case as \( a(x - x_1)^2 \). If \( b^2 - 4ac < 0 \), the quadratic will have no (real) roots and therefore it cannot be factored (over the real numbers).

**Examples 5.2.4.** (1) For the quadratic polynomial \(-2x^2 + 12x - 10\), the discriminant is \( b^2 - 4ac = (12)^2 - 4(-2)(-10) = 144 - 80 = 64 > 0 \). The polynomial has two distinct real roots \( x_1 = \frac{-12 + \sqrt{64}}{2} = \frac{-12 + 8}{2} = 1 \) and \( x_2 = \frac{-12 - \sqrt{64}}{2} = \frac{-12 + 8}{2} = 5 \). The quadratic can now be factored as \(-2(x - 1)(x - 5)\).

(2) For the polynomial \(-3x^2 + 6x - 3\), \( b^2 - 4ac = (6)^2 - 4(-3)(-3) = 36 - 36 = 0 \). The polynomial has one (double) real root \( x_1 = x_2 = \frac{6}{6} = 1 \) and the polynomial can now be factored as \(-3(x - 1)^2\).

(3) For the polynomial \(-3x^2 + 6x - 4\), \( b^2 - 4ac = (6)^2 - 4(-3)(-4) = 36 - 48 = -12 < 0 \). The polynomial has no real roots and it cannot be factored over the reals.

In some cases, a polynomial of degree 4 or more can made quadratic after a change of variables.

**Example 5.2.1.** Consider the polynomial \( x^4 - 3x^2 + 2 \) of degree 4. If we let \( t = x^2 \), the polynomial can be written as the quadratic \( t^2 - 3t + 2 \) which in turn can be factored as \((t - 1)(t - 2)\) (as above). Replacing \( t \) back with \( x^2 \) in the second form gives the following complete factorization of \( x^4 - 3x^2 + 2 \):

\[
x^4 - 3x^2 + 2 = (t - 1)(t - 2) = (x^2 - 1)(x^2 - 2) = (x - 1)(x + 1)\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right).
\]
Try it yourself 5.2.1. When possible, factor each of the following quadratic polynomials.

1. \(2x^2 - 3x + 1\)
2. \(x^4 - 4x^2 + 3\)
3. \(2x^2 - 3x + 5\)
4. \(x^2 - 8x + 16\)
5. \(x^6 - 8x^3 + 7\)

### 5.2.1.2 Completing the square

In general, completing the square in a quadratic consists of converting the quadratic \(ax^2 + bx + c\) into the form

\[a(x + \alpha)^2 + \beta\]

for some real numbers \(\alpha\) and \(\beta\) that we must look for. We explain this with an example.

**Example 5.2.2.** Consider the quadratic polynomial \(x^2 + 4x + 5\). The two terms containing the variable \(x\), namely \(x^2 + 4x\), can be thought of as beginning of a perfect square of the form \((x + \alpha)^2\). To find the value of \(\alpha\), note that \((x + \alpha)^2 = x^2 + 2\alpha x + \alpha^2\). Comparing with \(x^2 + 4x\), we get that \(2\alpha = 4\) and therefore \(\alpha = 2\). So the missing term in \(x^2 + 4x\) to complete the square is \(2^2 = 4\). Now add and subtract 4 in \(x^2 + 4x + 5\):

\[x^2 + 4x + 5 = (x^2 + 4x + 4 - 4) + 5 = (x + 2)^2 + 1.\]

In general, to complete the square in an expression of the form \(ax^2 + bx\):

1. we start by factoring out the coefficient \(a\) of \(x^2\):

\[ax^2 + bx = a\left(x^2 + \frac{b}{a}x\right).\]

2. Divide the coefficient of \(x\) in the expression \(x^2 + \frac{b}{a}x\) by 2 and get \(\frac{b}{2a}\).

3. The missing term to complete the square in \(x^2 + \frac{b}{a}x\) is then \(\left(\frac{b}{2a}\right)^2\).

4. Add and subtract \(\left(\frac{b}{2a}\right)^2\) to \(x^2 + \frac{b}{a}x + \frac{c}{a}\).
Let us work out another example.

**Example 5.2.3.** Suppose we want to complete the square in the quadratic $-2x^2 + 3x - 7$. First we factor out the constant $-2$ in $-2x^2 + 3x$:

$$-2x^2 + 3x - 7 = -2 \left( x^2 - \frac{3}{2}x \right) - 7.$$

Next we divide the coefficient of $x$ in $(x^2 - \frac{3}{2}x)$ with 2: $\frac{-3}{2} = -\frac{3}{4}$. We then take the square of the result: $\left( -\frac{3}{4} \right)^2 = \frac{9}{16}$. This is the missing term to complete the square. We add and subtract this number to $x^2 - \frac{3}{2}x$:

$$-2x^2 + 3x - 7 = -2 \left( x^2 - \frac{3}{2}x - \frac{9}{16} \right) - 7 + \frac{9}{16} = -2 \left( x - \frac{3}{4} \right)^2 - \frac{47}{8}.$$

**Try it yourself 5.2.2.** Complete the square in each of the following quadratic polynomials.

1. $x^2 - 3x + 3$
2. $-3x^2 - 4x - 2$
3. $2x^2 - 2x + 1$

### 5.2.2 Factoring a cubic and beyond

To factor a polynomial of degree 3 or more, the first attempt should be to check quickly if this can be done by regrouping (see above). If regrouping does not lead to a useful factorization, then one should try to "guess" one rational root (i.e., a root of the form $x = \frac{a}{b}$ where $a$ and $b$ are integers) by plugging values for $x$ and then use long division to find the other roots. In general, guessing a root of a polynomial can be very hard, but math problems that require factoring a polynomial or solving polynomial equations are often designed so that you don't usually have to guess very hard since there will be a limited choices for a possible (rational) roots. The following Theorem gives a finite set of values that a polynomial could have as rational roots (if a rational root exists).

**Theorem 5.2.1.** *(The rational root test)* Given a polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ with integer coefficients (each $a_i$ is an integer), then if $x = \frac{a}{b}$ is a rational root of the polynomial, we must have:
• \( a \) is a divisor of the constant coefficient \( a_0 \) of the polynomial

• \( b \) is a divisor of the leading coefficient \( a_n \) of the polynomial

**Remark 5.2.1.** It is important to understand that the rational root test does not guarantee the existence of a rational root. All what it says is that if there is such a root, then it should have the required form described above. It could very well happen that none of the rational numbers \( \frac{a}{b} \) with \( a \) and \( b \) as described above is a root of the polynomial.

Before we look at an example, let us look on how finding one root helps us to find the others. Suppose we found a rational root \( \lambda \) of a polynomial \( P(x) \), then after doing the long division of \( P(x) \) with \( (x - \lambda) \), we can write \( P(x) = (x - \lambda)Q(x) + R(x) \) where \( Q(x) \) and \( R(x) \) are the quotient and the remainder of the division respectively. Since the degree of the remainder is always strictly less than the degree of the divisor, the degree of \( R(x) \) is zero (since the degree of \( (x - \lambda) \) is one) and so \( R(x) \) is just a constant \( C \) and we have:

\[
P(x) = (x - \lambda)Q(x) + C.
\]

Since \( P(\lambda) = 0 \), replacing \( x \) with \( \lambda \) in (1) gives \( 0 = (\lambda - \lambda)Q(\lambda) + C \). In other words \( C = 0 \). Equation (1) becomes \( P(x) = (x - \lambda)Q(x) \) and that proves that \( (x - \lambda) \) is a factor of \( P(x) \). This provides a proof of a well known fact called the **Factor Theorem**.

**Theorem 5.2.2. (Factor Theorem)** \( x = \lambda \) is a root of the polynomial \( P(x) \) if and only if \( (x - \lambda) \) is a factor of \( P(x) \).

**Remark 5.2.2.** If \( \frac{a}{b} \) is a rational root of the polynomial \( P(x) \), then \( P(x) = \left(x - \frac{a}{b}\right)Q(x) \) for some polynomial \( Q(x) \). Note that

\[
P(x) = \left(x - \frac{a}{b}\right)Q(x) \\
= \left(\frac{bx - a}{b}\right)Q(x) \\
= (bx - a)R(x)
\]

where \( R(x) = \frac{1}{b}Q(x) \). This shows that \( (bx - a) \) is a factor of \( P(x) \).

Let us now make a summary on how to factor a polynomial.
Factoring of Polynomials with integer coefficients.
The following steps are used to factor a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ with $a_n, \ldots, a_1, a_0 \in \mathbb{Z}$.

**Step 1.** "Guess" a rational root $\lambda$ of $P(x)$. Such a root must be of the form $\lambda = \frac{a}{b}$ where $a$ is a divisor of $a_0$ and $b$ is a divisor of $a_n$.

**Step 2.** Using polynomial long division, divide $P(x)$ with $(b x - a)$. By the Factor Theorem and Remark 5.2.2 above, we know that $(b x - a)$ is a factor of $P(x)$, so the remainder of the division must be zero and $P(x)$ can be factored as $(b x - a) Q(x)$ for some polynomial $Q(x)$.

**Step 3.** Repeat Steps 1 and 2 for the polynomial $Q(x)$. The degree of the polynomial $Q(x)$ found in Step 2 is one less the degree of $P(x)$ and any zero of $Q(x)$ is at the same time a zero for $P(x)$ since $P(x) = (b x - a) Q(x)$.

**Step 4.** Stop when the polynomial $Q(x)$ is of degree zero or one or when $Q(x)$ can no longer be factored.

Let us see how this works within an example.

**Example 5.2.4.** Suppose we want to completely factor the polynomial $P(x) = x^4 + 5x^3 + 5x^2 - 5x - 6$ and to find its roots. By the rational root test, a rational root (if there is one) of $P(x)$ should be of the form $\frac{a}{b}$ where $a$ is a divisor of $-6$ and $b$ is a divisor of 1. Since the only (integer) divisors of 1 are $\pm 1$, the set of possible rational roots of $P(x)$ is $\{\pm 1, \pm 2, \pm 3, \pm 6\}$. Since $P(-1) = (-1)^4 + 5(-1)^3 + 5(-1)^2 - 5(-1) - 6 = 0$, $x = -1$ is a root. By the Factor Theorem, $(x - (-1)) = (x + 1)$ is a factor of $P(x)$. Next we do the long division of $P(x)$ with $x + 1$.

\[
\begin{array}{c|ccccc}
  & x^3 + 4x^2 + x - 6 \\
\hline
x + 1 & x^4 + 5x^3 + 5x^2 - 5x - 6 \\
& - x^4 - x^3 \\
\hline
& 4x^3 + 5x^2 \\
& - 4x^3 - 4x^2 \\
\hline
& x^2 - 5x \\
& - x^2 - x \\
\hline
& - 6x - 6 \\
& 6x + 6 \\
\hline
& 0
\end{array}
\]

As expected, the remainder is zero and $P(x)$ can be factored as

\[
P(x) = (x + 1) Q(x)
\]

We now repeat the same steps for the polynomial $Q(x) = x^3 + 4x^2 + x - 6$. As before, the possible rational roots (if any) belong to the set $\{\pm 1, \pm 2, \pm 3, \pm 6\}$. Since $Q(1) = (1)^3 + 4(1)^2 + (1) - 6 = 0$, $x = 1$ is a root of $Q(x)$.
and by the Factor Theorem, \((x - 1)\) is a factor of \(Q(x)\). We now divide \(Q(x)\) with \(x - 1\):

\[
\begin{array}{c|ccccc}
& x^2 + 5x + 6 \\
- \quad (x - 1) & x^3 + 4x^2 + x - 6 \\
\hline
& - x^3 + x^2 \\
& 5x^2 + x \\
& - 5x^2 + 5x \\
& 6x - 6 \\
& - 6x + 6 \\
\hline
& 0
\end{array}
\]

and \(Q(x)\) can be factored as

\[Q(x) = (x - 1)(x^2 + 5x + 6)\] \hspace{1cm} (3)

Combining (6) and (3), we get

\[P(x) = (x + 1)(x - 1)(x^2 + 5x + 6)\] \hspace{1cm} (4)

At this point, the new quotient \(x^2 + 5x + 6\) is quadratic so we have the choice between continuing on guessing rational roots of \(x^2 + 5x + 6\) or to simply find the roots by the quadratic formula:

\[x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-5 + \sqrt{25 - 4(1)(6)}}{2(1)} = -2, \quad x_2 = \frac{-5 - \sqrt{25 - 4(1)(6)}}{2(1)} = -3\]

and the quadratic \(x^2 + 5x + 6\) can now be factored as follows:

\[x^2 + 5x + 6 = (x + 2)(x + 3)\] \hspace{1cm} (5)

We get a complete factorization of \(P(x)\) when we combine (6), (3), (4) and (5):

\[P(x) = (x + 1)(x - 1)(x + 2)(x + 3).\] \hspace{1cm} (6)

The roots of \(P(x)\) are \(-3, -2, -1\) and 1.

**Example 5.2.5.** Let us next look at the factorization and the roots of the polynomial

\[P(X) = 2x^3 + 7x^2 + 10x + 6.\]

Since the divisors of 6 are \(\pm 1, \pm 2, \pm 3, \pm 6\) and the divisors of 2 are \(\pm 1, \pm 2\), a possible rational root must be chosen from the set

\[
\left\{ \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm 3 \right\}.
\]
By inspection, we find that $P\left(-\frac{3}{2}\right) = 0$. Remark 5.2.2 implies that $2x + 3$ is a factor of $P(x)$. Divide $P(x)$ with $2x + 3$:

\[
\begin{array}{c|cc}
& x^2 + 2x + 2. \\
\hline
2x + 3 & 2x^3 + 7x^2 + 10x + 6 \\
& -2x^3 - 3x^2 \\
\hline
& 4x^2 + 10x \\
& -4x^2 - 6x \\
\hline
& 4x + 6 \\
& -4x - 6 \\
\hline
& 0
\end{array}
\]

So $P(x) = (x^2 + 2x + 2)(2x + 3)$. For the quadratic $x^2 + 2x + 2$, $b^2 - 4ac = 2^2 - 4(1)(2) = -4 < 0$, which means that the quadratic has no real roots and therefore it cannot be factored any further over the reals. The only real root of $P(x)$ is then $x = -\frac{3}{2}$.

**Try it yourself 5.2.3.** Factor completely each of the following polynomials

1. $2x^3 - 5x^2 - 9x + 18$
2. $x^4 - 4x^3 + 5x^2 - 8x + 6$
3. $2x^5 - 2x^4 - 7x^3 + 7x^2 - 4x + 4$
Chapter 6

Solving equations

Solving an equation involving the variable $x$ means finding all possible values of $x$ that satisfy the equation. Equations can take different forms and the techniques used in solving them vary with the nature of the equation (polynomial, exponential, logarithmic, ...). In this chapter we will only deal with polynomial equations and equations involving absolute values. Once exponential, logarithmic and trigonometric functions are reviewed, we will be able to deal with equations involving these functions.

6.1 Polynomial equations

Generally speaking, these are probably the easiest to deal with. Linear, quadratic and cubic equations are just particular cases of polynomial equations. In general, a polynomial equation is one that can written under the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$  \hspace{1cm} (7)

where each $a_i$ is a real number and $n$ is a positive integer. For $n = 1, 2$ and $3$ in (7), we obtain linear, quadratic and cubic equations respectively.

Most of the techniques needed for this section were already developed in Chapter 5. The main idea behind solving a polynomial equation (of degree 2 or more) of the form $P(x) = 0$ is to factor the polynomial $P(x)$ completely (see Chapter 5) and then solve for $x$ in each factor.
6.1.1 Linear equations

These are equations of the form \( p(x) = 0 \) where \( p(x) \) is a polynomial of degree 1. In other words, they are equations of the form \( ax + b = 0 \) where \( a, b \in \mathbb{R} \) and \( a \neq 0 \). The solution of such an equation is, of course, \( x = -\frac{b}{a} \) (you see now why \( a \) has to be nonzero in order for the solution to exist).

**Examples 6.1.1.**

(1) \( \sqrt{2}x - \sqrt{5} = \sqrt{7} \iff \sqrt{2}x = \sqrt{5} + \sqrt{7} \iff x = \frac{\sqrt{5} + \sqrt{7}}{\sqrt{2}} \). Note that if this was a multiple choice question, then you will probably not see \( \frac{\sqrt{5} + \sqrt{7}}{\sqrt{2}} \) as one of the choices since you will have to rationalize first:

\[
x = \frac{\sqrt{5} + \sqrt{7}}{\sqrt{2}} = \frac{\sqrt{2}(\sqrt{5} + \sqrt{7})}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{10} + \sqrt{14}}{2}.
\]

(2) To solve the equation \( t + \frac{1}{2} - t + 2 = \frac{2t + 15}{3} - \frac{3t}{4} \) for the variable \( t \), we rewrite it under the form \( at = b \). We start by writing both sides of the equation in common denominators:

\[
\frac{t + 1}{2} + \frac{-t + 2}{3} = \frac{2t + 15}{6} - \frac{3t}{4}
\]

For the right side:

\[
\frac{2t + 15 - 3t}{6} = \frac{2(2t + 15) - 3(3t)}{12} = \frac{-5t + 30}{12}.
\]

So the original equation can now be written as

\[
\frac{t + 10}{6} = \frac{-5t + 30}{12}.
\]

Cross multiplying gives: \( 12(t + 10) = 6(-5t + 30) \iff 12t + 120 = -30t + 180 \iff 42t = 60 \iff t = \frac{60}{42} = \frac{10}{7} \).

**Try it yourself 6.1.1.** Solve each of the following equations.

(1) \( \frac{3}{2}x + \frac{7}{5} = \frac{3}{5}x - \frac{3}{5} \)  
(2) \( \frac{2x - 1}{5} + \frac{-x + 1}{8} = \frac{2x + 1}{8} \)  
(3) \( \sqrt{3}x + \sqrt{2} = 2 - \sqrt{7} \)

**Remark 6.1.1.** Although we don’t state it, but when you try to solve an equation, you are actually assuming that there is a solution. If after some steps in attempting to find a solution, you end up with a nonsense like \( 1 = 2 \), this indicates that your initial assumption that a solution exists was wrong and in fact, there is no solution. For example, if you try to solve the linear equation \( 1 + x = 3x - 2x + 2 \) the usual way, then you will end up with the equation \( 1 = 2 \). Of course, the statement "\( 1 = 2 \)" is utterly false, which means that there is no value of \( x \) that could ever make it true. Then the equation \( 1 + x = 3x - 2x + 2 \) has no solution.

