A DECOMPOSITION OF $E_0$-SEMIGROUPS

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Abstract. Any $E_0$-semigroup of a von Neumann algebra can be uniquely decomposed as the direct sum of an inner $E_0$-semigroup and a properly outer $E_0$-semigroup. This decomposition is stable under conjugacy and cocycle conjugacy.

Introduction

An $E_0$-semigroup of a von Neumann algebra $M$ is a $\sigma$-weakly continuous semigroup $\hat{\rho} = \{\rho_t : t \in \mathbb{R}_+\}$ of unital normal *-endomorphisms of $M$ with $\rho_0 = Id$. To avoid the classical case of semigroups of automorphisms, it is preferable to assume that $\rho_t(M) \neq M$ for all positive $t$.

The study of $E_0$-semigroups was initiated by R. Powers in late eighties [Po1], and since then many beautiful results have been obtained, especially in the case of the factor of type $I\infty$ (see [Ar1], [Po2] and the references therein). Special classes of $E_0$-semigroups acting on arbitrary von Neumann algebras were also considered and successfully investigated [Ar2], [Ar3].

The aim of this paper is to analyze the structure of $E_0$-semigroups that act on arbitrary von Neumann algebras, and to show that the study of such semigroups can be reduced to the study of two canonical classes of $E_0$-semigroups.

The outline of the paper is as follows: in Section 1 we state background definitions and results that will be needed in the sequel. In Section 2 we introduce the class of inner $E_0$-semigroups and the class of properly outer $E_0$-semigroups and we prove the main result of this paper: any $E_0$-semigroup can be decomposed as the direct sum of an inner $E_0$-semigroup and a properly outer $E_0$-semigroup. This decomposition is unique. In the last section we discuss the stability of this decomposition under conjugacy and cocycle conjugacy.

Finally, we emphasize that all von Neumann algebras we consider in this paper have separable preduals, and all Hilbert spaces are separable.

1. Preliminaries

In this section, we collect basic notations and facts about inner endomorphisms of von Neumann algebras. For a detailed treatment of this subject, we refer the reader to [Ro], [Fl].

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Let $M$ be a von Neumann algebra with center $Z(M)$ and set of projections $\mathcal{P}(M)$, and let $\text{End}(M)$ be the semigroup of all unital normal $\ast$-endomorphisms of $M$. For $\rho, \sigma \in \text{End}(M)$, we consider the set of intertwiners

$$\text{Hom}_M(\rho, \sigma) = \{a \in M : \sigma(x)a = a\rho(x) \text{ for all } x \in M\}.$$ 

Recall ([(Ro]) that a Hilbert space in $M$ is a norm-closed linear subspace $H$ of $M$ satisfying the following relations:

1. $u^*u \in \mathbb{C} \cdot 1$, for every $u \in H$;
2. $xH \neq \{0\}$, for every $x \in M$, $x \neq 0$.

A Hilbert space $H$ in $M$ becomes a genuine (complex) Hilbert space with respect to the following inner product:

$$\langle u, v \rangle_M \cdot 1 = v^*u, \quad u, v \in H.$$ 

An orthonormal basis for $H$ is then given by a family $\{u_i\}_{i \in I}$ of isometries of $M$ satisfying the Cuntz relations [Cu]:

$$u_i^*u_j = \delta_{i,j} \cdot 1 \quad \text{and} \quad \sum_{i \in I} u_iu_i^* = 1$$

where, if $|I| = \infty$, then the last sum is understood as convergence with respect to the strong topology of $M$.

As noted by J. Roberts ([Ro, Lemma 2.2]), any Hilbert space $H$ in $M$ gives rise to an endomorphism $\rho_H$ of $M$, uniquely determined by the condition

$$H \subseteq \text{Hom}_M(\text{Id}, \rho_H).$$

In fact, if $\{u_i\}_{i \in I}$ is an orthonormal basis for $H$, then $\rho_H$ has the following form:

$$\rho_H(x) = \sum_{i \in I} u_i x u_i^*, \quad x \in M.$$ 

Endomorphisms implemented in this way by Hilbert spaces in von Neumann algebras are generically called inner endomorphisms. In order to distinguish among inner endomorphisms implemented by Hilbert spaces having different dimensions, we introduce the following definition:

**Definition 1.1.** Let $M$ be a von Neumann algebra and $k \in \mathbb{N} \cup \{\infty\}$. A $k$-inner endomorphism $\rho$ of $M$ is an endomorphism implemented by a $k$-dimensional Hilbert space $H$ in $M$. An orthonormal basis of $H$ will be referred to as an implementing set of $\rho$. The set of all $k$-inner endomorphisms of $M$ will be denoted by $\text{End}_k(M)$.

