ON THE STRUCTURE OF ENDOMORPHISMS OF VON NEUMANN ALGEBRAS

REMUS FLORICEL

Abstract. We investigate the structure of unital endomorphisms of von Neumann algebras. We introduce and characterize the classes of \( k \)-inner and properly outer endomorphisms, and we show that any unital endomorphism can be decomposed as the direct sum of a family of \( k \)-inner endomorphisms and of a properly outer endomorphism. This decomposition is stable under conjugacy and cocycle conjugacy.

Introduction

The theory of endomorphisms of von Neumann algebras has become a very active area of research in recent years. A program for classification of shift type endomorphisms (and semigroups of endomorphisms) acting on factors of type \( I_\infty \) and type \( II_1 \) was initiated by R. Powers [15] and W. Arveson [1], and since then many interesting results having connections with other areas of research have been obtained: the study of endomorphisms of a factor of type \( I_\infty \) is strictly related to the study of non-degenerate representations of Cuntz algebras ([2], [3], [12]), while the study of conjugacy and cocycle conjugacy classes of binary shifts acting on the hyperfinite \( II_1 \) factor ([15]) has led to new results on polynomials over finite fields and Toeplitz matrices (see [16] and the references therein).

Endomorphisms of factors of type III have also been studied by R. Longo [14], M. Izumi [8] and others, in connection with superselection sectors in algebraic quantum field theory. In this way, the statistical dimension of a sector [14] has been related to the Jones-Kosaki index for subfactors [9], [11].

Nevertheless, little is known about the general structure of an arbitrary endomorphism that acts on a von Neumann algebra. Our purpose, in this paper, is to investigate this structure by making use of a decomposition result. Conceptually, the decomposition of endomorphisms we propose bears a resemblance to Kallman’s decomposition of automorphisms of von Neumann algebras into inner parts and freely acting parts ([10]).

This paper is organized as follows. In Section 2 we introduce the classes of \( k \)-inner, properly outer and freely acting endomorphisms, and we discuss their properties and some examples. In Section 3 we prove our main theorem: an endomorphism of a von Neumann algebra decomposes uniquely as the direct sum of a family of \( k \)-inner endomorphisms and a properly outer endomorphism. We also discuss the structure of ergodic endomorphisms. In Section 4 we show that this decomposition is stable under conjugacy and cocycle conjugacy.

We close our introduction with a few remarks on notation, most of which is

1991 Mathematics Subject Classification. Primary 46L40; Secondary 46E25, 20C20.

Key words and phrases. von Neumann algebras, endomorphisms, conjugacy.
standard. It this paper, we consider only separable Hilbert spaces. Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with center $Z(M)$ and lattice of projections $\mathcal{P}(M)$, and let $\text{End}(M)$ be the semigroup of all unital normal $^*$-endomorphisms of $M$. For simplicity, the elements of $\text{End}(M)$ will be often called endomorphisms. We note that an injective endomorphism is automatically normal [18].

For $\rho, \sigma \in \text{End}(M)$, we consider the space of intertwiners between $\rho$ and $\sigma$, 
\[
\text{Hom}_M(\rho, \sigma) = \{ u \in M : \sigma(x)u = up(x), \ x \in M \}.
\]
The fixed-point algebra of $\rho \in \text{End}(M)$ is defined as $M^\rho = \{ x \in M : \rho(x) = x \}$. If $M^\rho$ is one-dimensional, then we say that $\rho$ is an ergodic endomorphism of $M$.

1. Preliminaries on $k$-inner and properly outer endomorphisms

We begin this section by recalling briefly some elementary definitions and properties about inner endomorphisms of von Neumann algebras. For a detailed treatment of this matter, we refer the reader to [17], [7].

Let $M$ be a von Neumann algebra. By a Hilbert space in $M$ we mean a norm-closed linear subspace $H$ of $M$ that satisfies the following conditions:
\begin{enumerate}
  \item $u^*u \in \mathbb{C} \cdot 1$, for every $u \in H$;
  \item $xH \neq \{0\}$, for every $x \in M, x \neq 0$.
\end{enumerate}
A Hilbert space $H$ in $M$ has a natural inner product defined by
\[
(u, v)_M \cdot 1 = v^*u, \ u, v \in H,
\]
and this inner product makes $H$ into a genuine (complex) Hilbert space. An orthonormal basis for $H$ is then given by a family of isometries \{u\}_i=1^k of $M$ satisfying the Cuntz relations [6]:
\begin{align}
  u_i^*u_j = \delta_{i,j} \cdot 1 \quad \text{and} \quad \sum_{i=1}^k u_iu_i^* = 1,
\end{align}

where, if $k = \infty$, then the last sum is understood as convergence with respect to the strong topology of $M$.