6.1.2 Quadratic equations

A quadratic equation in the variable \( x \) is an equation of the form \( ax^2 + bx + c = 0 \) where \( a, b \) and \( c \) are in \( \mathbb{R} \) and \( a \neq 0 \). In chapter \[5\] above, we saw that the solutions to a quadratic equation depend on the sign of its discriminant \( b^2 - 4ac \):
• if $b^2 - 4ac > 0$, then the equation has two distinct real roots given explicitly by

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$ 

• if $b^2 - 4ac = 0$, then the equation has a double real root: $x_1 = x_2 = \frac{-b}{2a}$.

• if $b^2 - 4ac < 0$, then the equation has no real roots (it will have complex roots that we are not going to discuss here).

Note that in the particular case of the simple equation $x^2 = A$ where $A \geq 0$, the solutions are given by $x = \pm \sqrt{A}$.

**Examples 6.1.2.** Solve each of the following equations.

1. $2x^2 - 3x + 1 = 0$
2. $x^4 - 4x^2 + 3 = 0$
3. $2x^2 - 3x + 5 = 0$
4. $x^2 - 8x + 16 = 0$

**Solution.**

1. Since the discriminant: $b^2 - 4ac = (-3)^2 - 4(2)(1) = 1$ is positive, the equation has two distinct real roots: $x_1 = \frac{-(-3) + \sqrt{(-3)^2 - 4(2)(1)}}{2(2)} = \frac{3 + 1}{4} = 1$ and $x_2 = \frac{-(-3) - \sqrt{(-3)^2 - 4(2)(1)}}{2(2)} = \frac{1}{2}$.

2. Although the equation is of degree 4, it can be made quadratic using the change of variable $z = x^2$:

$$z^2 - 4z + 3 = 0$$

and since $z = x^2$, $z$ must be positive. The discriminant of the new equation in $z$ is $b^2 - 4ac = (-4)^2 - 4(1)(3) = 4 > 0$, the equation has two distinct real roots: $z_1 = \frac{-(-4) - \sqrt{(-4)^2 - 4(1)(3)}}{2(1)} = \frac{4 - 2}{2} = 1$ and $z_2 = \frac{-(-4) + \sqrt{(-4)^2 - 4(1)(3)}}{2(1)} = \frac{4 + 2}{2} = 3$. Both roots are acceptable since they correspond to positive values. Don't forget, the goal was to solve to $x$ not for $z$. For $z = 1$, $x^2 = 1$ and thus $x = \pm 1$. For $z = 3$, $x^2 = 3$ which gives $x = \pm \sqrt{3}$.

There are then four distinct real roots of the equation $x^4 - 4x^2 + 3 = 0$, namely $x = \pm 1$ and $x = \pm \sqrt{3}$.

3. $b^2 - 4ac = (-3)^2 - 4(2)(5) = -31 < 0$, the equation has no real roots.

4. $b^2 - 4ac = (-8)^2 - 4(1)(16) = 0$, the equation has one (double) real root given by $x = \frac{-(-8)}{2(1)} = 4$. 

65
Remark 6.1.2. In some cases, it is easier to "guess" the roots of a quadratic equation rather than using the quadratic formula. First note that if we add the two roots of a quadratic equation we get:

\[
\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}
\]

and if we multiply the two roots of a quadratic equation we get:

\[
\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{b^2 - (\sqrt{b^2 - 4ac})^2}{4a^2} = \frac{b^2 - b^2 + 4ac}{4a^2} = \frac{c}{a}.
\]

This means that to find the roots of \(ax^2 + bx + c\) (if they exist), one has to look for two numbers \(x_1\) and \(x_2\) such that

\[x_1 + x_2 = -\frac{b}{a}\] and \[x_1x_2 = \frac{c}{a}.
\]

Example 6.1.1. For the quadratic equation \(x^2 - 3x + 2 = 0\), we look for two numbers \(x_1\) and \(x_2\) such that their sum is \(-\frac{3}{1} = 3\) and their product is \(\frac{2}{1} = 2\). Clearly, one of them is 1 and the other is 2.

Try it yourself 6.1.2. When possible, solve each of the following equations.

1. \(2x^2 - 3x = x^2 + 2x - 4\)  
2. \(9x^2 - 12x + 4 = 0\)  
3. \(x^4 - 5x^2 = x^2 - 5\)  
4. \(2x^2 - 5x + 7 = 0\)

6.1.3 Cubic equations and beyond

At this point, the reader is encouraged to review the techniques of factorization reviewed in chapter 5. Always start by rearranging the equation if necessary to put it in the form above and then proceed to factor the polynomial on the left hand side of the equation.

Let us look at some examples.

Example 6.1.2. Suppose we want to solve the equation \(x^4 - x^2 = -x^3 - x + 2\). We start by rearranging the equation by putting all the term on one side: \(x^4 - x^2 + x^3 + x - 2 = 0\). Next we factor the polynomial \(P(x) = x^4 - x^2 + x^3 + x - 2 = 0\) completely. Note first that the possible rational roots of \(P(x)\) must belong to the set \(\{\pm 1, \pm 2\}\) by the rational root test (Theorem 5.2.1 above). By inspection, we find that \(x = 1\) is a root.
We divide $P(x)$ with $x - 1$ using the long division of polynomials (see section 5.1.1).

$$
\begin{array}{r}
\underline{x - 1) x^4 + x^3 - x^2 + x - 2} \\
x^4 + x^3 \\
\underline{- x^4 + x^3} \\
2x^3 - x^2 \\
\underline{- 2x^3 + 2x^2} \\
x^2 + x \\
\underline{- x^2 + x} \\
2x - 2 \\
\underline{- 2x + 2} \\
0
\end{array}
$$

So $P(x) = (x^3 + 2x^2 + x + 2)(x - 1)$. Next we need to factor the Polynomial $x^3 + 2x^2 + x + 2$. This could be done as before by inspecting one possible rational root and using the long division but in this particular case, this could be done easily by regrouping terms:

$$x^3 + 2x^2 + x + 2 = x^2(x + 2) + (x + 2) = (x + 2)(x^2 + 1).$$

The quadratic polynomial $x^2 + 1$ cannot be factored any further since its discriminant $b^2 - 4ac = -4 < 0$. This means that the complete factorization of $P(x)$ is given by $P(x) = (x - 1)(x + 2)(x^2 + 1)$ and the only real roots of the equation $x^4 - x^2 + x^3 + x - 2 = 0$ are $x = 1$ and $x = -2$.

**Example 6.1.3.** Let us solve the equation

$$-6x^3 + 29x^2 + 7x - 10 = 0. \tag{1}$$

By inspection, $x = 5$ is one root, so the polynomial $P(x) = -6x^3 + 29x^2 + 7x - 10$ is divisible by $x - 5$:

$$
\begin{array}{r}
\underline{x - 5) - 6x^2 - x + 2} \\
- 6x^3 + 29x^2 + 7x - 10 \\
\underline{6x^3 - 30x^2} \\
-x^2 + 7x \\
\underline{x^2 - 5x} \\
2x - 10 \\
\underline{- 2x + 10} \\
0
\end{array}
$$

So $x^4 - x^2 + x^3 + x - 2 = (x - 5)(-6x^2 - x + 2)$. Using the quadratic formula, we compute the two solutions of $-6x^2 - x + 2 = 0$: $x_1 = \frac{1}{2}$ and $x_2 = -\frac{2}{3}$. Equation (1) has then three real solutions, namely $x = -\frac{2}{3}$, $x = \frac{1}{2}$ and $x = 5$. 

67
Try it yourself 6.1.3. Solve each of the following equations.

(1) \( x^3 + 2x^2 - 8x + 5 = 0 \)  
(2) \( x^3 + 6x^2 - 4x - 24 = 0 \)  
(3) \( x^4 - 5x^2 = x^2 - 5 \)  
(4) \( -x^5 + 5x^4 + 3x^3 - 15x^2 + 4x - 20 = 0 \)

6.2 Radical equation

A radical equation is an equation in which one (or more) expression containing a variable is inside a radical, like a square root or a cubic root. In section 3.2, we saw that if \( n \) is a non-negative integer, then the solution to the equation \( x^n = b \) depends on the parity of \( n \) and the sign of \( b \). If \( n \) is even, then the solution \( x = \sqrt[n]{b} \) exists only if \( b \geq 0 \). If \( n \) is odd, the solution \( x = \sqrt[n]{b} \) exists for any real number \( b \).

Warning 6.2.1. With radical equations, don’t take the "face value" of your solutions. Because of the restrictions mentioned at the beginning of this section, what looks like a perfectly valid value of the variable may not actually be a solution. Always check your answer to any radical equation by plugging your solution back into the original equation, and make sure that the equation "works".

Solving a radical equation consists often of "undoing" what was done to the variable. For example, to get rid of a square root, you have to take the square, and to get rid of a cubic root, you take the cube and so on. Let us see how this works with some examples.

Example 6.2.1. Let us solve for the variable \( x \) in the equation \( \sqrt{3x + 2} - 1 = 2 \). We start by isolating the expression containing the square root: \( \sqrt{3x + 2} = 3 \). Now we take the square on both sides to get rid of the square root on the left: \( (\sqrt{3x + 2})^2 = 3^2 \Leftrightarrow 3x + 2 = 9 \Leftrightarrow 3x = 7 \Leftrightarrow x = \frac{7}{3} \). Don’t forget now to check your answer by plugging \( x = \frac{7}{3} \) back in the original equation:

\[
\sqrt{3 \times \frac{7}{3} + 2} - 1 = \sqrt{9} - 1 = 3 - 1 = 2.
\]

Example 6.2.2. In this example, we solve a radical equation involving two radical expressions:

\[\sqrt{2x - 1} = 2 - \sqrt{x} \]

Start by squaring both sides: \( 2x - 1 = (2 - \sqrt{x})^2 = 4 - 4\sqrt{x} + x \Leftrightarrow x - 5 = -4\sqrt{x} \). At this point, we square both sides one more time: \( (x - 5)^2 = 16x \Leftrightarrow x^2 - 26x + 25 = 0 \Leftrightarrow (x - 1)(x - 25) = 0 \Leftrightarrow x = 1, x = 25 \). Now we plug back in the original equation to check these values. For \( x = 1 \), \( \sqrt{2(1) - 1} = 13 \) and \( 2 - \sqrt{1} = 2 - \sqrt{1} = 1 \), so the original equation is satisfied. As for \( x = 25 \), the left hand of the equation gives \( \sqrt{2(25) - 1} = 7 \) while the right hand side gives \( 2 - \sqrt{25} = -3 \). Therefore, the only solution to the original equation is \( x = 1 \).

Example 6.2.3. To solve the equation \( \sqrt{6x - 5} + 3 = -2 \), we start by isolating the expression containing the variable \( x \): \( \sqrt{6x - 5} = -5 \). Next we cube both sides of the equation: \( (\sqrt{6x - 5})^3 = (-5)^3 \Leftrightarrow 6x - 5 = -125 \).
−125 ⇔ x = −\frac{120}{6} = −20. Substituting x with −20 in the original equation, we get on the left hand side:
\[\sqrt{6(-20)} - 5 + 3 = \sqrt{-120} + 3 = -5 + 3 = -2\] which is the same as the right hand side of the equation. So x = −20 is the only solution.

Try it yourself 6.2.1. Solve each of the following radical equations.

(1) \sqrt{3x^2 + 10x - 5} = 0
(2) \sqrt{3x - 5} = 2 - \sqrt{x - 1}
(3) \sqrt{2x + 3} = -3

6.3 Solving equations with absolute values

Recall from section 3.3.1 in chapter 3 that if x is a real number then |x| = −x if x < 0 and |x| = x if x ≥ 0. This means that if Q is any expression and A ≥ 0, then the equation |Q| = A is equivalent to Q = ±A. So to clear the absolute value from an equation, one has to split the equation into two different ones: Q = A and Q = −A. Let us work out some examples.

Example 6.3.1. Suppose you want to solve the equation |x + 2| + 1 = 4. Start by isolating the expression with absolute value: |x + 2| + 1 = 4 ⇔ |x + 2| = 3 ⇔ x + 2 = ±3. Two cases to consider.

• Case 1. x + 2 = −3. In this case, x = −5.
• Case 2. x + 2 = 3. In this case, x = 1.

Substituting back into the original equation, we conclude that both x = −5 and x = 1 are indeed solutions.

Example 6.3.2. Let us solve the equation |2x − 3| = x + 1. Two cases to consider.

• Case 1. 2x − 3 = −(x + 1). In this case, 2x − 3 = −x − 1 ⇔ x = \frac{2}{3}.
• Case 2. 2x − 3 = (x + 1). In this case, x = 4.

Substituting back into the original equation, we can easily verify that both x = \frac{2}{3} and x = 4 are solutions.

Try it yourself 6.3.1. Solve each of the following equations.

(1) 2|2x + 1| − 1 = 3
(2) |3x + 2| = −x + 2
Chapter 7

Solving inequalities

7.1 Introduction and basic properties

Solving an inequality involving a variable $x$ means finding all possible values of $x$ satisfying the inequality. Recall the following notations for real numbers $x$ and $y$:

- $x > y$ means "$x$ is greater than $y$".
- $x < y$ means "$x$ is less than $y$".
- $x \geq y$ means "$x$ is greater than or equal to $y$".
- $x \leq y$ means "$x$ is less than or equal to $y$".

Solving inequalities is an important tool you are definitely going to need in most of your math and science courses. For instance, problems concerning the analysis of a function like domain and range, intervals of increase or decrease, determining the concavity of the graph, often boil down to solving inequalities. Unlike equations, inequalities often have infinitely many solutions.

In addition to be able to solve an inequality and find its solution set, you are expected to write the solution set using interval or absolute value notations. Also, students are expected to be fully aware of all the properties of inequalities in particular those concerning the situations where the inequalities should
be reversed. The table below gives a summary of the basic properties of inequalities. Note that all the properties listed in the below table remain valid if we replace the inequality symbol \( \leq \) with any one of the symbols \(<\), \(>\) or \(\geq\).

<table>
<thead>
<tr>
<th>Properties of Inequalities.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x, y, z ) are real numbers.</td>
</tr>
<tr>
<td>If ( x \leq y ), then ( x \pm z \leq y \pm z )</td>
</tr>
<tr>
<td>If ( x \leq y ) and ( z \geq 0 ), then ( xz \leq yz )</td>
</tr>
<tr>
<td>If ( x \leq y ) and ( z \leq 0 ), then ( xz \geq yz )</td>
</tr>
<tr>
<td>If ( x \leq y &lt; 0 ), then ( \frac{1}{x} \geq \frac{1}{y} )</td>
</tr>
<tr>
<td>If ( 0 &lt; x \leq y ), then ( \frac{1}{x} \geq \frac{1}{y} )</td>
</tr>
<tr>
<td>If ( x &lt; 0 &lt; y ), then ( \frac{1}{x} &lt; \frac{1}{y} )</td>
</tr>
</tbody>
</table>

Note that in the last three properties in the above table, the inequality is reversed only when \( x \) and \( y \) have the same sign (both positive or both negative). For instance, \(-2 < 3\) (\(-2\) and \(3\) are of opposite signs) and \(\frac{1}{-2} < \frac{1}{3}\) since \(\frac{1}{-2} = -\frac{1}{2} < 0\) and \(\frac{1}{3} > 0\).

### 7.1.1 Sign Table

If \( f(x) \) is an expression containing a variable \( x \), then a "visual way" to solve an inequality of type \( f(x) \leq 0 \) or \( f(x) \geq 0 \) (or the corresponding strict inequalities) is to form the sign table of \( f(x) \). This is a table where the sign of the expression \( f(x) \) is displayed for all possible values of the variable \( x \). The first step in forming the sign table of \( f(x) \) is to find the roots of that expression, that is the values of \( x \) for which \( f(x) = 0 \).

A typical table looks as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\infty)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(+\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>+</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

where \( x_1, x_2 \) and \( x_3 \) are the roots of \( f(x) \). From the table we get for example that \( f(x) \geq 0 \) if and only if \( x \leq x_1 \) or \( x_2 \leq x \leq x_3 \). Using interval notations, the solution set of the inequality \( f(x) \geq 0 \) can be written as \( (-\infty, x_1] \cup [x_2, x_3] \).
7.2 Polynomial Inequalities

These are inequalities that can be reduced to one or the other of the forms \( P(x) \geq 0 \) or \( P(x) \leq 0 \) (or the corresponding strict inequalities) for some polynomial \( P(x) \). As we did for polynomial equations, we start by treating some basic cases, according to the degree of the polynomial \( P(x) \).

7.2.1 Linear Inequalities

As the name suggests, linear inequalities are inequalities that can be reduced to one or the other of the two forms \( ax + b \leq 0 \) or \( ax + b \geq 0 \) (or the corresponding strict inequalities) for some \( a, b \in \mathbb{R} \) with \( a \neq 0 \). The solution set depends on the sign of \( a \). Note that the root of the expression \( ax + b \) is \( x = -\frac{b}{a} \). This value of \( x \) divides the real line into two intervals \([ -\infty, -\frac{b}{a} ] \) and \( [ -\frac{b}{a}, +\infty ] \) and the sign table of \( ax + b \) is the following:

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\infty)</th>
<th>(-\frac{b}{a})</th>
<th>(+\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The sign of ( ax + b )</td>
<td>Opposite to the sign of ( a )</td>
<td>( \emptyset )</td>
<td>Same sign as ( a )</td>
</tr>
</tbody>
</table>

Examples 7.2.1. Solve each of the following inequalities:

1. \(-2x + \frac{x-1}{3} \geq 1\)  
2. \(\frac{x+1}{4} > \frac{3-2x}{3} - 2\)  
3. \(-7 + x \leq 2x + 3 < 13\)

Solution. (1) \(-2x + \frac{x-1}{3} \geq 1 \Rightarrow -2x + \frac{x-1}{3} - 1 \geq 0\). Next, we write the left hand side of the equation in common denominator:

\[
-2x + \frac{x-1}{3} - 1 \geq 0 \quad \Leftrightarrow \quad -\frac{6x + x - 1 - 3}{3} \geq 0 \quad \Leftrightarrow \quad -\frac{-5x - 4}{3} \geq 0 \quad \Leftrightarrow \quad -5x - 4 \geq 0
\]

since the denominator 3 in the last fraction is positive. Now, \(-5x - 4 \geq 0 \Leftrightarrow -5x \geq 4 \Leftrightarrow x \leq -\frac{4}{5}\). In interval term, the solution set is the interval \([ -\infty, -\frac{4}{5} ]\) that we can represent on the real line.
(2) As we did for part (1), we start by rearranging the expression so that the right hand side is zero:

\[
\frac{x + 1}{4} > \frac{3 - 2x}{3} - 2 \quad \iff \quad \frac{x + 1}{4} - \frac{3 - 2x}{3} + 2 > 0
\]

\[
\iff \quad \frac{3(x + 1) - 4(3 - 2x) + 2(12)}{12} > 0
\]

\[
\iff \quad \frac{11x + 15}{12} > 0
\]

\[
\iff \quad 11x + 15 > 0
\]

\[
\iff \quad x > -\frac{15}{11}
\]

The solution is then \(x > -\frac{15}{11}\). That can be represented by the interval \((-\frac{15}{11}, +\infty]\)

(3) Here we have two simultaneous inequalities \(-7 + x \leq 2x + 3 \text{ and } 2x + 3 < 13\). The first one can be reduced to (after simplifying) \(-x \leq 10\) or equivalently \(x \geq -10\). The second one is equivalent to \(x < 5\). The solution to \(-7 + x \leq 2x + 3 < 13\) is then the set of all real numbers \(x\) satisfying \(x \geq -10\) and \(x < 5\). This is the intersection of the intervals: \([-10, \infty[\text{ and } ]-\infty, 5[\). Representing the two intervals in different colors on the real line would help to reveal their intersection:

We conclude that the solution set is the interval \([-10, 5]\).