The above definition is consistent in the following sense:

**Proposition 1.2** ([FI]). Let $M$ be a von Neumann algebra and $k, l \in \mathbb{N} \cup \{\infty\}$. Let $\rho_1 \in \text{End}_k(M)$, $\rho_2 \in \text{End}_l(M)$, and let $\{u_i\}_{i=1}^{k+l}$, respectively $\{v_j\}_{j=1}^{k+l}$, be an implementing set of $\rho_1$, respectively $\rho_2$. Then the following conditions are equivalent:

1. $\rho_1 = \rho_2$;
2. $k = l$, and there exists a unitary matrix $(a_{ij})_{i,j=1}^{k+l}$ with coefficients in the center of $M$ such that $v_j = \sum_{i=1}^{k} u_i a_{ij}$, for every $j = 1, 2, \ldots, k$.

The existence of $k$-inner endomorphisms depends on the type of the von Neumann algebra on which the endomorphisms are acting. For example, if $M$ is a properly infinite von Neumann algebra, then for any $k \in \mathbb{N} \cup \{\infty\}$ there exists $k$-inner endomorphisms acting on $M$. If $M$ is a type $I_\infty$ factor, then any $\rho \in \text{End}(M)$
is a \( k \)-inner endomorphism, where \( k \) is the Powers index of \( \rho \), i.e. that \( k \) such that \( \text{Hom}_M(\rho, \rho) \) is isomorphic to a factor of type \( I_k \) ([Ar1], [Po1]). A finite von Neumann algebra does not have \( k \)-inner endomorphisms, except for \( k = 1 \). Obviously, the 1-inner endomorphisms are exactly the inner automorphisms of \( M \).

2. Two classes of \( E_0 \)-semigroups

The following proposition gives a characterization of an \( E_0 \)-semigroup whose elements are \( k \)-inner endomorphisms.

**Proposition 2.1.** Let \( M \) be a von Neumann algebra. If \( \{\rho_t : t \in \mathbb{R}_+\} \) is an \( E_0 \)-semigroup acting on \( M \) such that

\[
\{\rho_t : t \in \mathbb{R}_+^k\} \subset \bigcup_{k \geq 2} \text{End}_k(M),
\]

then \( \{\rho_t : t \in \mathbb{R}_+^k\} \subset \text{End}_{\infty}(M) \).

**Proof.** For any \( k \in \mathbb{N} \cup \{\infty\}, k \geq 2 \), we define

\[
A_k = \{t \in \mathbb{R}_+^k : \rho_t \in \text{End}_k(M)\}.
\]

By Proposition 1.2, the sets \( A_k \) are mutually disjoint sets, some of them possible empty, and

\[
(2.1) \quad \mathbb{R}_+^k = \bigcup_{k \geq 2} A_k.
\]

Let now \( k_0 = \min\{k | A_k \neq \emptyset\} \), and assume \( k_0 < \infty \). If \( t \in A_{k_0} \), then let \( t_1, t_2 > 0 \) be such that \( t = t_1 + t_2 \). By (2.1), there exist \( k_1, k_2 \in \{2, 3, \ldots, \infty\} \) such that

\[
\rho_{t_i} \in \text{End}_{k_i}(M), \quad i = 1, 2.
\]

We then have

\[
\rho_t = \rho_{t_1} \circ \rho_{t_2} \in \text{End}_{k_1k_2}(M).
\]

But \( \rho_t \in \text{End}_{k_0}(M) \), so, by Proposition 1.2, \( k_0 = k_1k_2 \). This contradicts the minimality of \( k_0 \). Thus \( k_0 = \infty \). \( \square \)

This proposition leads naturally to the following definition:

**Definition 2.2.** Let \( M \) be a von Neumann algebra. We say that an \( E_0 \)-semigroup \( \bar{\rho} = \{\rho_t : t \in \mathbb{R}_+\} \) of \( M \) is inner if

\[
\rho_t \in \text{End}_{\infty}(M), \quad \text{for all} \quad t \in \mathbb{R}_+^k.
\]

We shall denote by \( \text{Einn}(M) \) the set of all inner \( E_0 \)-semigroups acting on the Neumann algebra \( M \). We note that the existence of an inner \( E_0 \)-semigroup of \( M \) makes sense only if \( M \) is a properly infinite von Neumann algebra.