As noted by J. Roberts, the Hilbert spaces in $M$ implement a canonical class of endomorphisms of $M$:

**Proposition 1.1.** (cf. [17, Lemma 2.2]) Let $M$ be a von Neumann algebra and $H$ be a Hilbert space in $M$. Then there exists $\rho_H \in \text{End}(M)$, uniquely determined by the condition
\begin{equation}
H \subseteq \text{Hom}_M(Id, \rho_H).
\end{equation}
In fact, if $\{u_i\}_{i=1}^\infty$ is an orthonormal basis for $H$, then the endomorphism $\rho_H$ has the form
\begin{equation}
\rho_H(x) = \sum_{i=1}^k u_i x u_i^*, \ x \in M.
\end{equation}

Endomorphisms of $M$ implemented in this way by Hilbert spaces in $M$ are generically called inner endomorphisms.
1.1. \textit{k}-inner endomorphisms. Concerning the concept of inner endomorphism, several questions occur naturally. For example, can an inner endomorphism of $M$ be implemented by different Hilbert spaces in $M$? If the answer is affirmative, then what is the relation between two Hilbert spaces in $M$ that implement the same inner endomorphism? We will answer these questions presently. Before doing this, let us mention that if $M$ is a factor, then an inner endomorphism $\rho$ of $M$ is implemented by a unique Hilbert space in $M$, and this Hilbert space is exactly $\text{Hom}_M(\text{Id}, \rho)$ (cf. [13, Proposition 2.1]). If $M$ is an arbitrary von Neumann algebra, then $\text{Hom}_M(\text{Id}, \rho)$ is no longer a Hilbert space in $M$, but it has the following algebraic structure:

**Lemma 1.2.** Let $M$ be a von Neumann algebra and $\rho \in \text{End}(M)$ be an inner endomorphism implemented by a $k$-dimensional Hilbert space in $M$, $k \in \mathbb{N} \cup \{\infty\}$. Then the set $\text{Hom}_M(\text{Id}, \rho)$ is a free $\mathbb{Z}(M)$-bimodule of rank $k$.

**Proof.** Let $\{u_j\}_{j=1}^{k}$ be an orthonormal basis for the Hilbert space in $M$ that implements $\rho$. We define

$$S := \left\{ \sum_{j=1}^{k} u_j m_j : m_j \in \mathbb{Z}(M) \right\},$$

where, if $k = \infty$, then the elements of $S$ are understood as strongly summable series. Then it is easily seen that both $\text{Hom}_M(\text{Id}, \rho)$ and $S$ are $\mathbb{Z}(M)$-bimodules.

We claim that $S$ is a free $\mathbb{Z}(M)$-bimodule of rank $k$. Indeed, if $m_j \in \mathbb{Z}(M)$ are such that $\sum_{i=j}^{k} m_i u_j = 0$, then for each $i = 1, 2, \ldots, k$, we have

$$m_i = u_i^* \left( \sum_{j=1}^{k} u_j m_j \right) = 0.$$

Thus $S$ is a free $\mathbb{Z}(M)$-bimodule of rank $k$.

We also claim that $\text{Hom}_M(\text{Id}, \rho) = S$. The inclusion “$\subseteq$” is straightforward. For proving the converse inclusion, we fix $a \in \text{Hom}_M(\text{Id}, \rho)$. Then for any $x \in M$ and $j = 1, 2, \ldots, k$, we have

$$u_j^* ax = u_j^* \rho(x)a = xu_j^* a.$$

Therefore $m_j := u_j^* a \in \mathbb{Z}(M)$. We then have

$$a = \left( \sum_{j=1}^{k} u_j u_j^* \right) a = \sum_{j=1}^{k} u_j m_j \in S.$$ 

This ends the proof of Lemma 1.2. \hfill \Box

The following proposition gives an answer to the questions asked at the beginning of this subsection.

**Proposition 1.3.** Let $M$ be a von Neumann algebra, $\rho_1, \rho_2 \in \text{End}(M)$ be inner endomorphisms of $M$, and let $k, l \in \mathbb{N} \cup \{\infty\}$. Assume that $\rho_1$ is implemented by a $k$-dimensional Hilbert space in $M$ with orthonormal basis $\{u_j\}_{j=1}^{k}$, and that $\rho_2$ is implemented by a $l$-dimensional Hilbert space in $M$ with orthonormal basis $\{v_j\}_{j=1}^{l}$. Then the following conditions are equivalent:

(i) $\rho_1 = \rho_2$;
\[ v_j = \sum_{i=1}^{k} u_i a_{ij}. \]

(1.4)

**Proof.** (i) \(\Rightarrow\) (ii). Since the rank of a free module over a commutative ring is unique, Lemma 1.2 implies that \(k = l\). Moreover, for any \(x \in M\) we have

\[ xu^* v_j = u^*_i \rho_1(x) v_j = u^*_i \rho_2(x) v_j = u^*_i v_j x, \quad \text{for every } i, j. \]