Try it yourself 7.2.1. Solve each of the following inequalities.

\(1\) \( -x + \frac{2x - 1}{2} \leq -2 \quad \) \(2\) \( -\frac{x + 1}{3} < -\frac{1 + 2x}{4} + 1 \quad \) \(3\) \( -3x + 5 \geq -4x + 5 \geq 7x - 3 \)

7.2.2 Quadratic Inequalities

These are inequalities that can be reduced to the form \(ax^2 + bx + c \leq 0\) or \(ax^2 + bx + c \geq 0\) (or the corresponding strict inequalities) for some \(a, b, c \in \mathbb{R}\) with \(a \neq 0\). The solutions of a quadratic inequality depends on the number of real solutions the corresponding quadratic equation \(ax^2 + bx + c = 0\) has.
(otherwise, it would have a third root in that interval different from \(x_1\) and \(x_2\)). In general, the polynomial \(ax^2 + bx + c\) has the same sign as the leading coefficient \(a\) if the variable \(x\) belongs to one or the other of the intervals \([-\infty, x_1]\) and \([x_2, +\infty]\) and it has the opposite sign of \(a\) if \(x\) belongs to the interval \([x_1, x_2]\).

The following is the sign table of \(ax^2 + bx + c\) in the case where \(b^2 - 4ac > 0\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-\infty)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(+\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The sign of (ax^2 + bx + c)</td>
<td>Sign of (a)</td>
<td>Opposite sign of (a)</td>
<td>Sign of (a)</td>
<td></td>
</tr>
</tbody>
</table>

If \(b^2 - 4ac = 0\), then the polynomial \(ax^2 + bx + c\) has one (double) real root given by \(x_1 = x_2 = -\frac{b}{2a}\). Recall that in this case, the polynomial can be factored as \(a\left(x + \frac{b}{2a}\right)^2\) and consequently it will always have the same sign as the leading coefficient \(a\) for any choice of \(x \in \mathbb{R}\) since \(\left(x + \frac{b}{2a}\right)^2 \geq 0\) for any \(x\).

Finally, if \(b^2 - 4ac < 0\), then the polynomial \(ax^2 + bx + c\) has no real roots and therefore it will never change signs. Again in this case, \(ax^2 + bx + c\) will have the same sign as the leading coefficient \(a\) for any choice of \(x \in \mathbb{R}\).

**Examples 7.2.2.** Solve each of the following quadratic inequalities.

1. \(x^2 - 4x \geq -3\)
2. \(2x^2 - x + 3 < 0\)
3. \(4x^2 - 4x + 1 \geq 0\)
4. \(-x^2 + 2x - 1 \geq x^2 - x\)

**Solution.**

1. \(x^2 - 4x \geq -3 \iff x^2 - 4x + 3 \geq 0\). The polynomial \(x^2 - 4x + 3\) can be factored as \((x-1)(x-3)\) and the roots are \(x_1 = 1\) and \(x_2 = 3\). We form the sign table of \(x^2 - 4x + 3\):

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-\infty)</th>
<th>1</th>
<th>3</th>
<th>(+\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x^2 - 4x + 3)</td>
<td>+</td>
<td>(\theta)</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

So \(x^2 - 4x + 3 \geq 0 \iff x \in \mathbb{R}\). The solution set of the inequality is \([-\infty, 1] \cup [3, +\infty]\):

(2) For the polynomial \(2x^2 - x + 3\), \(b^2 - 4ac = (\frac{1}{2})^2 - 4(2)(3) = -23 < 0\), so the polynomial has no real roots. It will then have the sign of the leading coefficient, which is positive in this case regardless of the value of
the variable $x$. Thus, $2x^2 - 3x + 5 > 0$ for all $x \in \mathbb{R}$ and there is no solution (no real number $x$ will satisfy the inequality) for the inequality in (2). We say in this case that the solution set is $\emptyset$, the empty set.

(3) For the polynomial $4x^2 - 4x + 1$, $b^2 - 4ac = (-4)^2 - 4(4)(1) = 0$ so $4x^2 - 4x + 1$ has the sign of $a = 4 > 0$. Therefore, the inequality $4x^2 - 4x + 1 \geq 0$ is satisfied for any real number $x$. The solution set is $[\infty, \infty]$.

(4) $-2x^2 + 3x - 1 \geq 0$ $\iff$ $x \in \left[\frac{1}{2}, 1\right]$. The solution set of the inequality is $\left[\frac{1}{2}, 1\right]$:

Try it yourself 7.2.2. Solve each of the following inequalities.

1. $-x^2 \geq 7x - 8$
2. $x^2 - 4x + 6 \leq 0$
3. $4x^2 + 3x - 4 < -2x^2 - 2x$

7.2.3 Cubic inequalities and beyond

Like the polynomial equations, solving a polynomial inequality can be done generally by factoring and studying the sign of each of the factors individually. Recall the rules of signs multiplication:

$+$ $\times$ $+$ $=$ $+$,
$+$ $\times$ $-$ $=$ $-$,
$-$ $\times$ $+$ $=$ $-$,
$-$ $\times$ $-$ $=$ $+$

and the "quotient rules" for the signs symbols:

$\frac{+}{+} = +, \quad \frac{+}{-} = -, \quad \frac{-}{+} = -, \quad \frac{-}{-} = +$

Let us work out some examples.

Example 7.2.1. Assume we want to solve the inequality $2x^3 - x^2 - 2x - 9 \leq 0$. The first step is to completely factor the polynomial $2x^3 - x^2 - 2x - 9$. Using the techniques of section 5.2.2 we get that $2x^3 - x^2 - 2x - 9 = (2x + 3)(x - 1)(x - 3)$ and the roots are $x = -\frac{3}{2}, x = 1, x = 3$. Next, we form the sign table of $2x^3 - x^2 - 2x - 9$. 
Based on the sign table of each of its factors and the signs multiplication rules.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
x & -\infty & -\frac{3}{2} & 1 & 3 & +\infty \\
\hline
2x + 3 & - & 0 & + & + & + \\
\hline
x - 1 & - & - & 0 & + & + \\
\hline
x - 3 & - & - & - & 0 & + \\
\hline
2x^3 - x^2 - 2x - 9 & - & 0 & + & 0 & + \\
\hline
\end{array}
\]

From the table, \(2x^3 - x^2 - 2x - 9 \leq 0 \iff x \leq -\frac{3}{2} \) or \( 1 \leq x \leq 3 \). In interval terms, the solution set can be written as \([-\infty, -\frac{3}{2}] \cup [1, 3]\).

**Example 7.2.2.** Let us look now at the inequality \(-x^4 + 3x^2 + x^3 - 4x + 4 > 0\). The complete factorization of the polynomial \(-x^4 + 3x^2 + x^3 - 4x + 4\) is \((-x^2 + x - 1)(x - 2)(x + 2)\) (see section 5.2.2 above). The quadratic \(-x^2 + x - 1\) has no real roots (since \(b^2 - 4ac = -3 < 0\)) and therefore it will always have the sign as its leading coefficient. We form the sign table of \(-x^4 + 3x^2 + x^3 - 4x + 4\):

\[
\begin{array}{|c|c|c|c|c|}
\hline
x & -\infty & -2 & 2 & +\infty \\
\hline
-x^2 + x - 1 & - & - & - & - \\
\hline
x + 2 & - & 0 & + & + \\
\hline
x - 2 & - & - & 0 & + \\
\hline
-x^4 + 3x^2 + x^3 - 4x + 4 & - & 0 & + & 0 & - \\
\hline
\end{array}
\]

So, \(-x^4 + 3x^2 + x^3 - 4x + 4 > 0 \iff x \in ]-2, 2]\):

\[
\begin{array}{c}
-2 \quad 2
\end{array}
\]

**Try it yourself 7.2.3.** Solve each of the following inequalities.

\( (1) \ x^4 - 11x^2 + 9x - 18 \quad (2) \ x^3 - 2x^2 - x - 6 \leq 0 \quad (3) \ 2x^4 - 21x^2 < -x^3 + 9x - 27 \)
7.3 Rational inequalities

An expression of the form \( \frac{P(x)}{Q(x)} \) with \( P(x) \) and \( Q(x) \) are two polynomials is called a **rational function**. Using the quotient rules for the signs symbols, we can determine the sign table of a rational expression \( \frac{P(x)}{Q(x)} \) from the tables of \( P(x) \) and \( Q(x) \). The values we include on the sign table of \( \frac{P(x)}{Q(x)} \) are the roots of \( P(x) \) and \( Q(x) \). Just remember that we cannot divide with 0, so the zeros of the denominator \( Q(x) \) are values of the variable \( x \) for which the rational function \( \frac{P(x)}{Q(x)} \) is undefined. Let us work out some examples.

**Example 7.3.1.** Let us solve the inequality

\[
\frac{x + 2}{x - 1} \geq 3.
\]

First note that \( x \) cannot take the value 1 since we would have zero at the denominator.

\[
\frac{x + 2}{x - 1} \geq 3 \iff \frac{x + 2 - 3}{x - 1} \geq 0 \iff \frac{x + 2 - 3(x - 1)}{x - 1} \geq 0 \iff \frac{-2x + 5}{x - 1} \geq 0.
\]

Now, we form the sign table of \( \frac{-2x + 5}{x - 1} \) showing the values \( x = \frac{5}{2} \) (zero of the numerator) and \( x = 1 \) (zero of the denominator) on it.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\infty)</th>
<th>1</th>
<th>( \frac{5}{2} )</th>
<th>( +\infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2x + 5)</td>
<td>+</td>
<td>+</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>( x - 1 )</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( \frac{-2x + 5}{x - 1} )</td>
<td>-</td>
<td></td>
<td>0</td>
<td>-</td>
</tr>
</tbody>
</table>

The double bar at \( x = 1 \) in the last row of the table indicates that the expression \( \frac{-2x + 5}{x - 1} \) is undefined for that value of \( x \). From the above sign table, we get

\[
\frac{x + 2}{x - 1} \geq 3 \iff 1 \leq x < \frac{5}{2}.
\]

In interval notation, the solution set is \( \left[ 1, \frac{5}{2} \right) \):

---

**Example 7.3.2.** Solve the inequality:

\[
x \leq \frac{x + 6}{x + 2}.
\]
Start by arranging the equation

\[ x \leq \frac{x + 6}{x + 2} \iff x - \frac{x + 6}{x + 2} \leq 0 \iff \frac{x(x + 2) - (x + 6)}{x + 2} \leq 0 \iff \frac{x^2 + x - 6}{x + 2} \leq 0. \]

The numerator is 0 when \( x = -3 \) and \( x = 2 \), and the denominator is 0 for \( x = -2 \). We now form the sign table of \( \frac{x^2 + x - 6}{x + 2} \) from the tables of \( x^2 + x - 6 \) and \( x + 2 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\infty)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(2)</th>
<th>(+\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 + x - 6 )</td>
<td>+</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>( x + 2 )</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( \frac{x^2 + x - 6}{x + 2} )</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td>-</td>
<td>0</td>
</tr>
</tbody>
</table>

Thus, \( x \leq \frac{x + 6}{x + 2} \iff x \leq -3 \) or \(-2 \leq x \leq 2\). The solution to the inequality is then \(-\infty, -3] \cup [-2, 2]\):

\[ \begin{array}{cccccccc}
-3 & -2 & 2 & 0 \end{array} \]

**Try it yourself 7.3.1.** Solve each of the following inequalities.

1. \( \frac{x - 2}{x + 1} \leq -2 \)  
2. \( \frac{3x}{x^2 - 4} \leq -1 \)  
3. \( 2x \leq \frac{x + 3}{x + 1} \)

### 7.4 Inequalities with absolute value

Inequalities like \( |2x + 3| \leq 1 \) pop up regularly in your math courses. The following table gives basic rules to solve inequalities involving absolute values.

<table>
<thead>
<tr>
<th>Inequalities with absolute value.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let ( Y ) be any expression, and ( A ) a positive real number, then</td>
</tr>
<tr>
<td>• (</td>
</tr>
<tr>
<td>• (</td>
</tr>
</tbody>
</table>

Note that the rules in the above table remain valid if the inequality signs \( \leq \) and \( \geq \) are replaced with the strict inequalities \( < \) and \( > \) respectively.
Example 7.4.1. Solve each of the following inequalities.

(1) \(|2x + 3| \leq 4\)  \hspace{1cm} (2) \(|x^2 - 4| \geq 1\)  \hspace{1cm} (3) \(1 < |2x + 3| \leq 2\)

Solution. (1) Using the rules of inequalities with absolute values, we get:

\(|2x + 3| \leq 4\) \iff \(-4 \leq 2x + 3 \leq 4\) (by the first rule in the above table)
\iff \(-7 \leq 2x \leq 1\) (by subtracting 3 from each side)
\iff \(-\frac{7}{2} \leq x \leq \frac{1}{2}\) (by dividing through with 2).

The solution set is the closed interval \([-\frac{7}{2}, \frac{1}{2}]\):

(2) \(|x^2 - 4| \geq 1\) \iff \(x^2 - 4 \geq 1\) or \(x^2 - 4 \leq -1\). The first inequality is equivalent to \(x^2 - 5 \geq 0\) and the second to \(x^2 - 3 \leq 0\). Note that the roots of \(x^2 - 5\) are \(\pm \sqrt{5}\) and those of \(x^2 - 3\) are \(\pm \sqrt{3}\). We form the sign table of both
inequities at the same time:

<table>
<thead>
<tr>
<th></th>
<th>(-\infty)</th>
<th>(-\sqrt{5})</th>
<th>(-\sqrt{3})</th>
<th>(\sqrt{3})</th>
<th>(\sqrt{5})</th>
<th>(+\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x^2 - 5)</td>
<td>+</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
<td>+</td>
</tr>
<tr>
<td>(x^2 - 3)</td>
<td>+</td>
<td>+</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
<td>+</td>
</tr>
</tbody>
</table>

We conclude that \(x^2 - 5 \geq 0\) \iff \(x \leq -\sqrt{5}\) or \(x \geq \sqrt{5}\), and \(x^2 - 3 \leq 0\) \iff \(-\sqrt{3} \leq x \leq \sqrt{3}\). It is very important

to note in this example that the inequality \(|x^2 - 4| \geq 1\) is equivalent to \(x^2 - 5 \geq 0\) or \(x^2 - 3 \leq 0\). We are not
looking for the "intersection" of the two solution sets, but rather their union. Combining the two solution sets together we get \(|x^2 - 4| \geq 1\) \iff \(x \leq -\sqrt{5}\) or \(-\sqrt{3} \leq x \leq \sqrt{3}\) or \(x \geq \sqrt{5}\). In interval form, the solution set is

\([-\infty, -\sqrt{5}] \cup [-\sqrt{3}, \sqrt{3}] \cup [\sqrt{5}, +\infty]\):

(3) Here we have two simultaneous (linear) inequalities \(1 < |2x + 3|\) and \(|2x + 3| \leq 2\). The "and" means that
we are looking for the intersection of the two solution sets. Let us start with the first inequality:

\(|2x + 3| > 1\) \iff \(2x + 3 > 1\) or \(2x + 3 < -1\)
\iff \(x > -1\) or \(x < -2\).
As for the second inequality:

\[
|2x + 3| \leq 2 \Leftrightarrow -2 \leq 2x + 3 \leq 2
\]

\[
\Leftrightarrow -5 \leq 2x \leq -1
\]

\[
\Leftrightarrow -\frac{5}{2} \leq x \leq -\frac{1}{2}
\]

Therefore, the solution to \(1 < |2x + 3| \leq 2\) is the overlapping part of \((-\infty, -2] \cup -1, +\infty\) and \([-\frac{5}{2}, \frac{1}{2}]\) (that is the overlapping part of the red and the blue). From the diagram, it is clear that this overlapping part is \([-\frac{5}{2}, -2[ \cup ]-1, \frac{1}{2}]\). In other words,

\[
1 < |2x + 3| \leq 2 \Leftrightarrow -\frac{5}{2} \leq x < -2 \text{ or } -1 < x \leq \frac{1}{2}.
\]

**Try it yourself 7.4.1.** Solve each of the following inequalities.

1. \(|-2x + 1| \leq 3\)  
2. \(|x^2 - 5| \geq 4\)  
3. \(2 \leq |3x + 4| < 5\)
Chapter 8

Functions

8.1 Introduction

Although there are many mathematically "involved" ways to introduce the notion of a function, all what you need to understand entering a first year math course is the basic definition. Think of a function as rule (or a machine) that takes one number as input from an available set of inputs, does something with it and produces exactly one output number.
The function \( f \)

For example, the rule \( f(x) = 2x + 1 \) takes the input number \( x \), multiplies it first by 2 and adds 1 to the answer, producing a unique output number \( 2x + 1 \) for each input value \( x \). The following table gives some input-output values for this function:

<table>
<thead>
<tr>
<th>Input ( x )</th>
<th>Output ( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3/2</td>
<td>-2</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1/2</td>
<td>2</td>
</tr>
</tbody>
</table>

More generally, a function from a set \( A \) to a set \( B \) is a correspondence rule between \( A \) and \( B \) that assigns to every element \( x \) of \( A \) a unique element \( y \) of \( B \) called the image of \( x \) and written as \( y = f(x) \).

