Quadratic Forms and the Chi-square Distribution

The purpose of these notes is to introduce the non-central chi-square distribution and its relation with quadratic forms. Within these notes you will find some suggested exercises. Solutions of these exercises are going to be posted on the Web page as well.

Construction of \( \chi^2 \)

- We are using the following construction of a \( \chi^2 \) random variable. Let \( Y_1, \ldots, Y_n \) be independent normal random variables with a common variance \( \sigma^2 \), then

\[
\frac{\sum_{i=1}^{n} Y_i^2}{\sigma^2} \sim \chi^2,
\]

with \( \nu = n \) degrees of freedom and the non-centrality parameter is

\[
nc = \frac{\sum_{i=1}^{n} \mu_i^2}{\sigma^2}.
\]

- The parameters come from the mean of the sum of squares, that is

\[
E\left\{ \sum_{i=1}^{n} Y_i^2 \right\} = n \sigma^2 + \sum_{i=1}^{n} \mu_i^2.
\]

Remark: If the variables were centered properly, i.e. \( \mu_i = E\{Y_i\} = 0 \), then \( \sum_{i=1}^{n} Y_i^2/\sigma^2 \) would have a \( \chi^2 \) distribution with \( n \) degrees of freedom.

Exercise 1: Let \( Y_1, \ldots, Y_{10} \) be a random sample from \( N(35, 25) \). (It is a normal population with mean \( \mu = 35 \) and standard deviation \( \sigma = 5 \).

Give the distribution of

\[
\frac{\sum_{i=1}^{10} Y_i^2}{\sigma^2}
\]

and of

\[
\frac{\sum_{i=1}^{10} (Y_i - 35)^2}{\sigma^2}.
\]
**Definition:** Let $A$ be a square matrix of size $n$ by $n$. Its trace is defined as the sum of the elements in its diagonal, that is

$$\text{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$ 

Here are a few properties of the trace: Let $A$, $B$ and $C$ be square matrices such that $A + B$, $AC$ and $CA$ are square matrices.

- $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$;
- $sA = s\text{tr}(A)$, where $s$ is a scalar;
- $\text{tr}(AC) = \text{tr}(CA)$.

**Symmetric Matrices**

**Symmetric Matrices:** Let $A$ be an $m \times m$ matrix. It is said to be symmetric if $a_{ij} = a_{ji}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, m$. Equivalently, it means that $A' = A$, where $A'$ is the transpose of $A$.

**Quadratic Forms**

**Definition:** We say that $Q$ is a quadratic form of $Y = [Y_1, \ldots, Y_n]'$ if we can write it as follows:

$$Q = Y'AY, \quad \text{where } A \text{ is symmetric.}$$

Equivalently, we have

$$Q = \sum_{i=1}^{n} a_{ij}Y_iY_j, \quad \text{where } a_{ij} = a_{ji}.$$ 

**Note:** We say that $A$ is the matrix of the quadratic form.
Properties of Quadratic Forms:

- if \( Q_1, \ldots, Q_k \) are quadratic forms of \( Y_1, \ldots, Y_n \), then

\[
Q = c_1 Q_1 + \ldots + c_k Q_k = Y' \left( \sum_{i=1}^k c_i A_i \right) Y
\]

is also a quadratic form of \( Y_1, \ldots, Y_n \).

- **Expectation of a Quadratic form:** Assume that \( Y_1, \ldots, Y_n \) are independent with \( E\{Y_i\} = \mu_i \) and \( \sigma^2\{Y_i\} = \sigma^2 \). We showed in class that

\[
E\{Y'AY\} = \text{tr}(A) \sigma^2 + \mu' A \mu,
\]

where \( \mu = [\mu_1, \ldots, \mu_n]' \).

Remarks:

- We will see later that if \( A \) is a projection matrix, then \( Y'AY/\sigma^2 \)

will have a non-central chi-square distribution with \( \text{tr}(A) \) degrees of freedom and its non-centrality parameter is

\[
\text{n.c.} = \frac{\mu' A \mu}{\sigma^2}.
\]

- If \( \text{n.c.} = 0 \), we simply say that it is a chi-square distribution.

- A symmetric matrix \( A \) (i.e. \( A' = A \)), which is also idempotent (i.e. \( A^2 = A \)) is called a **projection** matrix.

**Exercise 2:** Consider \( Q = Q_1 + Q_2 \), then show that \( \nu = \nu_1 + \nu_2 \), where \( Q, Q_1, Q_2 \) are quadratic forms of \( Y_1, \ldots, Y_n \) of independent random variables with a common variance. In other words, show that the degrees of freedom are additive. *Hint:* Use the properties of the trace of a matrix.

**Exercise 3:** Consider the following quadratic forms of \( Y_1, \ldots, Y_n \):

\[
Q_1 = \sum_{i=1}^n Y_i^2 \quad \text{and} \quad Q_2 = \left( \sum_{i=1}^n Y_i \right)^2.
\]

(a) Show that \( I \) (i.e. the identity matrix) is the matrix of \( Q_1 \) and that \( J \) (i.e. an \( n \) by \( n \) matrix of 1's) is the matrix of \( Q_2 \).

(b) Show that

\[
\sum_{i=1}^n (Y_i - \bar{Y})^2 = \sum_{i=1}^n Y_i^2 - \left( \sum_{i=1}^n Y_i \right)^2 / n.
\]
(c) Use the results in (a) and (b), to show that

\[
\sum_{i=1}^{n}(Y_i - \bar{Y})^2
\]

is a quadratic form of \(Y_1, \ldots, Y_n\) and show that its matrix is \(I - (1/n)J\).

**Exercise 4:** Let \(Y_1, \ldots, Y_n\) independent normals with a common variance. That is \(Y_i \sim N(\mu_i, \sigma^2)\).

(a) Show that for the total variability

\[
\frac{\text{SSTO}}{\sigma^2} \sim \chi^2,
\]

by showing that \(\text{SSTO} = Y'PY\), where \(P\) is a projection matrix, i.e. \(P' = P\) and \(P^2 = P\). The total variability is

\[
\text{SSTO} = \sum_{i=1}^{n}(Y_i - \bar{Y})^2 \quad \text{and} \quad \bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i.
\]

*Hint:* See exercise 3.

(b) Show that the degrees of freedom is \(\nu = n - 1\). *Hint:* Compute the trace of the matrix.

(c) Give the form of the non-centrality parameter of \(\text{SSTO}\).

(d) Consider that the means are equal, that is \(\mu_1 = \mu_2 = \ldots = \mu_n = \mu\). Show that

\[
\frac{\text{SSTO}}{\sigma^2} = \frac{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}{\sigma^2} \sim \chi^2(n - 1).
\]