Therefore \(u^*_i v_j \in \mathcal{Z}(M)\), for all \(i, j = 1, \ldots, k\). We then define

\[ a = [u^*_i v_j]_{i,j} \in M_k(\mathcal{Z}(M)), \]

and we claim that \(a\) is a unitary matrix. Indeed,

\[ (a^* a)_{i,j} = \sum_{s=1}^{k} (u^*_s v_i)^* u^*_s v_j \]

\[ = \sum_{s=1}^{k} v^*_s u_s u^*_s v_j \]

\[ = v^*_s v_j = \delta_{i,j} \cdot 1, \]

and similarly, \(aa^* = 1\). Finally, for any \(j = 1, 2, \ldots, k\), we have

\[ v_j = \left( \sum_{i=1}^{k} u_i u^*_i \right) v_j = \sum_{i=1}^{k} u_i a_{ij}. \]

(2) \(\Rightarrow\) (1). For any \(x \in M\), we have

\[ \rho_2(x) = \sum_{j=1}^{k} v_j x v_j^* = \sum_{j=1}^{k} \left( \sum_{i=1}^{k} u_i a_{ij} \right) x \left( \sum_{l=1}^{k} u_l a_{lj} \right)^* \]

\[ = \sum_{i,l} u_i x \left( \sum_{j=1}^{k} a_{ij} a_{lj}^* \right) u^*_l \]

\[ = \sum_{i,l} \delta_{i,l} u_i x u^*_l = \rho_1(x). \]

The proposition is proved. \(\square\)

In order to distinguish among inner endomorphisms implemented by Hilbert spaces of different dimensions, we introduce the following definition:

**Definition 1.4.** Let \(k \in \mathbb{N} \cup \{\infty\}\) and \(\rho \in \text{End}(M)\). We say that \(\rho\) is a \(k\)-inner endomorphism (or an inner endomorphism of dimension \(k\)) if \(\rho\) is implemented by a \(k\)-dimensional Hilbert space in \(M\). Equivalently, \(\rho\) is \(k\)-inner if and only if there exists a set of isometries \(\{v_i\}_{i=1}^{k} \subset \text{Hom}_M(\text{Id}, \rho)\) that satisfies relations (1.1). Such a set will be called an implementing set of \(\rho\). The set of all \(k\)-inner endomorphisms of \(M\) will be denoted by \(\text{End}_k(M)\).
Remark 1.5. Concerning the existence of $k$-inner endomorphisms, we note that if $M$ is a properly infinite von Neumann algebra, then $\text{End}_k(M)$ is non-empty, for all $k \in \mathbb{N} \cup \{\infty\}$. Moreover, if $M$ is a type $I_\infty$ factor, then every $\rho \in \text{End}(M)$ is a $k$-inner endomorphism of $M$ ([1]), where $k$ is the Powers index of $\rho$, i.e. that $k$ such that $\text{Hom}_M(\rho, \rho)$ is isomorphic to the factor of type $I_k$ [15].

If $M$ is a finite von Neumann algebra, then $\text{End}_k(M)$ is empty for $k \geq 2$. Obviously, the 1-inner endomorphisms are exactly the inner automorphisms of $M$.

1.2. Properly outer endomorphisms and freely acting endomorphisms.

The class of properly outer endomorphisms will be defined as the opposite of the classes of $k$-inner endomorphisms. Roughly speaking, a properly outer endomorphism is an endomorphism which does not have inner parts. More concretely, it can be defined in the following way:

**Definition 1.6.** Let $M$ be a von Neumann algebra and $\rho \in \text{End}(M)$. We say that $\rho$ is a properly outer endomorphism of $M$, if for any central projection $p \in M^p$, we have

$$\rho \mid_{M_p} \notin \bigcup_{k=1}^{\infty} \text{End}_k(Mp).$$

By appropriating the notion of freely acting automorphisms [10], we also define the corresponding class of endomorphisms:

**Definition 1.7.** An endomorphisms $\rho$ of a von Neumann algebra $M$ is said to be freely acting, if $\text{Hom}_M(\text{Id}, \rho) = \{0\}$.

It is easily seen that a freely acting endomorphism must be properly outer.

In the case of automorphisms, these two concepts are equivalent ([10]), and this equivalence plays an important role in the work of A. Connes on the classification of automorphisms of factors ([5]). However, at least in the case of factors, one can construct examples of properly outer endomorphisms that are not freely acting.

**Example 1.8.** In [8, Proposition 3.2], M. Izumi has shown implicitly the existence of an endomorphism $\rho$ of a type III factor $M$ that satisfies the following Lee-Yang fusion rule:

$$[\rho^2] = [\text{Id}] \oplus [\rho].$$

Here $[\rho]$ denotes the sector associated to $\rho$ [14]. Equation (1.5) implies the existence of two isometries $u_1, u_2$ of $M$ that satisfy the Cuntz relations (1.1) such that

$$u_1 \in \text{Hom}_M(\text{Id}, \rho^2) \text{ and } u_2 \in \text{Hom}_M(\rho, \rho^2).$$

We also note that the Hilbert space $\text{Hom}_M(\text{Id}, \sigma^2)$ is 1-dimensional. It then follows that the endomorphism $\rho^2$ is properly outer, but not freely acting.