**Notation.** A function \( f \) from a set \( A \) to a set \( B \) is usually denoted by \( f : A \to B \).

### 8.2 Graph of a function

A visual way to visually represent a function is to plot its **graph**: let the \( x \)-axis be the input and the \( y \)-axis be the output, the graph of the function \( f \) is the set of all points in the plane \( xOy \) of the form \( (x, f(x)) \). So a point on the graph has its first coordinate as the input value, and its second coordinate as the corresponding output value. Your graphic calculator displays the graph of a function by simply joining a "sufficiently
large number of these points.

**Example 8.2.1.** The diagram on the left in the following picture shows 9 points of the form \((x, f(x))\) form some function \(f\). The diagram on the right shows the curve (or the graph) we get when we join these points.

![Diagram showing 9 points and their graph](image)

**Example 8.2.2.** In the case of the function \(f(x) = 2x + 1\), the points \(A = (\frac{-3}{2}, -2)\), \(B = (-1, -1)\), \(C = (0, 1)\), \(D = (\frac{1}{2}, 2)\) are all points on the graph of \(f\). Joining these points, we get the graph of \(f(x) = 2x + 1\) as a straight line in the plane:

![Diagram showing the graph of \(f(x) = 2x + 1\)](image)

### 8.2.1 New graphs from old

In many instances, we can draw the graph of a given function very quickly by simply applying some basic plane transformations like a shift or a scale on a well known graph. A typical example is the vertex equation
\[ y = a(x - h)^2 + k \] of a parabola that we looked at in section [4.1.5]. The graph of any parabola can be obtained from the graph of the basic parabola \( y = x^2 \) by means of a horizontal and a vertical shift.

### 8.2.1.1 Horizontal and vertical shifts

A shift is a plane transformation that does not change the shape or size of the graph of the given function. There are two types of shifts. A vertical shift adds (or subtracts) a constant to the \( y \)-coordinate while keeping the \( x \)-coordinate unchanged. A horizontal shift adds (or subtracts) a constant to the \( x \)-coordinate while leaving the \( y \)-coordinate unchanged. Very often, we combine vertical and horizontal shifts to get a new graph. Adding a constant to the \( x \)-coordinate will result in a horizontal shift and adding a constant to the \( y \)-coordinate will result in a vertical shift. The following table explains the details.

<table>
<thead>
<tr>
<th>Given the function ( y = f(x) ) and a positive constant ( c \in \mathbb{R} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>• The graph of ( y = f(x + c) ) is obtained from the graph of ( y = f(x) ) by a horizontal shift of ( c ) units to the left.</td>
</tr>
<tr>
<td>• The graph of ( y = f(x - c) ) is obtained from the graph of ( y = f(x) ) by a horizontal shift of ( c ) units to the right.</td>
</tr>
<tr>
<td>• The graph of ( y = f(x) + c ) is obtained from the graph of ( y = f(x) ) by a vertical shift of ( c ) units upward.</td>
</tr>
<tr>
<td>• The graph of ( y = f(x) - c ) is obtained from the graph of ( y = f(x) ) by a vertical shift of ( c ) units downward.</td>
</tr>
</tbody>
</table>

**Example 8.2.3.** The graph of a function \( f \) is given below.

Then the graphs of \( f(x+1) \), \( f(x-1) \), \( f(x)-1 \) and \( f(x)+1 \) are given below. The graph of the original function
is also drawn (in dashed curve) to allow a visual comparison with the shifting.

Combining a horizontal and a vertical shift can be done at the same time. For instance, the following is the graph of $f(x - 1) + 1$:

**8.2.1.2 Horizontal and vertical scaling**

Unlike the horizontal and vertical shifts, a scale is a transformation that usually alters the shape and size of the graph of the function. Given a function $y = f(x)$ and a constant $a \in \mathbb{R}$, a horizontal scaling of the graph of $f$ is the graph of the function $f(ax)$ (obtained from the original one by replacing every occurrence of the variable $x$ with $ax$). A vertical scaling of the graph of $f$ is the graph of the function $af(x)$. Replacing $x$ with $cx$ in the function $f(x)$ results in a horizontal stretching or a horizontal compression (depending on the value of $c$) of the graph of $f$. Similarly, multiplying the function itself with the constant $c$ results in a vertical stretching or a vertical compression (depending on the value of $c$) of the graph of $f$. We will see what these terms mean with some specific example. The following table gives the values of the constant $c$ that correspond to a compression or a stretching of the graph.
Given the function $y = f(x)$ and a positive constant $c \in \mathbb{R}$

- If $0 < c < 1$, then the graph of $y = f(cx)$ is the graph of $y = f(x)$ stretched horizontally away from the $y$-axis.
- If $c > 1$, then the graph of $y = f(cx)$ is the graph of $y = f(x)$ compressed horizontally towards the $y$-axis.
- If $0 < c < 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ compressed vertically towards the $x$-axis.
- If $c > 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched vertically away from the $x$-axis.

Let us look at some examples.

**Example 8.2.4.** The graph of a function $f$ is given below.

Then the graphs of $f(0.5x)$, $f(2x)$, $0.5f(x)$ and $2f(x)$ are given below. The graph of the original function is also drawn (in dashed curve) to allow a visual comparison with the scaling.
Like in the case of shifting, one can combine a horizontal and a vertical scale on the graph. For instance, the following is the graph of $2f(0.5x)$ where $f$ is the function given in Example 8.2.4.

![Graph of $2f(0.5x)$]

**Example 8.2.5.** Consider the function $f(x) = 2x^2 - 4x + 5$. Note that

$$f(x) = 2(x^2 - 2x) + 5 = 2(x^2 - 2x + 1 - 1) + 5 \text{ by completing the square as in Section 5.2.1.2} = 2(x - 1)^2 + 3.$$

Now the graph of $2(x - 1)^2 + 3$ is a combination of a shifting and scaling of the graph of the parabola $y = x^2$. First take a horizontal shift to the left by one unit and get $(x - 1)^2$. Second scale this new function by a factor of 2 and get $2(x - 1)^2$, Finally, shift this last function vertically upward by three units and get our final form $2(x - 1)^2 + 3$:

### 8.2.1.3 Reflection

If $c$ is a positive constant, then we know that the graph of $cf(x)$ is a scaling of the graph of $f(x)$. But what if $c < 0$? What would the graph of $cf(x)$ look like in this case? In fact, all what we need to understand if the relationship between the graph of $f$ and that of $-f$ since if $c < 0$, then $c = -\alpha$ for some positive real number $\alpha$ and $cf = -(\alpha f)$. Remember that $(\alpha f)$ is a scaling of $f$ that we know how to deal with.

A point on the graph of the function $-f$ is of the form $(x, -f(x))$ which a reflection of the point $(x, f(x))$.
on the graph of \( f \) in the \( x \)-axis. So, the graph of \(-f\) is simply a reflection of the graph of \( f \) in the \( x \)-axis:

\[
\begin{align*}
-y(x) & \quad \text{and} \\
-f(x) & \quad \text{in the} \quad x \text{-axis:}
\end{align*}
\]

8.3 Domain and range

Consider the two functions \( f(x) = \frac{x^2 + 2}{x^2 - 5x + 4} \) and \( g(x) = \frac{1}{x^2 + 1} \). If you try \( x = 1 \) as an input value for \( f \), you see that the function does not produce a real number since you would be dividing with zero. The same happens with the value \( x = 4 \). We say that these two values are not in the domain of \( f \). For the function \( g(x) \), note that \( y = 2 \) cannot be an output value since that would mean that \( \frac{1}{x^2 + 1} = 2 \) or equivalently \( x^2 = -1 \) which has no solution in \( \mathbb{R} \).

This leads to the following two important questions about a function \( f \):

1. what real numbers \( x \) can be used as input values for \( f \)? and
2. what real numbers \( y \) can be considered as output values for \( f \)?

The domain of a function is the complete set of all possible input values \( x \) which will make the function "work", that is will output "real" \( y \)-values. The range of a function is the complete set of all possible output values of \( f \).

Before we start some examples, keep always in mind the following:
• The denominator of a fraction cannot be zero
• Any expression under an even root sign, like $\sqrt[2]{\cdot}$, $\sqrt[3]{\cdot}$, must be positive.

**Example 8.3.1.** Find the domain and the range of each of the following functions.

(1) $f(x) = \frac{2x-3}{3x-4}$  
(2) $g(x) = \sqrt{2x+3}$  
(3) $h(x) = \frac{1}{x^2+1}$  
(4) $k(x) = \frac{1}{\sqrt[3]{x-2}}$

**Solution.**

1. Starting with the function $f$, note that the only thing that can go wrong is when the denominator $(3x-4)$ is equal to zero. This happens when $3x-4 = 0$ or $x = \frac{4}{3}$. The domain is then all real numbers except $\frac{4}{3}$ which can be written in the interval form as $[-\infty, \frac{4}{3}] \cup [\frac{4}{3}, +\infty]$. To determine the range of $f$, we must look for all possible "outputs" values of $f$, that is all real numbers $y$ of the form $y = \frac{2x-3}{3x-4}$ for some $x$ in the domain:

$$y = \frac{2x-3}{3x-4} \iff 2x-3 = y(3x-4) \iff 2x-3 = 3xy-4y \iff 2x-3xy = 3-4y \iff x = \frac{3-4y}{2-3y}$$

In order for $x$ to exist (be a real number), $y \neq \frac{2}{3}$. So other than $\frac{2}{3}$, every real number is an output. The range is then $[-\infty, \frac{2}{3}] \cup [\frac{2}{3}, +\infty]$.

2. Since $g$ is defined using a square root, the expression under the radical sign must be greater than or equal to 0. This amounts to solving the linear inequality $2x+3 \geq 0$. Of course, the solution set is the values of $x$ such that $x \geq -\frac{3}{2}$. In interval notations, the domain is $[-\frac{3}{2}, +\infty]$. For the range, we look at all possible values $y$ of the form $y = \sqrt{2x+3}$. The square root is never negative, the range is $[0, +\infty]$ (note that 0 is in the range since $0 = g\left(\frac{-3}{2}\right)$).

3. The only problem the function $h$ could have is when its denominator is equal to 0. But look closely, can $x^2 + 1 = 0$ with $x \in \mathbb{R}$? clearly not since otherwise $x^2 = -1$ (or simply compute the discriminant of the quadratic $x^2 + 1$ and you will find a negative value). So there is no chance anything can go wrong for $h$ and the domain is $\mathbb{R}$ or $[-\infty, +\infty]$. For the range, look closely at an output numbers $y = \frac{1}{1+x^2}$. Since $1 + x^2 \geq 1$ for any real number $x$, $y = \frac{1}{1+x^2} \leq 1$ (remember, you change the inequality sign when you take the inverse). On the other hand, we have that $\frac{1}{1+x} > 0$ (this is a strict inequality since the numerator of the fraction is never 0). Therefore, the range is $[0, 1]$.

4. Two issues must be dealt with for the function $k$. First $x$ has to be non-negative because of $\sqrt{x}$ appearing in $k(x)$. Second, the denominator cannot be zero which happens when $\sqrt{x} = 2$ or equivalently $x = 4$. So, $x$ can take all non-negative values except 4. In interval terms, the domain can be
Example 8.4.1. Consider the two functions

\[ f(x) = \frac{1}{\sqrt{x-2}} \quad \text{and} \quad g(x) = \sqrt{2x+4}. \]

The domain of \( f \) is the set of all real numbers \( x \) with \( x - 2 > 0 \) (strict inequality since the denominator cannot be zero). This is equivalent to \( x > 2 \). The domain of \( f \) is \( D_f = ]2, +\infty[ \). Similarly, the domain of \( g \) is \( D_g = [-2, +\infty[ \).

The intersection of the two domains is \( D_f \cap D_g = ]2, +\infty[ \). We know that each of the functions \( f + g, f - g, f g \) has the interval \( ]2, +\infty[ \) as the domain. Moreover, for every \( x \in ]2, +\infty[ \), we have:

- \( (f + g)(x) = \frac{1}{\sqrt{x-2}} + \sqrt{2x+4} = \frac{1+\sqrt{x-2}+2\sqrt{x+4}}{\sqrt{2x+4}} = \frac{1+\sqrt{(x-2)(2x+4)}}{\sqrt{2x+4}} = \frac{1+\sqrt{2(x^2-4)}}{\sqrt{2x+4}}. \)
- \( (f - g)(x) = \frac{1}{\sqrt{x-2}} - \sqrt{2x+4} = \frac{1-\sqrt{x-2}\sqrt{2x+4}}{\sqrt{2x+4}} = \frac{1-\sqrt{(x-2)(2x+4)}}{\sqrt{2x+4}} = \frac{1-\sqrt{2(x^2-4)}}{\sqrt{2x+4}}. \)
- \( (fg)(x) = \left( \frac{1}{\sqrt{x-2}} \right) (\sqrt{2x+4}) = \frac{\sqrt{2x+4}}{\sqrt{x-2}}. \)

8.4 Operations on Functions

8.4.1 Addition, substraction, multiplication and division of functions

Addition, substraction, multiplication and division are examples of operations one can perform on real numbers. Similar operations can be done on functions. Let \( f \) and \( g \) be two functions (on the reals) with domains \( D_f \) and \( D_g \) respectively. In what follows we define new functions using \( f \) and \( g \).

1. **Addition of functions** \( f + g \) is the function defined by \((f + g)(x) = f(x) + g(x)\).

2. **Subtraction of functions** \( f - g \) is the function defined by \((f - g)(x) = f(x) - g(x)\).

3. **Multiplication of functions** \( fg \) is the function defined by \( f(x)g(x) \). **Division of functions** \( \frac{f}{g} \) is the function defined by \( \frac{f(x)}{g(x)} \) for \( x \) in the domain of \( f \) where \( g(x) \neq 0 \).

**Remark 8.4.1.** For each of the functions \( f + g, f - g, fg \), the domain is the intersection \( D_f \cap D_g \) of the domains of \( f \) and \( g \). For the function \( \frac{f}{g} \), the domain is \( D_f \cap D_g \) from which we remove the points where \( g(x) = 0 \).

**Example 8.4.1.** Consider the two functions \( f \) and \( g \) defined by \( f(x) = \frac{1}{\sqrt{x-2}} \) and \( g(x) = \sqrt{2x+4} \). The domain of \( f \) is the set of all real numbers \( x \) with \( x - 2 > 0 \) (strict inequality since the denominator cannot be zero). This is equivalent to \( x > 2 \). The domain of \( f \) is \( D_f = ]2, +\infty[ \). Similarly, the domain of \( g \) is \( D_g = [-2, +\infty[ \).

The intersection of the two domains is \( D_f \cap D_g = ]2, +\infty[ \). We know that each of the functions \( f + g, f - g, fg \) has the interval \( ]2, +\infty[ \) as the domain.
Example 8.4.2. Consider the \( f \) and \( g \) defined by \( f(x) = \sqrt{x^2 - 4} \) and \( g(x) = x + 3 \). The domain of \( f \) is determined by the inequality \( x^2 - 4 \geq 0 \):

\[
\begin{array}{|c|cccc|}
\hline
x & -\infty & -2 & 2 & +\infty \\
\hline
x^2 - 4 & + & 0 & - & 0 & + \\
\hline
\end{array}
\]

So \( D_f = (-\infty, -2] \cup [2, +\infty) \). The domain of the function \( g \) is clearly \( D_g = [-3, +\infty) \).

The intersection of the two domains is \( D_f \cap D_g = [-3, -2] \cup [2, +\infty) \). The function \( \frac{f}{g} \) is defined by \( \frac{f}{g}(x) = \frac{\sqrt{x^2 - 4}}{x+3} \) and its domain is \( [-3, -2] \cup [2, +\infty) \) from which we remove the value that makes \( g \) equals zero, namely \( x = -3 \). So the domain of the function \( \frac{\sqrt{x^2 - 4}}{x+3} \) is \( [-3, -2] \cup [2, +\infty) \).

8.4.2 Composition of functions

In the previous section, we saw that it is possible to combine two given functions and get a new one by adding, subtracting, multiplying or dividing the two functions.

There is another way of combining two functions to create a new one. It is called the **composition of functions**. It is a mechanism through which we substitute an entire function into another one. Unlike the previous ways of combining functions, the composition does not always exists. So let’s start by explaining the condition required for the existence of the composition.

Assume \( f : A \to B \), \( g : C \to D \) are two functions with \( B \subseteq C \) (so the codomain of \( f \) is a subset of the domain of \( g \)). For any \( x \in A \), \( f(x) \) is an element of \( B \). Since \( C \subseteq D \), \( f(x) \) belongs to the domain of \( g \). So we can take the image of \( f \) under the function \( f \), namely \( g(f(x)) \) and we get an element of the set \( D \). The expression \( g(f(x)) \) defines a function from the set \( A \) to the set \( D \) that we denote with \( g \circ f \) and we call the composition of the two functions. Formally, we have the following definition.

**Definition 8.4.1.** Let \( f : A \to B \) and \( g : C \to D \) be two functions. If \( C \subseteq D \) (the codomain of \( f \) is a subset of the domain of \( g \)), then the composition \( g \circ f : A \to D \) exists and it is the function defined by the rule \( g \circ f(x) = g(f(x)) \) for all \( x \in A \).
In more practical terms, the expression \( g(f(x)) \) means go to the function \( g \) first and everywhere there is an \( x \), replace it with \( f(x) \).

**Warning 8.4.1.**

- Don’t treat the composition \( g \circ f \) of the two functions \( f \) and \( g \) as the multiplication of the functions. These are two different operations producing completely different new functions. One clear difference between the two operations is the fact that multiplication \( fg \) is commutative: \( fg = gf \) while composition is not: \( g \circ f \neq f \circ g \) in general (whenever both functions \( g \circ f \) and \( f \circ g \) are well defined).
- It could very well happen that \( f \circ g \) exists but \( g \circ f \) doesn’t.

**Example 8.4.3.** Consider the two functions \( f(x) = \frac{1}{x+1} \) and \( g(x) = x^2 \). Clearly, the domain of the function \( f \) is \( D_f = ]-\infty, -1[ \cup ]1, +\infty[ \), that is all the real numbers except \(-1\) since we cannot have a zero as a denominator. The range of \( f \) is \( ]-\infty, 0[ \cup ]0, +\infty[ \) since the only value the function cannot take is 0 (the numerator is equal to 1). As for the function \( g \), the domain is clearly \( \mathbb{R} \) (all real numbers) and the range is all non-negative real numbers: \( [0, +\infty[ \). The range of \( g \) is then a subset of the domain of \( f \) and the range of \( f \) is a subset of the domain of \( g \) which makes both functions

\[
\begin{align*}
  f \circ g : \mathbb{R} &\rightarrow ]-\infty, 0[ \cup ]0, +\infty[ \quad \text{and} \quad g \circ f : ]-\infty, -1[ \cup ]1, +\infty[ \rightarrow ]0, +\infty[ \end{align*}
\]

well defined functions. If \( x \in \mathbb{R} \), then \((f \circ g)(x) = f(g(x)) = f(x^2) = \frac{1}{x^2 + 1}\). The domain of this function is clearly \( \mathbb{R} \) since the denominator is never 0. On the other hand, \((g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x+1}\right) = \left(\frac{1}{x+1}\right)^2 = \frac{1}{(x+1)^2}\). The domain of this function is the same as the domain of \( f \), namely \( ]-\infty, -1[ \cup ]1, +\infty[ \) (but this is not a general rule). Note that in this case, \( f \circ g \) and \( g \circ f \) are two different functions.