**Remark 2.3.** In [Ar1], W. Arveson showed that any \( E_0 \)-semigroup of a type \( I_\infty \) factor is inner. Moreover, we can easily prove that if \( M \) is a properly infinite factor acting on some Hilbert space \( \mathcal{H} \), then any inner \( E_0 \)-semigroup of \( M \) can be extended to an \( E_0 \)-semigroup of \( \mathcal{B}(\mathcal{H}) \) [Fl]. Therefore we expect to have, and we have large similarities between the class of inner \( E_0 \)-semigroups and the class of \( E_0 \)-semigroups acting on a type \( I_\infty \) factor. For example, the inner \( E_0 \)-semigroups can be completely characterized in terms of product systems in von Neumann algebras [Fl], similar to Arveson’s characterization of \( E_0 \)-semigroups of a type \( I_\infty \) factor [Ar1]. They also can be classified in a way similar to Powers’ classification [Po2]. We wish to discuss more about these results in a future publication.
We introduce the class of properly outer $E_0$-semigroups as follows:

**Definition 2.4.** We say that an $E_0$-semigroup $\tilde{\rho} = \{\rho_t : t \in \mathbb{R}_+\}$ acting on a von Neumann algebra $M$ is properly outer, if for any central projection $p \in M^\tilde{\rho} := \{x \in M : \rho_t(x) = x \text{ for all } t \in \mathbb{R}_+\}$, the $E_0$-semigroup $\tilde{\rho}|_{Mp} := \{\rho_t|_{Mp} : t \in \mathbb{R}_+\}$ is not an inner $E_0$-semigroup of the von Neumann algebra $Mp$.

**Remark 2.5.** We can infer from the above definition that any $E_0$-semigroup acting on a type $\Pi_1$ factor is properly outer. $E_0$-semigroups on the hyperfinite type $\Pi_1$ factor were constructed in [Po1].

**Proposition 2.6.** Let $M$ be a von Neumann algebra and $\tilde{\rho} = \{\rho_t : t \in \mathbb{R}_+\}$ be an $E_0$-semigroup acting on $M$. If
\[
Q(\tilde{\rho}) := \{p \in \mathcal{P}(M) \cap \mathcal{Z}(M) \cap M^\tilde{\rho} : \tilde{\rho}|_{Mp} \in \text{Einn}(Mp)\},
\]
then the set $Q(\tilde{\rho})$ is closed under taking suprema.

**Proof.** We prove this proposition by using a maximality argument, similar to the one in [Ka, Lemma 1.9]. Let $\{p_i\}_{i \in I} \subseteq Q(\tilde{\rho})$ be an arbitrary family of projections, and define $p = \bigvee_{i \in I} p_i$. Then $p$ is a central projection of $M$ sitting in $M^\tilde{\rho}$, and we want to show that
\[
\tilde{\rho}|_{Mp} \in \text{Einn}(Mp).
\]
For proving this, we choose by Zorn’s lemma a maximal (countable) family $\{q_j\}_{j \in J}$ of mutually orthogonal central projections of $M$ having the following property: for any $j \in J$, there exists $i \in I$ such that $q_j \leq p_i$. As in [Ka, Lemma 1.9], the maximality of the family $\{q_j\}_{j \in J}$ implies that
\[
p = \sum_{j \in J} q_j.
\]
Since a direct sum of inner $E_0$-semigroups is still an inner $E_0$-semigroup, in order to finish this proof it suffices to show that for any $j \in J$,
\[
(2.2) \quad \tilde{\rho}|_{Mq_j} \in \text{Einn}(Mq_j).
\]
For this, we fix $j \in J$ and $t > 0$, and let $i \in I$ be such that $q_j \leq p_i$. Since
\[
\rho_t|_{Mp_i} \in \text{End}_\infty(Mp_i),
\]
we can choose an implementing set
\[
\{u_k(t)\}_{k \in \mathbb{N}} \subseteq \text{Hom}_{Mp_i}(Id, \rho_t|_{Mp_i})
\]
of the endomorphism $\rho_t|_{Mp_i}$. By using (1.3), we have
\[
\rho_t(q_j) = \sum_{k=1}^\infty u_k(t)q_ju_k(t)^* = q_j \left( \sum_{j=1}^\infty u_k(t)u_k(t)^* \right) = q_jp_i = q_j.
\]
Therefore $\rho_t|_{Mq_j}$ is an endomorphism of $Mq_j$. Moreover, we can easily check that the $u_k(t)q_j$ satisfy the Cuntz relations (1.1) in $Mq_j$. It then follows that $\rho_t|_{Mq_j}$ is an $\infty$-inner endomorphism of the von Neumann algebra $Mq_j$ with implementing set $\{u_k(t)q_j\}_{k \in \mathbb{N}}$. This establishes (2.2). The proposition is proved. \(\square\)
Definition 2.7. Let $\tilde{\rho}$ be an $E_0$-semigroup acting on a von Neumann algebra $M$. We define

$$p(\tilde{\rho}) = \bigvee Q(\tilde{\rho}).$$

Thus $p(\tilde{\rho})$ is the largest central projection $p \in M^{\tilde{\rho}}$ such that $\tilde{\rho} |_{Mp}$ is an inner $E_0$-semigroup of the von Neumann algebra $Mp$.