**Observation 1.9.** Let $M$ be an infinite factor and $\rho \in \text{End}(M)$ be a properly outer endomorphism which is not freely acting. Conceptually, $\rho$ has no “unital inner parts”. However, it has a “non-unital inner part”, as well as a “non-unital freely acting part”. Indeed, since $M$ is a factor and $\rho$ is not freely acting, $\text{Hom}_M(\text{Id}, \rho)$ is a non-trivial Hilbert space with respect to the usual inner product. Let $\{u_i\}_{i \in I}$ be an orthonormal basis for $\text{Hom}_M(\text{Id}, \rho)$. We define the projection

$$p = \sum_{i \in I} u_i u_i^* \in \text{Hom}_M(\rho, \rho).$$
Then the “non-unital inner part” of $\rho$ is the non-unital endomorphism $\rho_1 : M \to M_\rho \subset M$ defined by

$$\rho_1(x) = \sum_{i \in I} u_i x u_i^*, \ x \in M.$$  

We can also check that the mapping $\rho_2 : M \to M(1-p)$ defined by

$$\rho_2(x) = \rho(x) - \rho_1(x), \ x \in M,$$

is a *-homomorphism, and we claim that $\rho_2$ is “freely acting” in the following sense:

if $a \in M(1-p)$ satisfies $\rho_2(x)a = ax$ for all $x \in M$, then $a = 0$. Indeed, if $a$ is as above, then $a \in \text{Hom}_M(\text{Id}, \rho)$, and since

$$\langle a, u_j \rangle_M = u_j^* a = u_j^* (1-p)a = u_j^* \left( 1 - \sum_{i \in I} u_i u_i^* \right) a = 0$$

for all $j \in I$, we obtain $a = 0$.

Thus $\rho$ can be decomposed as the sum of an “inner” *-homomorphism and a “freely acting” *-homomorphism.

We end this section by presenting some examples of freely acting endomorphisms.

**Example 1.10.** Let $M$ be a factor and $\rho \in \text{End}(M)$, $\rho(M) \neq M$. If $M$ is a type $\text{II}_1$ factor, or if $\rho$ is irreducible, i.e. $\text{Hom}_M(\text{Id}, \rho) = \mathbb{C} \cdot 1_M$, then $\rho$ is a freely acting endomorphism. Indeed, if $u \in \text{Hom}_M(\text{Id}, \rho)$, then $u^* u \in Z(M) = \mathbb{C} \cdot 1$, and $uu^* \in \text{Hom}_M(\text{Id}, \rho)$. The conclusion follows then easily.

**Example 1.11.** Let $\Gamma$ be a discrete group, and let $\text{VN}(\Gamma) = \{\lambda_g \mid g \in \Gamma\}$ be the group von Neumann algebra generated by the left regular representation $\lambda$ of $\Gamma$. If $\rho : \Gamma \to \Gamma$ is an injective unital endomorphism of the group $\Gamma$, then $\rho$ induces an endomorphism $\tilde{\rho} \in \text{End}(\text{VN}(\Gamma))$ that acts on generators as

$$\tilde{\rho}(\lambda_g) = \lambda_{\rho(g)} \cdot g \in \Gamma.$$  

By using a straightforward adaptation of Kallman’s original argument in [10, Theorem 2.2], we can easily check that $\tilde{\rho}$ is freely acting if and only if the set $\{\rho(g)hg^{-1} \mid g \in \Gamma\}$ is infinite, for every $h \in \Gamma$. We note that if $\Gamma$ is an ICC group, then $\tilde{\rho}$ is automatically a freely acting endomorphism (see Example 1.10).

As an immediate exemplification, let $\Gamma$ be the (non-ICC) group of all $2 \times 2$ matrices of the form $\begin{pmatrix} e & k \\ 0 & e' \end{pmatrix}$, where $e, e' = \pm 1$ and $k \in \mathbb{Z}$, and let $\rho$ be given by

$$\rho(\begin{pmatrix} e & k \\ 0 & e' \end{pmatrix}) = \begin{pmatrix} e & 2k \\ 0 & e' \end{pmatrix}.$$  

Then it is easily seen that $\{\rho(g)hg^{-1} \mid g \in \Gamma\}$ is infinite for all $h \in \Gamma$, so $\tilde{\rho}$ is freely acting.

2. DECOMPOSITION OF ENDOMORPHISMS

The following lemma is the main ingredient in the analysis of the structure of an arbitrary endomorphism. The argument we use in the proof has its roots in [10, Lemma 1.9] and [4, Proposition 1.5.1].
Lemma 2.1. Let $M$ be a von Neumann algebra, $\rho \in \text{End}(M)$ and $k \in \mathbb{N} \cup \{\infty\}$. If we define

$$Q_k(\rho) = \{p \in \mathcal{P}(M) \cap \mathcal{Z}(M) \cap M^\rho : \rho|_{M_p} \in \text{End}_k(M_p)\},$$

then the set $Q_k(\rho)$ is closed under taking suprema.