**Example 8.4.4.** In this example, let \( f(x) = \sqrt{x + 1} \) and \( g(x) = x^3 \). The domain of \( f \) consists of all real numbers \( x \) satisfying \( x + 1 \geq 0 \) or \( x \geq -1 \) and its range is \( ]0, +\infty[ \). For the function \( g \), both the domain and the range are equal to \( \mathbb{R} \). Since the range of \( g \) is not a subset of the domain of \( f \), the function \( f \circ g \) does not exist (undefined). For instance, \((f \circ g)(-2) = f(g(-2)) = f(-8) = \sqrt{-8 + 1} = \sqrt{-7} \) which is not a real number. However, if we restrict ourselves to the interval \([1, +\infty[ \), then \((f \circ g)(x) = f(g(x)) = f(x^3) = \sqrt{x^3 + 1} \) exists in this case since \( x^3 + 1 \geq 0 \). On the other hand, the range of \( f \) is a subset of the domain of \( g \), so \( g \circ f : ]-\infty, +\infty[ \rightarrow \mathbb{R} \) exists and for any \( x \geq -1 \), and we have \((g \circ f)(x) = g(f(x)) = g(\sqrt{x+1}) = (\sqrt{x+1})^3 = (x+1)^{\frac{3}{2}}\).

**Remarks 8.4.1.**

1. The composition of three or more functions is defined the same way as the case of two functions (just make sure in each case, the composition is well defined). If \( f, g, h \) are three functions, then

\[
(f \circ g \circ h)(x) = f(g(h(x)))
\]

Apply \( h \) first to \( x \) to get \( h(x) \), then apply \( g \) to \( h(x) \) to get \( g(h(x)) \) and apply \( f \) last to \( g(h(x)) \) to get \( f(g(h(x))) \).

2. If the range of a function \( f \) is a subset of its domain, then we can take the composition of \( f \) with itself.
For example, if \( f(x) = x^2 \), the range of \( f = [0, +\infty) \) is a subset of its domain (= \( \mathbb{R} \)) and we can consider the function \( f \circ f : \mathbb{R} \to [0, +\infty] \) defined by for any \( x \in \mathbb{R} \), \((f \circ f)(x) = f(f(x)) = f(x^2) = (x^2)^2 = x^4\).

8.5 Inverse function

If \( x \) is a (nonzero) real number, then one can talk about the "inverse" of \( x \) denoted by \( x^{-1} \). This is the unique real number with the property that \( xx^{-1} = 1 \). There is only one real number with no inverse, namely 0.

A similar notion of "invertibility" exists for functions. First we need to define the analogous for "1" for functions. If \( A \) is any set, the identity function of \( A \), denoted by \( 1_A \) is the function defined by \( 1_A(x) = x \) for all \( x \in A \) (hence the name "identity function"). Like the real number 1 for multiplication, the identity function is the "neutral element" for the composition functions in the following sense: if \( f : A \to B \) is any function, then \( f \circ 1_A = f \). Note that if \( f : A \to A \), then both \( f \circ 1_A \) and \( 1_A \circ f \) are well defined and we have \( f \circ 1_A = 1_A \circ f = f \).

Definition 8.5.1. Given a function \( f : A \to B \), we say that \( f \) is invertible if there exists a function \( g : B \to A \) such that \( f \circ g = 1_B \) and \( g \circ f = 1_A \). The inverse function \( g \) is usually denoted by \( f^{-1} \). So we have the following basic relations

\[
\text{If } f : A \to B \text{ is an invertible function, then } f \circ f^{-1} = 1_B \text{ and } f^{-1} \circ f = 1_A
\]

The following table gives the main properties of the inverse function.

<table>
<thead>
<tr>
<th>If } f : A \to B \text{ is an invertible function with inverse } f^{-1} : B \to A, \text{ then</th>
</tr>
</thead>
<tbody>
<tr>
<td>• } y = f(x) \Leftrightarrow x = f^{-1}(y)</td>
</tr>
<tr>
<td>• } f(f^{-1}(y)) = y \text{ for all } y \in B \text{ and } f^{-1}(f(x)) = x \text{ for all } x \in A</td>
</tr>
<tr>
<td>• } The domain of } f^{-1} \text{ is equal to the range of } f</td>
</tr>
<tr>
<td>• } The range of } f^{-1} \text{ is equal to the domain of } f</td>
</tr>
</tbody>
</table>

95
8.5.1 The graph of the inverse function

In many instances in your first year Calculus, a function \( y = f(x) \) from the real numbers to the real numbers is given by its graph rather than a formula. In this case, the function is invertible if and only if its graph has, for each value \( y \) in the range, a unique corresponding \( x \) value in the domain. In more simple terms, this means that the graph intersects any horizontal line at at most one point. This visual way to check invertibility of the function defined with a graph is known as the horizontal line test. If a function passes the horizontal line test, it is usually called one-to-one.

Example 8.5.1. The graphs of the functions \( f(x) \) and \( f(x) \) are given.

![Graphs of f(x) and g(x)](image)

Clearly, the graph of the function \( f \) intersects each horizontal line at either zero or one point (hence at most one point). So \( f^{-1} \) exists. Moreover, since \( f \) is from \( \mathbb{R} \) to \([0, +\infty[\), we know that \( f^{-1} \) is defined from \([0, +\infty[ \) to \( \mathbb{R} \). On the other hand, a horizontal line could clearly intersect the graph of \( g \) at up to four points. So the graph of \( g \) fails the horizontal line test and consequently, \( g \) is not invertible on its domain (\( \mathbb{R} \)).

Remark 8.5.1. The notion of "invertibility" of a function depends largely on the domain on which the function is defined. A function given by a formula of the form \( y = f(x) \) is not necessarily invertible on its entire domain, but often it is possible to find a portion of the graph that passes the horizontal line test and so the "restricted" function represented by that portion is invertible. For example, if the function \( g \) in

96
Example 8.5.2. In example 8.5.1 above, we saw that the graph of the function $f$ passes the horizontal line test and hence $f$ is an invertible function on its domain. To draw the graph of its inverse function $f^{-1}$,
we draw the reflection of the graph of \( f \) in the line \( y = x \):

\begin{center}
\begin{tikzpicture}
\draw[->] (-2,0) -- (2,0) node[right] {\( x \)};
\draw[->] (0,-2) -- (0,2) node[above] {\( y \)};
\draw[red, thick] plot[domain=-2:2,samples=100] function {x^2 + 1};
\draw[blue, thick] plot[domain=-2:2,samples=100] function {x};
\draw[green, thick] plot[domain=-2:2,samples=100] function {x^2 + 1} node[below right] {\( f^{-1}(x) \)};
\end{tikzpicture}
\end{center}

### 8.5.2 Finding the inverse function analytically

Although our initial definition of the notion of invertible function was a general one between two abstract sets, your first year calculus is mainly concerned with functions that map real numbers to real numbers. In this case, the functions are often explicitly defined with formulas, such as \( y = \frac{x}{x^2 + 1} \). As seen above, the function \( y = f(x) \) is invertible if and only if it is one-to-one, that is its graph passes the horizontal line test. Algebraically, this means that for each value \( y \) in the range, the equation \( y = f(x) \) has a unique solution for the variable \( x \). The following table gives the steps you need to follow in order to find a formula for \( f^{-1}(x) \).

**Steps to find the inverse of the function \( y = f(x) \) (if it exists).**

1. Replace \( f(x) \) with \( y \)
2. Solve for \( x \) in terms of \( y \) in \( y = f(x) \)
3. In your solution, replace every occurrence of \( x \) with \( y \) and every occurrence of \( y \) with \( x \)
4. Replacing \( y \) with \( f^{-1}(x) \) gives the formula for the inverse function.

**Examples 8.5.1.** For each of the following functions, determine the domain and the range. Then decide if the function is invertible on its domain. If you say it is not invertible, restrict the function to the largest domain on which it is invertible then find a formula of the inverse function. Specify clearly the domain
and range of the inverse function.

\[
(1) \quad f(x) = 2x + 3 \quad (2) \quad g(x) = \frac{2x-3}{3x-4} \quad (3) \quad h(x) = x^2 + 1
\]

**Solution.**  (1) The domain and the range of the function \(f\) are both equal to \(\mathbb{R}\). Note also that the graph of \(f\) is a line which clearly satisfies the horizontal line test. Start by replacing \(f(x)\) with \(y\):

\[
y = 2x + 3 \quad \Leftrightarrow \quad x = \frac{y-3}{2}.
\]

Now, interchange the roles of \(x\) and \(y\) in the above solution and get
\[
y = \frac{4y-3}{3y-2}.
\]

Since \(y \neq \frac{2}{3}\), \(x = \frac{4y-3}{3y-2}\) exists and is unique such that \(y = g(x)\). So \(g\) is invertible on its domain. Interchanging the roles of \(x\) and \(y\) in \(x = \frac{4y-3}{3y-2}\) gives \(y = \frac{4x-3}{3x-2}\) and so \(f^{-1}(x) = \frac{4x-3}{3x-2}\). The domain of \(f^{-1}\) is \(]-\infty, -\frac{2}{3}[ \cup ]-\frac{2}{3}, +\infty[\) and its range is \(]-\infty, -\frac{2}{3}[ \cup ]-\frac{2}{3}, +\infty[\) (see part (1) of Example 8.3.1 above). Fix a value \(y\) in the range of \(g\) (that is \(y \neq \frac{2}{3}\)) and let us see if we can compute a unique value \(x\) in the domain satisfying \(y = g(x)\):

\[
y = \frac{2x-3}{3x-4} \quad \text{(replace } g(x) \text{ with } y) \quad \Leftrightarrow \\
(3x-4)y = 2x-3 \quad \Leftrightarrow \\
3xy - 4y = 2x - 3 \quad \Leftrightarrow \\
3xy - 2x = 4y - 3 \quad \Leftrightarrow \\
x = \frac{4y-3}{3y-2}
\]

(3) The domain of \(h(x)\) is the all real numbers \(\mathbb{R}\) and since \(x^2 + 1 \geq 1\), the range is \([1, +\infty[\). The graph of the function \(h(x)\) is a parabola that we know how to draw (see section 4.1.5).
Clearly the graph does not satisfy the horizontal line test, so the function \( h(x) \) is not invertible on its domain. Algebraically, try to solve for \( x \) in the equation \( y = x^2 + 1 \), you find that \( x^2 = y - 1 \) or \( x = \pm \sqrt{y - 1} \). For any given value of \( y \), we actually have two different values for \( x \), one with negative square root and the other with the positive square root. However, if we restrict the function to either one of the intervals \( (-\infty, 0] \) or \([0, +\infty[ \), then the graph will clearly pass the horizontal line test and the function would be invertible. The inverse function on \( (-\infty, 0] \) is given by \( y = -\sqrt{x - 1} \) with domain \([1, +\infty[ \) and range \( (-\infty, 0] \). The inverse function on \([0, +\infty[ \) is given by \( y = \sqrt{x - 1} \) with domain \([1, +\infty[ \) and range \([0, +\infty[ \).

**Try it yourself 8.5.1.** For each of the following functions, determine the domain and the range. Then decide if the function is invertible on its domain. If you say it is not invertible, restrict the function to the largest domain on which it is invertible then find a formula of the inverse function. Specify clearly the domain and range of the inverse function.

1. \( f(x) = -3x + 4 \)  
2. \( g(x) = \frac{3x^2 + 2}{x^2 - 3} \)  
3. \( h(x) = -x^2 + 1 \)
Chapter 9

Trigonometric Functions

Primarily, trigonometry was founded in ancient Greece (around two millennia ago) to deal with the relationships between the measurement of the sides and angles of triangles. In fact, the origin of the word "Trigonometry" is in the Greek language, which means "the measurement of triangles". With the revolution Calculus had experienced in the 17th century, classical trigonometric identities started to be viewed and treated from functions perspective. This opened the door for a new array of physical phenomena that can be modeled by trigonometric functions.

Trigonometric Functions are probably the type of functions that often cause a lot of confusion and stress for students. But because of their periodical nature, they are also one of the most useful tools to represent many real life phenomena. This is why you are likely to see them popping up over and over not only in your math courses but in other Science and Engineering courses as well.

Your first year Calculus and/or Linear Algebra Instructors will likely review these functions and some of their properties but will not spend a lot of time doing that assuming that you have seen them in your High School Math courses. This chapter begins with a small discussion on angles and how they are measured. We then introduce the trigonometric functions in terms of ratios of the sides of a right angled triangle and in terms of coordinates of points on a circle and we explore their basic properties. After extending the notion of trigonometric functions to any real number, we explore more properties of these functions and their graphs.
9.1 the Measurement of Angles, Degrees versus Radians

An angle is defined to be the set of all points between two part-lines \( l_1 \) and \( l_2 \) meeting at a point \( O \) extending infinitely in one direction. The two part-lines are called the rays of the angle. If \( A \) is a point of the first ray \( l_1 \) and \( B \) is a point on \( l_2 \), then we denote the angle between the two rays by \( \angle AOB \). Very often, the angle \( \angle AOB \) is interpreted as the rotation of the ray \( l_1 \) onto ray \( l_2 \). In this case, we say that \( l_1 \) is the initial side of the angle and \( l_2 \) is the terminal side. Starting at the initial side of the angle, we say that the angle is positive if the rotation is counterclockwise and we say it is negative if the rotation is clockwise.

In the cartesian plane with \( x \) and the \( y \) coordinate axes meeting at the point \( O \) (the origin), then we get the standard position of an angle by taking the origin as the vertex of the angle and letting its the initial side \( l_1 \) coincides with the \( x \)-axis. By the above convection, if \( l_1 \) is rotated counterclockwise from the standard position, then it is considered positive and if it is rotated clockwise, it is considered negative.
9.1.1 Radian

You have probably always used degrees (°) to measure an angle. Calculus-based university courses rarely use degrees as measure of an angle, rather a second unit called Radian (rad) is widely used in these courses. Using radians would allows a more smooth extension of the domain of trigonometric functions to all real numbers. Also, later on you will see that known formulas involving the rate of change of trigonometric functions are only valid when the angles are expressed in radians rather than degrees. For now, let us, once and for all, try to understand this notion of radian so we can move on.

Imagine a straight radius of a circle is lifted off, properly banded and placed on the curve of the circumference of the circle. The angle at the center of circle bounded by the two end points of the curve, called a central angle, has a measure of one radian, written 1 rad for short.

9.1.2 Radian versus degree

The above definition of radian allows us to establish a relation between the two units of measurement of angles (radian versus degree). To start, consider a circle of radius \( r \). Recall that a complete tour of the circle corresponds to a central angle of 360°. Since the circumference of the circle is equal to \( 2\pi r \), it follows that the number of times one can "wrap" the radius around the circumference is \( 2\pi \) (just over 6 times) and the central angle corresponding to a full revolution is then equal to \( 2\pi \) radians. In other words, \( 2\pi \) radians is equivalent to 360° or, \( \pi \) radians is equivalent to 180°. This is the basic relation between the two units of measurement of an angle:

<table>
<thead>
<tr>
<th>Radian versus Degree.</th>
</tr>
</thead>
<tbody>
<tr>
<td>• If ( \theta ) is an angle measured in degree, then ( \frac{\pi}{180} \theta ) is the value of the angle in radians</td>
</tr>
<tr>
<td>• If ( \theta ) is an angle measured in radians, then ( \frac{180}{\pi} \theta ) is the value of the angle in degrees</td>
</tr>
</tbody>
</table>
Example 9.1.1. Each of the following angles are expressed in radians. Find their values in degrees.

(1) \( \frac{\pi}{12} \)  (2) \( \frac{\pi}{5} \)  (3) \( \frac{7\pi}{6} \)

Solution.  
(1) \( \frac{\pi}{12} \) rad = \( \frac{180}{\pi} \times \frac{\pi}{12} \) = 15°
(2) \( \frac{\pi}{5} \) rad = \( \frac{180}{\pi} \times \frac{\pi}{5} \) = 36°
(3) \( \frac{7\pi}{6} \) rad = \( \frac{180}{\pi} \times \frac{7\pi}{6} \) = 210°

Example 9.1.2. Each of the following angles are expressed in degrees. Find their values in radians.

(1) 75  (2) 135  (3) 225

Solution.  
(1) 75° = \( \frac{75}{180} \pi \) rad = \( \frac{25}{36} \pi \) rad
(2) 135° = \( \frac{135}{180} \) rad = \( \frac{3}{4} \pi \) rad
(3) 225° = \( \frac{225}{180} \) rad = \( \frac{5}{4} \pi \) rad

Definition 9.1.1. If 0 < \( \theta \) < 90°, we say that \( \theta \) an acute angle; \( \theta \) is called obtuse if 90° < \( \theta \) < 180°. Two angles \( \alpha \) and \( \beta \) are called complementary if \( \alpha + \beta = 90° \) and they are called supplementary if \( \alpha + \beta = 90° \).

If there is no unit attached to use angle, you can assume that the angle is measured in radians. In most Math courses, angles are almost always measured in radians. The following tables gives the values of some frequently used angles in radians and degrees.

<table>
<thead>
<tr>
<th>Radians</th>
<th>Degrees</th>
<th>Radians</th>
<th>Degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\pi}{6} )</td>
<td>30</td>
<td>( \frac{5\pi}{6} )</td>
<td>150</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
<td>45</td>
<td>( \pi )</td>
<td>180</td>
</tr>
<tr>
<td>( \frac{\pi}{3} )</td>
<td>60</td>
<td>( \frac{2\pi}{3} )</td>
<td>210</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>90</td>
<td>( \frac{5\pi}{4} )</td>
<td>225</td>
</tr>
<tr>
<td>( \frac{2\pi}{3} )</td>
<td>120</td>
<td>( \frac{\pi}{2} )</td>
<td>240</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>135</td>
<td>( 2\pi )</td>
<td>360</td>
</tr>
</tbody>
</table>

9.2 The Trigonometric Functions

There are two (equivalent) ways of defining Trigonometric Functions, the right-angled triangle definitions and the (unit) circle definition.
9.2.1 The right-angled triangle definitions

If \( \theta \) is one of the two acute angles shown in the right-angled triangle to the right, the table on the left defines the three basic trigonometric functions of the angle \( \theta \), namely the sine function (\( \sin x \)), the cosine function (\( \cos x \)) and the tangent function (\( \tan x \)). These functions are simply defined as ratios of the side lengths \( a, b \) and the hypotenuse \( c \) of the triangle.

