MAT 3378

Bonferroni-Holm

Bonferroni method for joint estimation: Suppose that we want to jointly estimate g parameters, say $\theta_1, \theta_2, \ldots, \theta_g$ with a family confidence level of $1-\alpha$. Suppose that CI_i is a random interval to is used to construct a confidence interval for θ_i, for $i = 1, \ldots, g$. We want

$$1 - \alpha = P(\{\theta_1 \in \mathrm{CI}_1\} \cap \ldots \cap \{\theta_g \in \mathrm{CI}_g\}).$$

Let α_i be the error rate for the ith estimation, i.e.

$$1 - \alpha_i = P(\theta_i \in \mathrm{CI}_i),$$

for $i = 1, \ldots, g$. We saw in class, that if the error rate for the ith estimation is set to $\alpha_i = \alpha/g$, then the familywise confidence level will be at least $1 - \alpha$.

The proof is based on Boole's inequality. We give the proof for completeness.

Proof: The family confidence level is

$$\begin{aligned}
& P(\{\theta_1 \in \mathrm{CI}_1\} \cap \ldots \cap \{\theta_g \in \mathrm{CI}_g\}) \\
= \; & 1 - P\left[(\{\theta_1 \in \mathrm{CI}_1\} \cap \ldots \cap \{\theta_g \in \mathrm{CI}_g\})^c\right] \quad \text{(complement rule)} \\
= \; & 1 - P\left[\{\theta_1 \in \mathrm{CI}_1\}^c \cup \ldots \cup \{\theta_g \in \mathrm{CI}_g\}^c\right] \quad \text{(DeMorgan rule)} \\
\geq \; & 1 - \sum_{i=1}^{g} P(\{\theta_i \in \mathrm{CI}_i\}^c) \quad \text{(Boole's Inequality)} \\
= \; & 1 - \sum_{i=1}^{g} \alpha_i.
\end{aligned}$$

Thus, if we use $\alpha_i = \alpha/g$, then

$$P(\{\theta_1 \in \mathrm{CI}_1\} \cap \ldots \cap \{\theta_g \in \mathrm{CI}_g\}) \geq 1 - \alpha.$$

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Bonferroni method for simultaneous testing: Suppose that we have g null hypotheses to test and that we want to control the familywise error rate, that is the probability that we will identify at least one significant result, when in fact all of the null hypotheses are true. Let p_i be the p-value for the ith test.

Bonferroni method: We reject the ith null hypothesis only when $p_i < \alpha/g$, for all $i = 1, \ldots, g$.

Remark: Equivalently, we could adjust the p-values as follows

$$p_i* = \begin{cases} g\, p_i, & \text{if } g\, p_i < 1 \\ 1, & \text{if } g\, p_i \geq 1 \end{cases}$$

for $i = 1, \ldots, g$. We reject the ith null hypothesis only when $p_i^* < \alpha$, for all $i = 1, \ldots, g$.

Bonferroni-Holm method for simultaneous testing: Holm in 1979, propose a method to sequentially test the g null hypotheses that is uniformly more efficient that Bonferroni. It is a simple adjustment of the above Bonferroni method.

Bonferroni-Holm method: Order the p-values in ascending order $p_{(1)} \leq p_{(2)} \leq \ldots \leq p_{(g)}$. Let $H_{(1)}, H_{(2)}, \ldots, H_{(g)}$ be the corresponding null hypotheses.

- Step 1: If $p_{(1)} < \alpha/g$, then we reject the corresponding $H_{(1)}$, else we do not reject H_i for $i = 1, \ldots, g$ and STOP.
- Step 2: If $p_{(2)} < \alpha/(g-1)$, then we reject the corresponding $H_{(2)}$, else we do not reject H_i for $i = 2, \ldots, g$ and STOP.
- Step 3: If $p_{(3)} < \alpha/(g-2)$, then we reject the corresponding $H_{(3)}$, else we do not reject H_i for $i = 3, \ldots, g$ and STOP.
- And so on.

Remarks:

- Equivalently, we could adjust the p-values as follows

$$p_{(i)}* = \begin{cases} g\, p_{(1)}, & \text{if } i = 1 \\ \max\{p_{(i-1)}, p_{(i)}\,(g - i + 1)\} & \text{if } i = 2, \ldots, g \end{cases}$$

  As always, if any adjusted p-value exceeds 1, it is set to 1.

- We reject $H_{(i)}$ only if $p_{(i)}* < \alpha$.

- Notice that if $p_{(i')}* \geq \alpha$ for some i', this means that we do not reject $H_{(i')}$, but since all adjusted p-values for $i > i'$ must be larger or equal to $p_{(i')}*$, then we do not reject the null hypotheses for $i = i', i'+1, \ldots, g$.