Proof. Let $\{p_i\}_{i \in I} \subseteq Q_k(\rho)$ and set $p = \bigvee_{i \in I} p_i$. We want to show that $\rho|_{M_p} \in \text{End}_k(M_p)$.

By applying Zorn’s Lemma, we can find a maximal (countable) family $\{q_j\}_{j \in J}$ of mutually orthogonal central projections of $M$, having the property that for any $j \in J$, there exists an $i \in I$ such that $q_j \leq p_i$. Then we have

$$\sum_{j \in J} q_j \leq p,$$

and we claim that we have equality. Indeed, if $\sum_{j \in J} q_j \neq p$, then there exists $i \in I$ such that $p_i \not\leq \sum_{j \in J} q_j$. Then it is easily seen that $(p - \sum_{j \in J} q_j)p_i$ is a nonzero central projection, and that $(p - \sum_{j \in J} q_j)p_i q_j = 0$, for all $l \in J$. This contradicts the maximality of the family $(q_j)_{j \in J}$. Thus $p = \sum_{j \in J} q_j$.

Since the direct sum of a family of $k$-inner endomorphisms is still a $k$-inner endomorphism, in order to complete this proof it is enough to show that

$$\rho|_{M_{q_j}} \in \text{End}_k(M_{q_j}) \quad \text{for all } j \in J.$$

For proving this, let $j \in J$ be fixed, and let $i \in I$ such that $q_j \leq p_i$. We choose an implementing set

$$\{u_l\}_{l = 1}^k \subseteq \text{Hom}_{M_{p_i}}(Id, \rho|_{M_{p_i}})$$

of the $k$-inner endomorphism $\rho|_{M_{p_i}}$. Then

$$\rho(q_j) = \sum_{l = 1}^k u_l q_j u_l^* = q_j \left(\sum_{l = 1}^k u_l u_l^*\right) = q_j p_i = q_j,$$

so $\rho|_{M_{q_j}} \in \text{End}(M_{q_j})$. Moreover, the set $\{u_l q_j\}_{l = 1}^k \subseteq \text{Hom}_{M_{q_j}}(Id, \rho|_{M_{q_j}})$ is an implementing set of the endomorphism $\rho|_{M_{q_j}}$, so $\rho|_{M_{q_j}} \in \text{End}_k(M_{q_j})$. \hfill \Box

Definition 2.2. Let $M$ be a von Neumann algebra. For any $\rho \in \text{End}(M)$ and $k \in \mathbb{N} \cup \{\infty\}$, we shall denote by $p_k(\rho)$ the projection $\bigvee Q_k(\rho)$. We also define the central projection

$$p(\rho) = \sum_{k \in I_\rho} p_k(\rho),$$

where $I_\rho = \{k \in \mathbb{N} \cup \{\infty\} \mid p_k(\rho) \neq 0\}$.

Lemma 2.3. If $\rho \in \text{End}(M)$ and $k, l \in \mathbb{N} \cup \{\infty\}$, $k \neq l$, then $p_k(\rho)$ and $p_l(\rho)$ are orthogonal projections.

Proof. It is enough to show that for any $p \in Q_k(\rho)$ and $q \in Q_l(\rho)$, we have $pq = 0$. Suppose that $pq \neq 0$. Since $\rho|_{M_{p}}$ is a $k$-inner endomorphism of $M_{p}$ and $\rho|_{M_{q}}$ is a $l$-inner endomorphism of $M_{q}$, it follows that

$$\rho|_{M_{pq}} \in \text{End}_k(M_{pq}) \cap \text{End}_l(M_{pq}).$$

This contradicts Proposition 1.3. Hence $pq = 0$. \hfill \Box
having the following properties:

We establish the existence of such a decomposition. For this purpose, we prove:

Theorem 2.5. Let \( M \) be a von Neumann algebra and \( \rho \in \text{End}(M) \). Then there exists a set \( I \subseteq \mathbb{N} \cup \{\infty\} \) such that \( M \) decomposes as a direct sum of von Neumann subalgebras

\[
M = \left( \sum_{k \in I} \mathcal{M}_k \right) \bigoplus M_0,
\]

and \( \rho \) decomposes as a direct sum of endomorphisms

\[
\rho = \left( \sum_{k \in I} \rho_k \right) \oplus \rho_0
\]

having the following properties:

1. \( \rho_k \in \text{End}_k(M_k) \), for every \( k \in I \);
2. \( \rho_0 \in \text{End}(M_0) \) is a properly outer endomorphism of \( M_0 \).

This decomposition is unique.