<table>
<thead>
<tr>
<th>Basic Trigonometric Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{a}{c} )</td>
</tr>
<tr>
<td>( \cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{b}{c} )</td>
</tr>
<tr>
<td>( \tan \theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{a}{b} = \frac{\sin \theta}{\cos \theta} )</td>
</tr>
</tbody>
</table>

Other trigonometric functions include the cotangent function (\( \cot \theta \)), the cosecant function (\( \csc \theta \)) and the secant function (\( \sec \theta \)). The definition of each is given in the following table.

<table>
<thead>
<tr>
<th>Other Trigonometric Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cot(\theta) = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta} )</td>
</tr>
<tr>
<td>( \csc \theta = \frac{1}{\sin \theta} )</td>
</tr>
<tr>
<td>( \sec \theta = \frac{1}{\cos \theta} )</td>
</tr>
</tbody>
</table>

9.2.2 The Unit circle definitions

How do we define these functions for a non acute angles or for negative angles? To answer this question, we define the trigonometric functions from the perspective of the coordinates of a point on a circle. Although the circle definitions of trigonometric functions can be done using a circle centered at the origin and with arbitrary radius, they are most simply defined using the unit circle. That is the circle in the cartesian plane centered at the origin \( O \) and of radius 1.

Given any angle \( 0 \leq \theta \leq 2\pi \), we consider the point \( P(x, y) \) on the unit circle such that the angle between
the positive $x$-axis and the line $OP$ is equal to $\theta$. Let $Q(x,0)$ be the projection of the point $P$ on the $x$-axis.

We define the trigonometric functions of the angle $\theta$ as follows

<table>
<thead>
<tr>
<th>Unit circle definitions of trigonometric functions.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta = y, \cos \theta = x, \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{y}{x}$</td>
</tr>
<tr>
<td>$\cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{x}{y}, \csc x = \frac{1}{\sin x} = \frac{1}{y}, \sec x = \frac{1}{\cos x} = \frac{1}{x}$</td>
</tr>
</tbody>
</table>

**Remark 9.2.1.** In the above definition, one can use a circle of radius $r$ in general (rather than the unit circle). In this case, if $P(x, y)$ is a point on the circle, then $\cos \theta = \frac{x}{r}$ and $\sin \theta = \frac{y}{r}$. The other definitions follow easily.

### 9.3 Basic properties of the Trigonometric functions

Immediate consequences of the above definitions are the following:

1. Since the coordinates of the point $P(x, y)$ on the unit circle satisfy $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$, then $-1 \leq \cos \theta \leq 1$ and $-1 \leq \sin \theta \leq 1$.

2. $\cos^2 \theta + \sin^2 \theta = x^2 + y^2 = 1$ since $P(x, y)$ is on the unit circle of equation $x^2 + y^2 = 1$. Therefore $\sin^2 \theta + \cos^2 \theta = 1$.
3. Knowing the signs of basic trigonometric functions is required in many situations in various math and physics courses. Given an angle \( \theta \in [0, 2\pi] \), the following table gives the signs of \( \sin \theta \), \( \cos \theta \) and \( \tan \theta \) in each of the four quadrants of the cartesian plane.

<table>
<thead>
<tr>
<th></th>
<th>( 0 &lt; \theta &lt; \frac{\pi}{2} )</th>
<th>( \frac{\pi}{2} &lt; \theta &lt; \pi )</th>
<th>( \pi &lt; \theta &lt; \frac{3\pi}{2} )</th>
<th>( \frac{3\pi}{2} &lt; \theta &lt; 2\pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \theta )</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( \cos \theta )</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( \tan \theta )</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

4. If the angle \( \theta \) is represented by the point \( P(x, y) \) in the cartesian plane, then the points \( Q(x, -y) \), \( S(-x, y) \) and \( R(-x, -y) \) represent the angles \( -\theta \), \( \pi - \theta \) and \( \pi + \theta \) respectively. Using the unit circle definitions of \( \sin \) and \( \cos \) and the following diagram:

![Diagram of the unit circle with angles and points labeled](image)

we immediately get the following trigonometric identities:

- \( \sin(-\theta) = -\sin \theta \)
- \( \cos(-\theta) = \cos \theta \)
- \( \tan(-\theta) = -\tan \theta \)
- \( \sin(\pi - \theta) = \sin \theta \)
- \( \cos(\pi - \theta) = -\cos \theta \)
- \( \tan(\pi - \theta) = -\tan \theta \)
- \( \sin(\pi + \theta) = -\sin \theta \)
- \( \cos(\pi + \theta) = -\cos \theta \)
- \( \tan(\pi + \theta) = \tan \theta \)
- \( \sin(2\pi - \theta) = \sin \theta \)
- \( \cos(2\pi - \theta) = \cos \theta \)
- \( \tan(2\pi - \theta) = \tan \theta \)

5. The last row in Table 9.3.2 above can be generalized as follows: \( \sin(\theta + n\pi) = \sin \theta \), \( \cos(\theta + n\pi) = \cos \theta \), \( \tan(\theta + n\pi) = \tan \theta \) for any integer \( n \). This should come as no surprise as \( n\pi \) represents \( n \) complete tours of the circle. We say in this case that \( \sin x \) and \( \cos x \) are **periodic functions** with **period** equal to \( 2\pi \).
9.3.1 Some remarkable angles.

Very often, you are not allowed to use a calculator on a test. The questions are usually designed so that you only need to memorize the trigonometric functions of few special angles that we call "remarkable".

**bullet** For \( \theta = 0 \), the corresponding point on the unit circle is the point \( P(1,0) \). Therefore, \( \cos 0 = 1 \) and \( \sin 0 = 0 \). Consequently, \( \tan 0 = 0 \) and \( \cot 0 \) is undefined.

• For \( \theta = \frac{\pi}{6} \) rad (or 30°), let \( P(x,y) \) be the corresponding point on the unit circle and consider the two points \( Q(x,0) \) and \( R(x,-y) \).

![Diagram](image-url)

The triangle \( OPR \) is clearly an equilateral triangle (since each of its interior angles is equal to 60°). Consequently, \( PQ = \frac{1}{2} PR = \frac{1}{2} OP = \frac{1}{2} \) and \( \sin \left( \frac{\pi}{6} \right) = \frac{QO}{OP} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \). By Pythagorean Theorem, \( OQ^2 = 1^2 - QP^2 = 1 - \frac{1}{4} = \frac{3}{4} \) and therefore \( OQ = \frac{\sqrt{3}}{2} \). Consequently, \( \cos \left( \frac{\pi}{6} \right) = \frac{OQ}{OP} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \). Now, \( \tan \left( \frac{\pi}{6} \right) = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \) and \( \cot \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2} \).
For $\theta = \frac{\pi}{4}$ rad (or 45°), the triangle $OPR$ is right angled at the point $O$.

By the Pythagorean theorem, $PR^2 = 1^2 + 1^2 = 2$ and $PR = \sqrt{2}$. Consequently, $PQ = \frac{1}{2}PR = \frac{\sqrt{2}}{2}$. Since $\triangle OQP$ is a $45° - 45° - 90°$ triangle, $OQ = PQ = \frac{\sqrt{2}}{2}$, we conclude that $\sin\left(\frac{\pi}{4}\right) = PQ = \frac{\sqrt{2}}{2}$ and $\cos\left(\frac{\pi}{4}\right) = OQ = \frac{\sqrt{2}}{2}$. Therefore, $\tan\left(\frac{\pi}{4}\right) = \cot\left(\frac{\pi}{4}\right) = 1$.

For $\theta = \frac{\pi}{3}$ rad (or 60°), we can use the work done above to compute the trigonometric functions of the angle $\frac{\pi}{6}$ = 30° together with the right-angled triangle definition of an acute angle (remember that 60° is acute). Let us consider a $30° - 60° - 90°$ triangle. Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, we assume that the hypotenuse is of length 2 and the other two sides of the right-angled triangle are 1 and $\sqrt{3}$.

We conclude that $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$. Now, $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ and $\cot\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}$.

For $\theta = \frac{\pi}{2}$ rad (or 90°), the corresponding point on the unit circle is the point $P(0, 1)$. Therefore, $\sin\left(\frac{\pi}{2}\right) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$. Consequently, $\tan\left(\frac{\pi}{2}\right)$ is undefined.

For $\theta = \pi$ rad (or 180°), the corresponding point on the unit circle is the point $P(-1, 0)$. So, $\sin\left(\frac{\pi}{2}\right) = 0$ and $\cos\left(\frac{\pi}{2}\right) = -1$. Consequently, $\tan(\pi) = 0$.

The following table gives a summary of the trigonometric ratios of the special angles computed above.
Using the above identities and the remarkable angles, we can get the trigonometric functions of much more angles than those shown in the above table. Here are few e xamples.

**Examples 9.3.1.**

1. \( \sin \left( -\frac{\pi}{6} \right) = -\sin \left( \frac{\pi}{6} \right) = -\frac{1}{2} \) and \( \cos \left( -\frac{\pi}{6} \right) = \cos \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2} \).

2. \( \sin \left( \frac{2\pi}{3} \right) = \sin \left( \pi - \frac{\pi}{3} \right) = \sin \left( \frac{\pi}{3} \right) = \frac{\sqrt{3}}{2} \) and \( \cos \left( \frac{2\pi}{3} \right) = \cos \left( \pi - \frac{\pi}{3} \right) = -\cos \left( \frac{\pi}{3} \right) = -\frac{1}{2} \).

Using the above established results and identities, you should be able to see how one get the sine and the cosine of each of the angles shown in the following diagram. Each of the angles is represented by its terminal point on the unit circle. The first coordinate of the terminal point gives the value of the sine of the angle and the second gives the value of its cosine. For instance, \( \sin \left( \frac{11\pi}{6} \right) = \frac{\sqrt{3}}{2} \) and \( \cos \left( \frac{11\pi}{6} \right) = -\frac{1}{2} \).
Try it yourself 9.3.1. Find the trigonometric functions of each of the following angles.

(a) $\frac{\pi}{6}$  
(b) $\frac{5\pi}{6}$  
(c) $\frac{3\pi}{4}$  
(d) $-\frac{7\pi}{6}$

9.3.2 Other trigonometric identities.

In addition to the trigonometric identities and properties we have seen so far, the following table gives some more useful trigonometric identities that you will more likely use in your Linear Algebra course (especially to establish formulas for complex numbers) and Calculus II and III courses.

<table>
<thead>
<tr>
<th>Other Trigonometric Identities.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)$, $\sin(\alpha - \beta) = \sin(\alpha) \cos(\beta) - \cos(\alpha) \sin(\beta)$</td>
</tr>
<tr>
<td>$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$, $\cos(\alpha - \beta) = \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta)$</td>
</tr>
<tr>
<td>$\sin(2\alpha) = 2\sin(\alpha) \cos(\alpha)$, $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha)$</td>
</tr>
</tbody>
</table>

The last two formulas in the above table are called the **Double angle Formulas**.

Using identities in Table 9.3.3 and the trigonometric functions for $\pm \frac{\pi}{2}$, we get:

$$\sin \left( \theta + \frac{\pi}{2} \right) = \sin(\theta) \cos \left( \frac{\pi}{2} \right) + \cos(\theta) \sin \left( \frac{\pi}{2} \right)$$

$$= \cos(\theta).$$

Similarly, one can verify the following identities.

<table>
<thead>
<tr>
<th>Table 9.3.4.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \left( \frac{\pi}{4} + \theta \right) = \cos(\theta)$, $\sin \left( \frac{\pi}{4} - \theta \right) = \cos(\theta)$</td>
</tr>
<tr>
<td>$\cos \left( \frac{\pi}{4} + \theta \right) = -\sin(\theta)$, $\cos \left( \frac{\pi}{4} - \theta \right) = \sin(\theta)$</td>
</tr>
</tbody>
</table>
Examples 9.3.2.

1. \( \cos \left( \frac{\pi}{6} \right) = \cos \left( \frac{\pi}{12} \right) \). Therefore, \( \cos^2 \left( \frac{\pi}{12} \right) = \frac{\cos(\pi/6) + 1}{2} = \frac{\sqrt{3} + 1}{2} = \frac{\sqrt{3} + 2}{4} \). In theory, we have two possible values of \( \cos \left( \frac{\pi}{12} \right) \), namely \( \pm \frac{\sqrt{3} + 2}{4} \). But since \( 0 \leq \frac{\pi}{12} \leq \frac{\pi}{2} \), we know (from Table 9.3.1 above) that \( \cos \left( \frac{\pi}{12} \right) > 0 \). We conclude that \( \cos \left( \frac{\pi}{12} \right) = \frac{\sqrt{3} + 2}{4} \).

Alternatively,

\[
\cos \left( \frac{\pi}{12} \right) = \cos \left( \frac{\pi}{3} - \frac{\pi}{4} \right) = \cos \left( \frac{\pi}{3} \right) \cos \left( \frac{\pi}{4} \right) + \sin \left( \frac{\pi}{3} \right) \sin \left( \frac{\pi}{4} \right) = \frac{1}{2} \sqrt{2} + \frac{\sqrt{3}}{2} \sqrt{2} = \frac{\sqrt{3} + 2}{4}.
\]

2. \( \sin \left( \frac{7\pi}{12} \right) = \sin \left( \frac{\pi}{3} + \frac{\pi}{4} \right) = \sin \left( \frac{\pi}{3} \right) \cos \left( \frac{\pi}{4} \right) + \cos \left( \frac{\pi}{3} \right) \sin \left( \frac{\pi}{4} \right) = \frac{\sqrt{3}}{2} \frac{1}{2} + \frac{1}{2} \frac{\sqrt{2}}{2} = \frac{\sqrt{3} + \sqrt{2}}{4} \).

Exercises 9.3.1. Find the trigonometric functions of each of the following angles.

(a) \( \frac{5\pi}{12} \)  
(b) \( \frac{7\pi}{12} \)  
(c) \( \frac{3\pi}{8} \)  
(d) \( -\frac{\pi}{12} \)

9.4 Graphs of trigonometric functions.

In this section, all angles are assumed to be measured in radians.

We started out this chapter dealing with trigonometric functions of "angles". But if you think of any real number as being an amount of "radians", then really the domain of trigonometric functions can be expanded to (almost) all real numbers, just like many other functions we treated so far. For instance, when we write \( \sin \left( \frac{\pi}{3} \right) \), we are actually evaluating the sine of the angle \( \frac{\pi}{3} \) radians, or about 134°. This gives a new perspective on these ratios and allows to draw their graphs outside of the interval \([0, 2\pi]\).

9.4.1 Graphs of the sine and cosine functions

Let us start by looking at the two functions \( y = \sin x \) and \( y = \cos x \). From the discussions in the above sections, the following observations are drawn.

1. It is clear that if \( x \) is any real number (thought of as an angle in radian), then we can compute values for \( \sin x \) and \( \cos x \). Hence, the domain of each of these two functions is all real numbers \( \mathbb{R} \).
2. Since $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$ (see Section 9.3), the range of each of $\sin x$ and $\cos x$ is the interval $[-1, 1]$.

3. Note that $\sin(-2\pi) = 0$, $\sin(-\pi) = 0$, $\sin 0 = 0$, $\sin \pi = 0$, $\sin(2\pi) = 0$ and so on. In fact, $\sin x$ is zero if and only if $x$ is any integer multiple (positive or negative) of $\pi$:

\[ \sin x = 0 \Leftrightarrow x = k\pi, \ k \in \mathbb{Z}. \]

Graphically, this means that the graph of the function $y = \sin x$ intersects the $x$-axis at the points $x = k\pi, \ k \in \mathbb{Z}$.

4. We have $\sin\left(\frac{\pi}{2}\right) = 1$, $\sin\left(\frac{3\pi}{2}\right) = -1$, $\sin\left(\frac{5\pi}{2}\right) = 1$ and so on. In general, we have:

\[ \sin\left(\frac{\pi}{2} + k\pi\right) = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases} \]

5. For the function $\cos x$, we have

\[ \cos x = 0 \Leftrightarrow x = \frac{\pi}{2} + k\pi, \ k \in \mathbb{Z} \]

and

\[ \cos (k\pi) = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases} \]

6. As mentioned above, the functions $\sin x$ and $\cos x$ are periodic with period $2\pi$:

\[ \sin(x + 2k\pi) = \sin x \text{ and } \cos(x + 2k\pi) = \cos x \text{ for any } k \in \mathbb{Z}. \]

Graphically, this means that once the portions of the graphs of $\sin x$ and $\cos x$ in the interval $[0, 2\pi]$ are plotted, we get the full graphs of these two functions by extending the portions in both directions. The diagram below shows the graph of $y = \sin(x)$ with the bold part in red being the main graph on the interval $[0, 2\pi]$. 

![Graph of sin(x) and cos(x)](image-url)
Similarly, the graph of the function $y = \cos(x)$ is given.

![Graph of the function $y = \cos(x)$](image)

### 9.4.1.1 Sinusoidal function

A sinusoidal function is a function that behaves like a sine or a cosine function in the sense that the function is obtained by shifting, stretching or compressing the sine or the cosine function.

**Definition 9.4.1.** A **sinusoidal function** is a function that can be written under the standard form:

$$f(x) = A \cos(B(x - \phi)) + D$$

for some constants $A, B, \phi$ and $D$.

**Remark 9.4.1.** In case you are wondering why we call it sinusoidal function since its standard form involves the cosine function, note that one of the identities in Table 9.3.4 above gives us $\cos(\theta) = \sin\bigg(\frac{\pi}{2} - \theta\bigg)$, so the standard form of a sinusoidal function can be expressed as:

$$A \cos(B(x - \phi)) + D = A \sin\left(\frac{\pi}{2} + B(x - \phi)\right) + D$$

$$= A \sin(B(x + \frac{3\pi}{2} - \phi)) + D$$

$$= A \sin(B(x - \psi)) + D$$

for some constants $A, B, \psi$ and $D$ with $\psi = \phi - \frac{\pi}{2}$. This means that the standard form of a sinusoidal function can be defined as one or the other of the forms: $f(x) = A \cos(B(x - \phi)) + D$ or $f(x) = A \sin(B(x - \psi)) + D$. We will adopt the first form in these notes.