Proof. We establish the existence of such a decomposition. For this purpose, we take \( I = I_\rho \), where \( I_\rho \) is as in Definition 2.2. For any \( k \in I \), we define

\[
M_k = M_{p_k(\rho)}, \quad \rho_k = \rho|_{M_k},
\]

as well as

\[
M_0 = M(1 - p(\rho)), \quad \rho_0 = \rho|_{M_0}.
\]

Then it is easily seen that setting gives the required decomposition.

We show that the above decomposition is unique. For proving this, let \( J \subseteq \mathbb{N} \cup \{\infty\} \) be a set of indices, and let \( \{q_l\}_{l \in J} \subseteq M^\rho \) be a family of mutually orthogonal central projections of \( M \) such that

1. \( \rho|_{M_{q_l}} \in \text{End}_l(M q_l) \), for every \( l \in J \);
2. \( \rho|_{M(1 - q)} \) is a properly outer endomorphism of \( M(1 - q) \),

where \( q = \sum_{l \in J} q_l \). By construction, we have \( J \subseteq I_\rho \), and we claim that \( J = I_\rho \).

Firstly, we prove that if \( k \in I_\rho \), then \( p_k(\rho)q_l = 0 \), for every \( l \in J \), \( l \neq k \). Assume that \( p_k(\rho)q_l \neq 0 \). Since \( M p_k(\rho)q_l = M p_k(\rho) \cap M q_l \), we obtain that

\[
\rho|_{M p_k(\rho)q_l} \in \text{End}_k(M p_k(\rho)q_l) \cap \text{End}_l(M p_k(\rho)q_l)
\]

and it follows from Proposition 1.3 that \( k = l \).

Secondly, we want to show that for any \( l \in J \), we have \( p_l(\rho) = q_l \). Indeed, the construction of the projections \( p_l(\rho) \) guarantees that \( q_l \leq p_l(\rho) \), for every \( l \in J \). Assume that there exists \( l \in J \) such that \( p_l(\rho) \neq q_l \). Then

\[
p_l(\rho)(1 - q_l) = p_l(\rho) - q_l \neq 0,
\]

and since \( \rho|_{M(1-q)} \) is a properly outer endomorphism of \( M(1 - q) \), we obtain that

\[
\rho|_{M p_l(\rho)(1-q)} \not\in \text{End}_l(M p_l(\rho)(1 - q)).
\]
On the other hand, since $M_p(\rho)(1-q) \subset M_p(\rho)$, we have that
\[ \rho|_{M_p(\rho)(1-q)} \in \text{End}_k(M_p(\rho)(1-q)). \]
This is a contradiction. Thus $p_l(\rho) = q_l$, for every $l \in J$.

Finally, we prove that $I_\rho = J$. Assume that there exists $k \in I_\rho \setminus J$. It then follows that $p_k(\rho) \leq 1 - q$, and since $\rho|_{M(1-q)}$ is properly outer, we obtain that $\rho|_{M_p(\rho)} \notin \text{End}_k(M_p(\rho))$.

This contradicts the definition of the projection $p_k(\rho)$. Therefore $I_\rho = J$, and the theorem is proved. \hfill \square

An immediate application of the above theorem gives the structure of ergodic endomorphisms of von Neumann algebras.

**Corollary 2.6.** Let $M$ be a von Neumann algebra and $\rho \in \text{End}(M)$ be an ergodic endomorphism. Then $\rho$ is either a $k$-inner endomorphism ($k \geq 2$), or a properly outer endomorphism. Moreover, if $\rho$ is a $k$-inner endomorphism, then $M$ must be a factor.

**Proof.** If $\rho$ is an ergodic endomorphism, then either $p(\rho) = p_k(\rho) = 1$, for some $k \in \mathbb{N} \cup \{\infty\}$, $k \geq 2$, or $p(\rho) = 0$. The last statement follows from the fact that, if $\rho$ is a $k$-inner endomorphism, then $\mathcal{Z}(M)$ is a subset of $M^\rho$. \hfill \square

**Observation 2.7.** Although there are no ergodic inner automorphisms acting on von Neumann algebras, one can construct ergodic $k$-inner endomorphisms ($k \geq 2$).

The construction is based on the following remark: if $\rho \in \text{End}_k(M)$, ($k \in \mathbb{N}$, $k \geq 2$), and if $\{u_i\}_{i \in \mathbb{N}}$ is an implementing set of $\rho$, then
\[ M^\rho = \mathcal{O}_k' \cap M, \]
where $\mathcal{O}_k$ is the Cuntz algebra generated by $\{u_i\}_{i \in \mathbb{N}}$ [6]. The proof of this equality is straightforward, and we leave the details to the diligent reader. We note that if $M$ is a type $I_{\infty}$ factor, then this result was also obtained by M. Laca (see [12, Proposition 3.1]).