The graph of a sinusoidal function in standard form $f(x) = A \cos(B(x - \phi)) + D$ can be easily drawn using what we know so far about the properties of sine and cosine functions and the shifting and scaling of graphs done in Section 8.2.1. It might be a good idea that you review that section at this point.

First some observations.
If \( |\phi| < 1 \), then \(-|A| \leq \cos(B(x - \phi)) \leq |A|\) and therefore \(-|A| + D \leq \cos(B(x - \phi)) \leq |A| + D\). We conclude that the minimum value of the function \( f(x) \) is \(-|A| + D\) and its maximum value is \(|A| + D\). In particular the average value of \( f(x) \) is \( \frac{(-|A| + D) + (|A| + D)}{2} = D\).

The difference between the maximum and minimum values of \( f(x) \) is \((-|A| + D) + (|A| + D) = 2|A|\). Therefore

\[
|A| = \frac{(\text{Maximum value of } f(x)) - (\text{Minimum value of } f(x))}{2}
\]

\(|A|\) called the amplitude of \( f(x) \). Graphically, the amplitude of a periodic function is half the distance between the highest point and the lowest point of the function. For example, from the graphs of \( \sin(x) \) and \( \cos(x) \) above, clearly both have an amplitude of 1.

Since \( \cos(\alpha) = \cos(-\alpha) \), we can always assume that \( B > 0 \):

\[
\cos(B(x - \phi)) = \cos(-B(x - \phi)).
\]

If \( 0 < B < 1 \), the graph of \( \cos(Bx) \) is a horizontal stretch of the graph of \( y = \cos(x) \) away from the \( y \)-axis. If \( B > 1 \), the graph of \( \cos(Bx) \) is a horizontal compression of the graph of \( y = \cos(x) \) toward the \( y \)-axis. In addition, note that the period of the function \( f(x) = \cos(Bx) \) is \( \frac{2\pi}{B} \) since \( f\left(\frac{2\pi}{B}\right) = \cos\left(\frac{2\pi}{B}\right) = \cos(2\pi) = 1 \), so the value \( \frac{2\pi}{B} \) for the function \( \cos(Bx) \) is like the value \( 2\pi \) for the regular cosine function \( y = \cos(x) \). Since the period of the function \( f(x) = \cos(Bx) \) is \( T = \frac{2\pi}{B} \), the horizontal distance between two successive maxima or two successive minima is equal to \( T \). Moreover, The horizontal distance between a maxima and the successive minima is \( \frac{T}{2} \).

If \( 0 < A < 1 \), the graph of \( A\cos(x) \) is a vertical compression towards the \( x \)-axis of the graph of \( y = \cos(x) \). If \( 0A > 1 \), the graph of \( A\cos(x) \) is a vertical stretching away from the \( x \)-axis of the graph of \( y = \cos(x) \). If \( A < 0 \), the graph of \( A\cos(x) \) is a reflection in the \( x \)-axis of the graph of \( y = (-A)\cos(x) \) (note that \(-A \) is positive in this case).

If \( \phi > 0 \), the graph of \( y = \cos(x - \phi) \) is just a horizontal shift to the right by \( \phi \) units of the graph of \( y = \cos(x) \). If \( \phi < 0 \), the graph of \( y = \cos(x - \phi) \) is just a horizontal shift to the left by \(|\phi| \) units of the graph of \( y = \cos(x) \).

If \( D > 0 \), the graph of \( y = \cos(x) + D \) is just a vertical shift upwards by \( D \) units of the graph of \( y = \cos(x) \). If \( D < 0 \), the graph of \( y = \cos(x) + D \) is just a vertical shift downwards by \(|D| \) units of the graph of \( y = \cos(x) \).
For a sinusoidal function in standard form $f(x) = A\cos(B(x-\phi)) + D$:

- The domain is all real numbers.
- The range is $[-|A| + D, |A| + D]$.
- $|A|$ is the amplitude.
- $T = \frac{2\pi}{B}$ is the period.
- $D$ is the average value of the function.
- $\phi$ the phase shift.

Taking these observations into consideration, the following are easy steps to draw the graph of $f(x) = A\cos(B(x-\phi)) + D$. We will assume that $A$ is a positive constant.

- Start by drawing the three horizontal lines $y = D$, $y = -A + D$ and $y = A + D$. The line $y = D$ is the average line of the function, it will split the graph into symmetrical upper and lower halves. The two lines $y = -A + D$ and $y = A + D$ determine a horizontal strip inside which the graph of the sinusoidal function is drawn. The line $y = -A + D$ will "host" the peak points of the graph and the line $y = -A + D$ will "host" the minimal ones.

- For $x = \phi$, $f(x) = A + D$ (maximum value), so the point $(\phi, A + D)$ is a peak point for the graph. The graph starts at the point $(\phi, A + D)$, crosses the average line $y = D$ at the point $(\phi + \frac{T}{4}, D)$ on its way down to the successive minima $(\phi + \frac{T}{2}, -A + D)$. From the minima point, the graph goes up again, meets the average line $y = D$ once more at the point $(\phi + \frac{3T}{4}, D)$ on its way up to the successive peak point $(\phi + T, A + D)$.

- Now you have the portion of the graph on the interval $[\phi, \phi + T]$ of length equal to the period $T$.

- Once this portion of the graph is drawn, we obtain the full graph by simply repeating the portion on the intervals $[\phi + T, \phi + 2T]$, $[\phi + 2T, \phi + 3T]$, etc...
Example 9.4.1. Consider the sinusoidal function $f(x) = 3 \cos(2(x-1)) + 2$. We start by rewiring the function in standard form: $f(x) = 3 \cos(2(x-1)) + 2$. By comparing with the general standard form notations, we have that $A = 3$, $B = 2$, $\phi = 1$ and $D = 2$. We conclude that the amplitude is $A = 3$, the average value is $D = 2$, the period is $T = \frac{4\pi}{2} = \pi$ and the phase shift is $\phi = 1$. The maximum value of the function is $A + D = 5$ and the minimum value is $-A + D = -1$. We know that the graph will oscillate between the horizontal lines $y = -1$ and $y = 5$. We start a cycle of the graph at the maxima point $(\phi, A+D) = (1, 2)$. The graph decreases to the minima point $(\phi + \frac{T}{2}, -A+D) = (1 + \frac{\pi}{2}, -1)$ crossing the average line $y = 2$ at the point $(\phi + \frac{T}{4}, D) = (1 + \frac{\pi}{4}, 2)$. From the minima point, the graph goes up to the next peak point $(\phi + T, A+D) = (1 + \pi, 5)$ crossing the average line $y = 2$ at the point $(\phi + \frac{3T}{4}, D) = (1 + \frac{3\pi}{4}, 2)$. This completes a full cycle of the graph, shown in bold red in the diagram below. From the full cycle, one can extend the graph in both directions.
9.4.2 The graph of $\tan x$

Unlike the sine and cosine functions, the tangent function $y = \tan x$ is not defined for all $x$. The tangent function is defined as a ratio $\tan x = \frac{\sin x}{\cos x}$, so it is undefined at the points where $\cos x = 0$. These points are of the form $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$ (see above). In other words, the graph of $y = \tan x$ has vertical asymptotes at each of the lines $x = -\frac{3\pi}{2}$, $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, etc... On the other hand, $\tan x = 0 \Leftrightarrow \sin x = 0 \Leftrightarrow x = k\pi$, $k \in \mathbb{Z}$, so the graph of $\tan x$ intersects the x-axis at each of the integer multiples of $\pi$. Note also that by Table 9.3.2 above, $\sin(\pi + x) = -\sin(x)$ and $\cos(\pi + x) = -\cos(x)$ which means that $\tan(\pi + x) = \frac{-\sin(x)}{-\cos(x)} = \tan(x)$. The function $y = \tan(x)$ is then periodic with period $\pi$:

$$\tan(x + k\pi) = \tan x \text{ for any } k \in \mathbb{Z}$$

The graph of $y = \tan(x)$ looks as follows:

![Graph of tan(x)](image)

**Try it yourself 9.4.1.** Sketch the graph of each of the following functions.

(1) $2 \cos(2(x + \pi)) - 1$  (2) $\sin(x - 1) + 2$  (3) $\tan(x - 1)$  (4) $-3 \cos(2x + \pi)$  

9.5 Solving Trigonometric Equations

A trigonometric equation is an equation involving one or more of the trigonometric functions of the variable $x$. Solving a trigonometric equations is not an easy task in general. The good news is that you will
only deal with "manageable" ones in your first year Calculus or Linear Algebra courses. The following steps are often all what you need to know in order to solve trigonometric equations encounter in your first year math courses.

1. Using the above trigonometric identities, try to reduce the equation to a one containing a single trigonometric function and one angle.

2. Rewrite the equation so it has the form \( f(x) = A \) where \( f \) is one of the basic trigonometric functions and \( A \) is a constant.

3. Solve for the variable \( x \).

4. Take into account restrictions on the solution given in the question, if any. A rule of thumb when it comes to solving trigonometric equations is to find the solutions in an interval of length equal to the period of the main trigonometric function involved.

Let us look at some examples.

**Examples 9.5.1.** Solve each of the following trigonometric equations.

1) \( 2 \sin(2x) + 1 = 0 \) on \([0, 2\pi]\)  
2) \( 2 \sin^2 x + 5 \cos x - 5 = 0 \)  
3) \( \cos(x) = \sin(2x) \) on \([0, 3\pi]\)

**Solution.** 1) There is only one trigonometric function involved in this equation. We start by isolating \( \sin(2x) \):

\[
2 \sin(2x) + 1 = 0 \Leftrightarrow \sin(2x) = -\frac{1}{2}.
\]

Now, there are only two angles \( \theta \) in \([0, 2\pi]\) satisfying \( \sin \theta = -\frac{1}{2} \), namely \( \theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \) and \( \theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \).

This could be easily seen either by using the graph of the function \( y = \sin x \) on \([0, 2\pi]\) or by looking at the unit circle:

So we get:

\[
\sin(2x) = -\frac{1}{2} \Leftrightarrow 2x = \frac{7\pi}{6} \text{ or } 2x = \frac{11\pi}{6} \Leftrightarrow x = \frac{7\pi}{12} \text{ or } x = \frac{11\pi}{12}.
\]
The solutions of the equation $2\sin(2x) + 1 = 0$ in the interval $[0, 2\pi]$ are $x = \frac{7\pi}{12}$ and $x = \frac{11\pi}{12}$.

2). Unlike the first equation, $2\sin^2 x + 5\cos x - 5 = 0$ involves two trigonometric functions. We need first to arrange the equation so it contains only one of the trigonometric functions. Replacing $\sin^2 x$ with $1 - \cos^2 x$ in the equation gives

$$2(1 - \cos^2 x) + 5\cos x - 5 = 0 \iff -2\cos^2 x + 5\cos x - 3 = 0 \iff 2\cos^2 x - 5\cos x + 3 = 0.$$  \hspace{1cm} (1)

With a change of variables $t = \cos x$, the last equation in (1) can be transformed into the following quadratic equation in the variable $t$: $2t^2 - 5t + 3 = 0$. Using the quadratic formula (see section 5.2.1 above), we find the two solutions $t_1 = 1$ and $t_2 = \frac{3}{2}$. But $t = \cos x \in [-1, 1]$, so the only possible solution is $t_1 = 1$ ($t_2 = \frac{3}{2} \notin [-1, 1]$). Now, $t_1 = \cos x = 1 \iff x = \ldots, -2\pi, 0, 2\pi, 4\pi, 6\pi, \ldots$. Since there are no restrictions on $x$, the general solution of the equation $2\sin^2 x + 5\cos x - 5 = 0$ is $x = 2k\pi$ with $k \in \mathbb{Z}$.

3). We use the double angle identity given Table 9.3.3 above to rewrite our equation:

$$\cos x = \sin(2x) \iff \cos x = 2\sin x \cos x \iff 2\sin x \cos x - \cos x = 0 \iff \cos x(2\sin x - 1) = 0.$$  \hspace{1cm} (2)

The last equation in (2) implies that either $\cos x = 0$ or $2\sin x - 1 = 0 \iff \sin x = \frac{1}{2}$. In the interval $[0, 3\pi]$, there are three solutions to the equation $\cos x = 0$, namely $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, $x = \frac{5\pi}{2}$ and there are three solutions as well to the equation $\sin x = \frac{1}{2}$: $x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$ and $x = \frac{7\pi}{6}$. So there are in total six possible solutions to the equation $\cos x = \sin(2x)$ in the interval $[0, 3\pi]$.

**Try it yourself 9.5.1.** Solve each of the following trigonometric equations.

1) $\sqrt{3}\sin(3x) + 1 = 0$ on $[0, 2\pi]$ \hspace{1cm} 2) $3\cos^2 x - 2\sin x + 2 = 0$ \hspace{1cm} 3) $\sin x = \sin(2x)$ on $[0, 5\pi]$
Chapter 10

Logarithmic and Exponential functions

10.1 Exponential functions

Population growth, compound interest and radioactive decay are just few examples of phenomenon that can be modeled by a type of functions widely used in Science and engineering called the exponential functions.

Definition 10.1.1. Given a positive real number $a$, the function defined by $f(x) = a^x$ for any real number $x$ is called the exponential function with base $a$.

Remark 10.1.1. Very often, we compute the powers of a negative number like $(-2)^2 = 4$. But as a function, $a^x$ is defined only for positive bases $a$.

Given $a > 0$, the following are immediate consequences of the definition of an exponential function.

1. $a^x$ is defined for any $x \in \mathbb{R}$, which means that the domain of the exponential function is $\mathbb{R}$.

2. $a^x > 0$ (strict inequality) for any $x \in \mathbb{R}$, even for $x \leq 0$. For example, $2^{-3} = \frac{1}{2^3} = \frac{1}{8} > 0$. This means that the range of the function $a^x$ is the interval $]0, +\infty[.$

3. For $0 < a < 1$, $a^x$ decreases as $x$ increases. For example, $\left(\frac{1}{2}\right)^3 = \frac{1}{8} < \left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

4. For $a > 1$, it is the other way around: as $x$ increases, $a^x$ increases. For example, $2^2 = 4 < 2^3 = 8$. 

121
The last two properties suggest the following shapes of the graph of $a^x$ depending on the base $a$.

\[ y = a^x, \quad a > 1 \]

\[ y = a^x, \quad 0 < a < 1 \]

Note that in both cases, the graph passes through the point (0, 1). This is expected since $a^0 = 1$.

**Example 10.1.1.** The graphs of four functions $f, g, h$ and $k$ are given in the diagram below.

[Diagram showing four graphs labeled $f$, $g$, $h$, and $k$]

If we know that the four functions are $(0.1)^x$, $(0.5)^x$, $2^x$ and $5^x$. Which one is which?

**Solution.** First notice that $f$ and $g$ are decreasing functions and $h$ and $k$ are increasing. So, one of the two functions $f$ and $g$ is $(0.1)^x$ and the other one is $(0.5)^x$. Similarly, one of the two functions $h$ and $k$ is $2^x$ and the other one is $5^x$. But for $x \leq 0$, $(0.1)^x \geq (0.5)^x$ and for $x \geq 0$, $(0.1)^x \leq (0.5)^x$ (for example $(0.1)^{-2} = \frac{1}{(0.1)^2} = 10^2 = 100$ and $(0.5)^{-2} = \frac{1}{(0.5)^2} = 2^2 = 4$). This means that the graph of $(0.1)^x$ is above the graph of $(0.5)^x$ for $x \leq 0$ and it is below the graph of $(0.5)^x$ for $x \geq 0$. Therefore, $g(x) = (0.1)^x$ and $f(x) = (0.5)^x$. A similar argument shows that $h(x) = 5^x$ and $k(x) = 2^x$.

**Example 10.1.2.** Draw the graphs of the functions $f(x) = 3^x$ and $g(x) = 3^{-x}$ in the same system of axes.

**Solution.** We know that $f(x) = 3^x$ is an increasing exponential function as its base is greater than 1. As for the function $g(x)$, note that $g(x) = 3^{-x} = \frac{1}{3^x} = \left(\frac{1}{3}\right)^x$, this is a decreasing exponential function of base
The graphs look as follows.

10.1.1 Basic algebraic properties of the exponential function

Given a positive real number $a$, we saw in Chapter 3 the main properties of integer and rational exponents of $a$. Now that we have introduced the notion of exponential function, we extend these properties to any real exponent of $a$.

In the following table, $x$, $y$ are any real numbers and $a > 0$.

<table>
<thead>
<tr>
<th>Basic properties of the exponential function $a^x$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>• $a^{x+y} = a^x a^y$</td>
</tr>
<tr>
<td>• $a^{x-y} = \frac{a^x}{a^y}$</td>
</tr>
<tr>
<td>• $a^{-x} = \frac{1}{a^x}$</td>
</tr>
<tr>
<td>• $a^0 = 1$</td>
</tr>
<tr>
<td>• $a^x = a^y \Leftrightarrow x = y$</td>
</tr>
</tbody>
</table>

Example 10.1.3. Write each of the following expressions as one single exponential function.

(1) $\frac{8^{2x-3}}{2^{2x+1}}$ (2) $3^{x-2}27^{2-x}$ (3) $\frac{(3^{1-2x})^3}{27^{x-3}}$

Solution. Using the above properties, we have

(1)

\[
\frac{8^{2x-3}}{2^{2x+1}} = \frac{(2^3)^{2x-3}}{2^{2x+1}} = 3^{3(2x-3)} \cdot \frac{1}{3^{2x+1}} = 3^{6x-9} \cdot \frac{1}{3^{2x+1}} = \frac{3^{6x}}{3^7} = 3^{6x-7}
\]
In this case, the exponentiation is picked, namely that the goal is to solve for $x$.

Example 10.1.2. A word on $e$

We have seen that the exponential function $y = a^x$ is defined for any positive base $a$. In most applications you will encounter, a very particular base for the exponential is picked, namely $a \approx 2.718281828...$ This
is such an important number that we give it its own "letter": $e \approx 2.718281828\ldots$ and the corresponding function $y = e^x$ is often referred to as the natural exponential function. All the properties of exponential functions listed in the above table apply in particular to $e^x$ and since $e > 1$, the graph of $y = e^x$ is an increasing one.

10.2 The Logarithmic Functions

Many students start at the wrong foot with logarithms in high school. In most cases, students would simply memorize the Logarithmic rules without even understanding the definition of Logarithms. Guess what, they are coming to hunt you again in University, in more sophistication and depth. Applying the rules blindly is just a recipe for failure. So play it right this time and make sure you have a solid base on this important notion.