To construct ergodic $k$-inner endomorphisms, we start with a Cuntz algebra $\mathcal{O}_k$ with $k$-generators $\{u_i\}_{i \in \mathbb{N}}$. Let $\phi$ be a factor state of $\mathcal{O}_k$, and let $\pi_\phi$ be the GNS representation of $\mathcal{O}_k$ with respect to $\phi$. Then the canonical endomorphism $\sigma$ of $\mathcal{O}_k$, defined by
\[ \sigma(x) = \sum_{i=1}^{k} u_i x u_i^*, \quad x \in \mathcal{O}_k, \]
can be extended to a $k$-inner endomorphism of $\pi_\phi(\mathcal{O}_k)^\prime\prime$. Since $\pi_\phi(\mathcal{O}_k)^\prime\prime$ is a factor, it follows from (2.1) that this $k$-inner endomorphism is ergodic.

3. **Reduction of Conjugacy and Cocycle Conjugacy**

The concepts of conjugacy and outer conjugacy are central to the classification theory of automorphisms of the hyperfinite $\text{II}_1$ factor ([4]). An equivalent of these concepts can also be formulated at the level of the theory of endomorphisms of von Neumann algebras (cf. [15]): given two von Neumann algebras $M$ and $N$, the endomorphisms $\rho \in \text{End}(M)$ and $\sigma \in \text{End}(N)$ are said to be conjugate, if there exists a $^*$-isomorphism $\theta$ of $N$ onto $M$ such that $\rho \circ \theta = \theta \circ \sigma$. They are called cocycle conjugate, if there exists a unitary $u$ of $N$ such that the endomorphisms
Ad(u) ◦ σ and ρ are conjugate.

Theorem 2.5 allows us to reduce the classification up to conjugacy of arbitrary endomorphisms to the classification up to conjugacy of the class of k-inner endomorphisms and of the class of properly outer endomorphisms.

**Theorem 3.1.** Let M be a von Neumann algebra and ρ, σ ∈ End(M) be two endomorphisms of M decomposed as in Theorem 2.5:

\[ ρ = \left( \sum_{k \in I_ρ} \rho_k \right) ⊕ ρ_0, \quad σ = \left( \sum_{l \in I_σ} σ_l \right) ⊕ σ_0. \]

Then ρ and σ are conjugate if and only if the following conditions are satisfied:

1. \( I_ρ = I_σ; \)
2. \( ρ_k \) and \( σ_k \) are conjugate endomorphisms, for every \( k \in I_ρ; \)
3. \( ρ_0 \) and \( σ_0 \) are conjugate endomorphisms.

**Proof.** (⇒) Suppose that ρ and σ are conjugate, and let \( θ ∈ Aut(M) \) be such that

\[ ρ \circ θ = θ \circ σ. \]

Then for any fixed \( l ∈ I_σ, \) we have

\[ \theta^{-1} \circ ρ \circ θ \big|_{Mθ(p_l(σ))} = σ_l. \]

Since \( σ_l(p_l(σ)) = p_l(σ), \) we deduce from the above relation that \( θ(p_l(σ)) ∈ M^p. \)

Let now \( \{u_i\}_{i=1}^l \subset Hom_{Mθ(p_l(σ))}(I_σ, σ_l) \) be an implementing set of the l-inner endomorphism \( σ_l. \) Then for any \( x ∈ Mθ(p_l(σ)) \) and \( i, j = 1, 2, \ldots, l, \) we have

\[ ρ(θ(x))θ(u_i) = θ(u_i)θ(x), \]

as well as

\[ θ(u_i)^*θ(u_j) = δ_{i,j} \cdot θ(p_l(σ)) \quad \text{and} \quad \sum_{i=1}^k θ(u_i)^*θ(u_i) = θ(p_l(σ)). \]

Therefore \( \{θ(u_i)\}_{i=1}^l \) is an implementing set of the endomorphism \( ρ \big|_{Mθ(p_l(σ))}. \)

Hence

\[ ρ \big|_{Mθ(p_l(σ))} ∈ End(Mθ(p_l(σ))). \]

It then follows from Definition 2.2 that \( l ∈ I_θ \) and that \( θ(p_l(σ)) ≤ p_l(ρ). \)

On the other hand, since \( θ \) is an automorphism, by repeating the above calculation for \( ρ = θ \circ σ \circ θ^{-1}, \) we obtain \( I_ρ = I_θ \) and

\[ θ(p_k(σ)) = p_k(ρ), \quad \text{for every} \ k ∈ I_ρ. \]

In particular, for any \( k, \) \( θ \big|_{Mθ(p_l(σ))} \) is a *-isomorphism of \( Mθ(p_l(σ)) \) onto \( Mθ(p_l(ρ)) \) which implements the conjugacy between \( ρ_k \) and \( σ_k. \) By construction, \( θ \big|_{Mθ(p_l(σ))} \) is also a *-isomorphism of \( M(1 − p(σ)) \) onto \( M(1 − p(ρ)) \) which implements the conjugacy between \( ρ_0 \) and \( σ_0. \)

(⇐) We assume that the endomorphisms ρ and σ satisfy conditions (i), (ii), and (iii). For any \( k ∈ I_ρ, \) let \( θ_k : Mθ(p_l(σ)) → Mθ(p_k(ρ)) \) be a *-isomorphism such that \( ρ_k \circ θ_k = θ_k \circ σ_k. \) Also, let \( θ_0 : M(1 − p(σ)) → M(1 − p(ρ)) \) be a *-isomorphism such that \( ρ_0 \circ θ_0 = θ_0 \circ σ_0. \) Thus, if we define

\[ θ = \left( \sum_{k ∈ I} θ_k \right) ⊕ θ_0, \]

then \( θ ∈ Aut(M) \) and \( ρ \circ θ = θ \circ σ, \) as we can easily see. □

A stronger characterization can be provided in the case of cocycle conjugacy.
Theorem 3.2. Let ρ and σ be two endomorphisms of M decomposed as in Theorem 2.5:

$$\rho = (\sum_{k \in I_{\rho}} \rho_k) \oplus \rho_0, \quad \sigma = (\sum_{l \in I_{\sigma}} \sigma_l) \oplus \sigma_0.$$  

Then ρ and σ are cocycle conjugate if and only if the following conditions are satisfied:

(i) \(I_{\rho} = I_{\sigma}\);

(ii) \(M_{p_k}(\rho)\) and \(M_{p_k}(\sigma)\) are *-isomorphic von Neumann algebras, for every \(k \in I_{\rho}\);

(iii) \(\rho_0\) and \(\sigma_0\) are cocycle conjugate endomorphisms.

Proof. (⇒) Let \(u\) be a unitary of \(M\) such that the endomorphisms \(\rho\) and \(\text{Ad}(u) \circ \sigma\) are conjugate. If we denote by \(u_l = u p_l(\sigma)\), \(l \in I_{\sigma}\), and by \(u_0 = u(1 - p(\sigma))\), then the endomorphism \(\text{Ad}(u) \circ \sigma\) decomposes as

$$\text{Ad}(u) \circ \sigma = \left( \sum_{l \in I_{\sigma}} \text{Ad}(u_l) \circ \sigma_l \right) \oplus \left( \text{Ad}(u_0) \circ \sigma_0 \right),$$

where \(\text{Ad}(u_l) \circ \sigma_l \in \text{End}_l(M_{p_l}(\sigma))\) for all \(l \in J\), and \(\text{Ad}(u_0) \circ \sigma_0\) is a properly outer endomorphism of \(M(1 - p(\sigma))\). The required result follows then from Theorem 3.1.

(⇐) Let \(k \in I_{\rho}(\sigma)\) be fixed. We claim that the \(k\)-inner endomorphisms \(\rho_k\) and \(\sigma_k\) are *-isomorphic for all \(k \in I_{\rho}\). Moreover, since \(\text{Ad}(\rho_k)\) and \(\text{Ad}(\sigma_k)\) are cocycle conjugate endomorphisms, \(xu = \text{Ad}(\rho_k)\circ \sigma_k \circ \theta_k^{-1}(x)\) for any \(x \in M_{p_k}(\rho)\) we have

$$\rho_k(x) = \sum_{i=1}^{k} u_i x u_i^*$$

$$= \text{Ad}(w_k) \left( \sum_{i=1}^{k} \theta_k(v_i) x \theta_k(v_i)^* \right)$$

$$= \text{Ad}(w_k) \circ \theta_k \circ \sigma_k \circ \theta_k^{-1}(x).$$
Thus $\rho_k$ and $\sigma_k$ are cocycle conjugate endomorphisms, for every $k \in I$.  

On the other hand, since $\rho_0$ and $\sigma_0$ are cocycle conjugate, there exists a *-isomorphism $\theta_0 : M(1-p(\sigma)) \to M(1-p(\rho))$ and a unitary $w_0$ of $M(1-p(\rho))$ such that $\rho_0 = \text{Ad}(w_0) \circ \theta_0 \circ \sigma_0 \circ \theta_0^{-1}$.

Therefore, if we define 

$$w = \left( \sum_{k \in I} \otimes w_k \right) \oplus w_0$$

and 

$$\theta = \left( \sum_{k \in I} \otimes \theta_k \right) \oplus \theta_0,$$

then $w$ is a unitary of $M$, $\theta \in \text{Aut}(M)$ and $\rho = \text{Ad}(w) \circ \theta \circ \sigma \circ \theta^{-1}$.  

We deduce from the above corollary that, if $M$ is a properly infinite von Neumann algebra, then for any $k \in \mathbb{N} \cup \{\infty\}$ there exists a unique $k$-inner endomorphism of $M$, up to cocycle conjugacy. However, the conjugacy class of a $k$-inner endomorphism is no longer the whole set $\text{End}_k(M)$, even if $M$ is a type I$_\infty$ factor ([12], [2]). We wish to investigate the conjugacy properties of some classes of $k$-inner endomorphisms in a future publication.

References


Department of Mathematics and Statistics, University of Ottawa, Ottawa, ON, Canada, K1N 6N5

E-mail address: floricel@aix1.uottawa.ca