In the previous section, we saw that the exponential function $f(x) = a^x$ is either strictly increasing (if $a > 1$) or strictly decreasing (if $0 < a < 1$) on its domain (all real numbers). In both cases, the "horizontal line test" discussed in section 8.5.1 above shows that $a^x$ has an inverse function. This inverse function of $a^x$ is important enough to give it its own name, the logarithmic function with base $a$ that we usually denote by $\log_a(x)$.

Recall that if $f$ is an invertible function with inverse $f^{-1}$, then $y = f^{-1}(x)$ is equivalent to $x = f(y)$. This the same as saying $f^{-1}(f(x)) = x$, $f(f^{-1}(x)) = x$. In the context of exponential and logarithmic functions, these relations translate to the following in terms of these functions.

<table>
<thead>
<tr>
<th>Exponential and Logarithmic functions.</th>
</tr>
</thead>
<tbody>
<tr>
<td>For positive real numbers $a$ and $x$:</td>
</tr>
<tr>
<td>$y = \log_a(x) \iff a^y = x$</td>
</tr>
<tr>
<td>$\log_a(a^x) = x$ and $a^{\log_a(x)} = x$</td>
</tr>
</tbody>
</table>

In plain English, these equivalent properties say that $\log_a(x)$ is simply the power of $a$ needed to get $x$. In many Calculus textbooks, this is the definition given to $\log_a(x)$.

Examples 10.2.1.
1. \( \log_2 16 = 4 \) since \( 2^4 = 16 \) (the power of 2 needed to get 16 is 4)
2. \( \log_2 \left( \frac{1}{32} \right) = -5 \) since \( 2^{-5} = \frac{1}{32} \)
3. \( \log_{\frac{1}{2}} \left( \frac{1}{8} \right) = 3 \) since \( \left( \frac{1}{2} \right)^3 = \frac{1}{8} \)
4. \( \log_a(1) = 0 \) for any \( a > 0 \) since \( a^0 = 1 \)
5. \( \log_a(a) = 1 \) for any \( a > 0 \) since \( a^1 = a \).

Let \( a \) be a positive real number. Using properties of inverse functions (see section 8.5 above), the following conclusions can be drawn.

- The domain of \( \log_a(x) \) is equal to the range of the exponential function \( a^x \):
  
  \[ \text{The domain of } \log_a(x) \text{ is } ]0, \infty[. \]

  The expression \( \log_a(x) \) is only defined if \( x > 0 \)

- The range of \( \log_a(x) \) is equal to the domain of the exponential function \( a^x \):
  
  \[ \text{The range of } \log_a(x) \text{ is } ]-\infty, \infty[. \]

- The graph of \( \log_a(x) \) is the reflection of the graph of \( a^x \) in the line \( y = x \). Like the exponential function, the graph of \( \log_a(x) \) depends on the value of \( a \).

\[ y = \log_a(x), \ a > 1 \]
\[ y = \log_a(x), \ 0 < a < 1 \]

Note that the graph of \( \log_a(x) \) always crosses the \( x \)-axis at the point \( (1, 0) \).

10.2.1 Other properties of the Logarithmic function

Properties of exponential functions we have seen so far can be translated into properties of Logarithmic functions. Given positive real numbers \( a, x \) and \( y \), let \( \alpha = \log_a(xy), \beta = \log_a x \) and \( \gamma = \log_a y \). By definition of \( \log_a \), we have \( a^\alpha = xy, a^\beta = x \) and \( a^\gamma = y \).
• \( a^\alpha = x y = a^\beta a^\gamma \). Therefore, \( \alpha = \beta + \gamma \). In other words,

\[
\log_a (xy) = \log_a x + \log_a y.
\]

This is called **The Product Rule for Logarithms**.

• \( \frac{x}{y} = a^\beta - a^\gamma \). So \( \beta - \gamma \) is the power of \( a \) needed to get \( \frac{x}{y} \). By definition of \( \log_a \), this means that \( \beta - \gamma = \log_a \left( \frac{x}{y} \right) \) or equivalently,

\[
\log_a \left( \frac{x}{y} \right) = \log_a x - \log_a y.
\]

This is called **The Quotient Rule for Logarithms**.

• Let \( r \) be any real number, \( \xi = \log_a (x^r) \). By definition of \( \log_a \), \( a^\xi = x^r = (a^\beta)^r = a^{r\beta} \). This means that \( \xi = r\beta \) or

\[
\log_a (x^r) = r \log_a x.
\]

This is called **The Power Rule for Logarithms**.

• Let \( b \) be a positive real number. Then one can talk about \( \log_b x \) and \( \log_b y \) (same \( x \)). We will extract a relation between these two expressions. Let \( \delta = \log_b x \) and \( \sigma = \log_b y \), then \( x = b^\delta \) and \( y = b^\sigma \). Therefore, \( x = b^\delta = (a^\sigma)^\delta = a^{\sigma\delta} \). By definition, \( \log_a x = \sigma\delta = \log_a b \log_b x \), so

\[
\log_b x = \frac{\log_a x}{\log_a b}.
\]

This is called **The change of base rule for Logarithms**.

• \( \log_a x = \log_a y \Leftrightarrow a^{\log_a x} = a^{\log_a y} \Leftrightarrow x = y \)

We gather together all the above established Logarithmic properties in the following tables. These properties are to be memorized as they will (more likely) not be provided on any form of testing.

<table>
<thead>
<tr>
<th>Laws of Logarithms</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log_a (1) = 0 ), ( \log_a (a) = 1 ), ( \log_a (xy) = \log_a (x) + \log_a (y) )</td>
</tr>
</tbody>
</table>

**Table 10.2.1.**

| \( \log_a \left( \frac{x}{y} \right) = \log_a (x) - \log_a (y) \), \( \log_a (x^r) = r \log_a (x) \), \( \log_b x = \frac{\log_a x}{\log_a b} \) |

\( \log_a x = \log_a y \Leftrightarrow x = y \)

**Warning 10.2.1.** A very common mistake is to treat the logarithmic function as a linear one. Be very careful, \( \log_a (x + y) \neq \log_a x + \log_a y \) in general.
Example 10.2.1. Without using a calculator, evaluate each of the following expressions.

(1) \( \log_3 9 + 2 \log_3 \left( \frac{1}{27} \right) \)  
(2) \( \log_8 2 + \log_{64} 8 - \log_2 \sqrt{2} \)  
(3) \( \log_5 10 + \log_5 20 - 3 \log_5 2 \)  
(4) \( 3^{\log_3 5 - \log_3 4} \)

Solution.  
(1) There are two ways to proceed. Firstly, Note that \( 2 \log_3 \left( \frac{1}{27} \right) = \log_3 \left( \frac{1}{27} \right)^2 \) (by the Power Rule), so

\[
\log_3 9 + 2 \log_3 \left( \frac{1}{27} \right) = \log_3 9 + \log_3 \left( \frac{1}{27} \right)^2 \quad \text{(by the Power Rule)}
\]

\[
= \log_3 \left( \frac{3^2}{(3^3)^2} \right) \quad \text{(by the Product Rule)}
\]

\[
= \log_3 \left( \frac{1}{3^4} \right) \quad \text{(since \( \frac{3^2}{(3^3)^2} = \frac{3^2}{3^6} = \frac{1}{3^4} \))}
\]

\[
= \log_3 3^{-4} \quad \text{(by the Power Rule)}
\]

\[
= -4 \log_3 3 \quad \text{(since \( \log_3 3 = 1 \))}
\]

Another way to find the value of the expression is to compute each of the two terms separately and then add the results. The first term can be evaluated as follows: \( \log_3 9 = \log_3 3^2 = 2 \log_3 3 = 2 \). As for the second term: \( 2 \log_3 \left( \frac{1}{27} \right) = 2 \log_3 \left( \frac{1}{3^3} \right) = 2 \log_3 (3^{-3}) = -6 \log_3 3 = -6 \). we conclude that \( \log_3 9 + 2 \log_3 \left( \frac{1}{27} \right) = 2 - 6 = -4 \).

(2) Here we have an expression containing logarithmic terms with different bases.

\[
\log_8 2 + \log_{64} 8 - \log_2 \sqrt{2} = \log_8 \left( 2^{\frac{1}{3}} \right) + \log_{64} \left( 64^{\frac{1}{5}} \right) - \log_2 \left( 2^\frac{1}{2} \right)
\]

\[
= \frac{1}{3} \log_8 2 + \frac{1}{5} \log_{64} 64 - \frac{1}{5} \log_2 2
\]

\[
= \frac{1}{3} + \frac{1}{5} - \frac{1}{5}
\]

\[
= \frac{19}{36}
\]

(3) All the logarithms are with the same base 5. We regroup all the terms in one logarithm with base 5 using the Product and quotient logarithmic rules.

\[
\log_5 10 + \log_5 20 - 3 \log_5 2 = \log_5 (5 \times 20) - 3 \log_5 (2^3)
\]

\[
= \log_5 (5 \times 20) - \log_5 (2^3)
\]

\[
= \log_5 \left( \frac{100}{8} \right)
\]

\[
= \log_5 \left( \frac{25}{2} \right).
\]

128
(4) We start by rearranging the exponent: \( \log_3 5 - \log_3 4 = \log_3 \left( \frac{5}{4} \right) \). Therefore,
\[
3^{\log_3 5 - \log_3 4} = 3^{\log_3 \left( \frac{5}{4} \right)} = \frac{5}{4}.
\]

**Try it yourself 10.2.1.** Evaluate each of the following expressions.

1. \( \log_2 8 - 3 \log_2 \left( \frac{1}{16} \right) \)
2. \( \log_6 62 - \log_6 3 + \log_6 \sqrt{2} \)
3. \( \log_5 20 + 2 \log_5 \left( \frac{1}{125} \right) - 2 \log_5 3 \)
4. \( 2^{\log_2 3 + 3 \log_2 4} \)

### 10.3 Special Logarithmic Functions

If you look closely at your calculator, you will most likely see two logarithm buttons in different levels (or colors). One is marked "log" and the other is marked "ln" with neither one has the base written in. The base can be determined by looking at the inverse function written above the key and accessed by the 2nd key. Above the "log" button, you will see "10^x", which indicates that "log" actually stands for \( \log_{10} \).

Similarly, above the "ln" button, the function \( e^x \) is marked with (as introduced above) \( e \approx 2.718281828... \) indicating that \( \ln \) actually stands for \( \log_e \).

The function \( y = \log x \) stands for \( \log_{10} x \) and \( \ln x \) stands for \( \log_e x \).

The function \( y = \log x \) is referred to as the common logarithmic function and the function \( y = \ln x \) is referred to as the natural logarithmic function.

The natural logarithmic function \( \ln x \) is nothing but a particular logarithmic function with base \( e \approx 2.718281828... > 1 \), but it is probably the one that you will use more often. Although all the Logarithmic rules apply to \( \ln x \), we stress the importance of this function by including separately its graph and main properties in the following table.
Example 10.3.1. Express each of the following as a single logarithm.

\[
(1) \log_3 27 - \frac{1}{2} \log_3 9 + 3 \log_3 2 \quad \text{and} \quad (2) 3 \ln 4 - \ln 5 + 2 \ln 3.
\]

Solution.

\[
(1) \log_3 27 - \frac{1}{2} \log_3 9 + 3 \log_3 2 = \log_3 3^3 - \frac{1}{2} \log_3 3^2 + \log_3 2^3
\]
\[
= 3 \log_3 3 - 2 \left( \frac{1}{2} \right) \log_3 3 + 3 \log_3 2
\]
\[
= 3 \log_3 3 - \log_3 3 + 3 \log_3 2
\]
\[
= 2 \log_3 3 + 3 \log_3 2
\]
\[
= \log_3 3^2 + \log_3 2^3 = \log_3 (3^2)(2^3) = \log_3 72.
\]

\[
(2) 3 \ln 4 - \ln 5 + 2 \ln 3 = \ln (4^3) - \ln 5 + \ln (3^2)
\]
\[
= \ln \left( \frac{4^3}{3^2} \right) + \ln 3^2
\]
\[
= \ln \left( \frac{64}{5} \right) + \ln \left( \frac{576}{5} \right)
\]

Remark 10.3.1. As explained above, your calculator has only two buttons for evaluating logarithmic expressions, one for the natural logarithm \( \ln \) and another for the common logarithm \( \log \). This means that expressions like \( \log_2 6 \) or \( \log_5 3 \) cannot be evaluated using your calculator unless you know how to convert them into expressions containing \( \ln \) or \( \log \) only. The change of base Rule is handy in this case. For example, to find the value of \( \log_5 3 \), we use the change of base formula for natural logarithm: 

\[
\log_5 3 = \frac{\ln 3}{\ln 5} \approx 1.46497352.
\]

10.3.1 The Logarithms as tools to solve equations

One important feature of the logarithmic functions is their ability to serve as a tool in solving some algebraic equations, other than polynomial ones.

To solve an equation of type \( a^x = b \) for the variable \( x \), we take the natural logarithm \( \ln \) of both sides: 

\[
\ln a^x = \ln b.
\]

Using the properties of \( \ln \), we get \( x \ln a = \ln b \) and so \( x = \frac{\ln b}{\ln a} \). Similarly, the solution of an equation of type \( \log_a x = b \) is \( x = a^b \) since by definition \( b = \log_a x \) is the power of \( a \) needed to get \( x \).

Remarks 10.3.1.
1. We used the natural logarithm to solve the equation \( a^x = b \) for \( x \). Note that you could use any other logarithm \( \log_t \) for a positive real number \( t \) for that purpose. The advantage of using \( \ln \) is the ability to evaluate the solution \( x = \frac{\ln b}{\ln a} \) numerically on your calculator directly.

2. Let us be honest, more likely your professor will not ask you to solve a simple equation like \( a^x = b \) on a test, but rather a bit more complicated one that could be brought (after some steps) to this simple form. In most cases, the tricky part in solving exponential or logarithmic equations is to simplify them to the forms \( a^x = b \) or \( \log_a x = b \).

3. Like in the case of radical equations, some (or all) of the values you find do not satisfy the original equation. Plug back each of the values you find in the original equation and verify it is indeed satisfied.

**Example 10.3.2.** Solve each of the following equations for the variable \( x \).

1. \( \log_2(x + 3) = 4 \)
2. \( \log_4(x) + \log_4(2x + 4) = 2 \)
3. \( 3(2^{-3x+4}) = 2\left(5^{2x+3}\right) \)
4. \( \ln x + \ln(x + 5) = \ln 2 + \ln 7 \)

**Solution.**

1. 
\[
\log_2(x + 3) = 4 \quad \Rightarrow \quad x + 3 = 2^4 = 16 \\
\quad \Rightarrow \quad x = 13.
\]

We still need to verify our solution by plugging it back into the original equation: \( \log_2(13 + 3) = \log_2 16 = \log_2 2^4 = 4\log_2 2 = 4 \). So \( x = 13 \) is indeed the solution.

2. The Left Hand Side of the equation can be written in one logarithmic expression with base 4 (using the Logarithmic Product Rule) and the Right Hand Side can also be written in term of \( \log_4 \) using the basic relation \( x = \log_a x^a \) for any real number \( x \).

\[
\log_4(x) + \log_4(2x + 4) = 2 \quad \Rightarrow \quad \log_4(x(2x + 4)) = 2 = \log_4 4^2 \\
\quad \Rightarrow \quad x(2x + 4) = 16 \\
\quad \Rightarrow \quad 2x^2 + 4x - 16 = 0 \\
\quad \Rightarrow \quad x = -4, \ x = 2
\]

Note that \( x = -4 \) is not a solution since \( \log_4 x \) is not defined in this case. For \( x = 2 \), the Left Hand Side of the original equation becomes

\[
\log_4(2) + \log_4(2(2) + 4) = \log_4(2) + \log_4 8 = \log_4(2 \times 8) = \log_4(16) = \log_4 4^2 = 2 \log_4 4 = 2
\]

which is equal to the Right Hand Side of the equation. The only solution is then \( x = 2 \).
(3) We start by rearranging the equation.

\[
3 \left(2^{-3x+4}\right) = 2 \left(5^{2x+3}\right) \quad \iff \quad 3 \left(\left(2^{-3}\right)^4\right) = 2 \left(5^2\right)^3
\]

\[
\iff 48 \left(2^{-3}\right)^x = 250 \left(5^2\right)^x
\]

\[
\iff 48 \left(\frac{1}{8^x}\right) = 250 \left(25\right)^x
\]

\[
\iff 8^x \left(250 \left(25\right)^x\right) = 48
\]

\[
\iff 250 \left(8 \times 25\right)^x = 48
\]

\[
\iff 200^x = \frac{48}{250} = \frac{24}{125}.
\]

Now this last equation is of the form \(a^x = b\) which has \(x = \log_b a\) as solution. Therefore, \(x = \log_{200} \left(\frac{24}{125}\right)\). Note that this is not the only way to solve this equation. We present another way. Start by taking the natural logarithm on both sides of the equation:

\[
\ln \left(3 \left(2^{-3x+4}\right)\right) = \ln \left(2 \left(5^{2x+3}\right)\right) \quad \iff \quad \ln 3 + \ln \left(2^{-3x+4}\right) = \ln 2 + \ln \left(5^{2x+3}\right)
\]

\[
\iff \ln 3 + (-3x + 4) \ln 2 = \ln 2 + (2x + 3) \ln 5
\]

\[
\iff \ln 3 - (3 \ln 2) x + 4 \ln 2 = \ln 2 + (2 \ln 5) x + 3 \ln 5
\]

\[
\iff \ln 3 + 4 \ln 2 - (\ln 2 + 3 \ln 5) = (2 \ln 5 + 3 \ln 2) x
\]

\[
\iff \ln \left(\frac{48}{250}\right) = (\ln 200) x
\]

\[
\iff x = \frac{\ln \left(\frac{24}{125}\right)}{\ln 200}
\]

(4)

\[
\ln x + \ln (x + 5) = \ln 2 + \ln 7 \quad \Rightarrow \quad \ln x(x + 5) = \ln (2 \times 7) = \ln 14
\]

\[
\Rightarrow \quad x(x + 5) = 14
\]

\[
\Rightarrow \quad x^2 + 5x - 14 = 0
\]

\[
\Rightarrow \quad x = -7, \ x = 2
\]

As in part (1), \(x = -7\) is not solution, but \(x = 2\) is.

Try it yourself 10.3.1. Solve each of the following equations for the variable \(x\).

(1) \(\log_3(2x + 5) = -1\). (2) \(\log x + \log(x - 3) = 1\) (3) \(4 \left(2^{x-2}\right) = 5 \left(3^{-x+1}\right)\) (4) \(\ln(x + 7) + \ln x = \ln 2 + \ln 4\)