The two classes of $E_0$-semigroups introduced above can be characterized in terms of the projection $p(\tilde{\rho})$: an $E_0$-semigroup $\tilde{\rho}$ is inner if and only if $p(\tilde{\rho}) = 1$. It is properly outer if and only if $p(\tilde{\rho}) = 0$.

Moreover, the properties of the projection $p(\tilde{\rho})$ allow us also to decompose the $E_0$-semigroup $\tilde{\rho}$ as follows:

**Theorem 2.8.** Let $M$ be a von Neumann algebra and let $\tilde{\rho} = \{\rho_t : t \in \mathbb{R}_+\}$ be an $E_0$-semigroup acting on $M$. Then $M$ can be decomposed as a direct sum of von Neumann subalgebras

$$M = M_1 \oplus M_2,$$

and, for any $t \in \mathbb{R}_+$, $\rho_t$ can be decomposed as a direct sum of endomorphisms

$$\rho_t = \rho'_t \oplus \rho''_t,$$

such that

1. $\{\rho'_t : t \in \mathbb{R}_+\}$ is an inner $E_0$-semigroup acting on $M_1$,
2. $\{\rho''_t : t \in \mathbb{R}_+\}$ is a properly outer $E_0$-semigroup acting on $M_2$.

This decomposition is unique.

**Proof.** First of all, we show the existence of such a decomposition. Indeed, if we define

$$M_1 := Mp(\tilde{\rho}), \quad M_2 := M(1 - p(\tilde{\rho})),$$

and

$$\rho'_t = \rho_t |_{M_1}, \quad \rho''_t = \rho_t |_{M_2},$$

then it is easily seen that this setting gives the required decomposition.

Next, we prove that such a decomposition can be uniquely realized. For this, let $q \in M^{\tilde{\rho}}$ be a central projection of $M$ such that

1. $\tilde{\rho} |_{Mq} \in \text{Einn}(Mq)$;
2. $\tilde{\rho} |_{M(1-q)}$ is a properly outer $E_0$-semigroup of $M(1-q)$.

The maximality of the projection $p(\tilde{\rho})$ implies that $q \leq p(\tilde{\rho})$, and we claim that we have equality. Indeed, if $q \neq p(\tilde{\rho})$, then $p(\tilde{\rho})(1 - q)$ is a non-zero central projection in $M^{\tilde{\rho}}$. By using (b), we obtain that

$$\tilde{\rho} |_{Mp(\tilde{\rho})(1-q)} \notin \text{Einn}(Mp(\tilde{\rho})(1-q)).$$

On the other hand, since $Mp(\tilde{\rho})(1-q) \subset Mp(\tilde{\rho})$, relation (a) and an argument similar to that used in the proof of Proposition 2.6 imply that $\tilde{\rho} |_{Mp(\tilde{\rho})(1-q)}$ is an inner $E_0$-semigroup acting on $Mp(\tilde{\rho})(1-q)$. This is a contradiction. The theorem is proved.
3. Conjugacy and cocycle conjugacy

We recall the notions of conjugacy and cocycle conjugacy of $E_0$-semigroups acting on von Neumann algebras (cf. [Pol1], [Ar1]):

**Definition 3.1.** Let $\tilde{\rho} = \{\rho_t : t \in \mathbb{R}_+\}$ and $\tilde{\sigma} = \{\sigma_t : t \in \mathbb{R}_+\}$ be $E_0$-semigroups acting on the von Neumann algebra $M$, respectively $N$. We say that $\tilde{\rho}$ and $\tilde{\sigma}$ are conjugate if there exists a *-isomorphism $\theta$ of $M$ onto $N$ such that

$$\theta \circ \rho_t = \sigma_t \circ \theta, \text{ for all } t \in \mathbb{R}_+.$$  

$\tilde{\rho}$ and $\tilde{\sigma}$ are said to be cocycle conjugate if there exists a strongly continuous family of unitaries $\{u_t : t \in \mathbb{R}_+\}$ of $N$ such that

(i) $u_{s+t} = u_s \sigma_s(u_t), \ s, t \in \mathbb{R}_+$;
(ii) $\tilde{\rho}$ is conjugate to the $E_0$-semigroup $\{\text{Ad}(u_t) \circ \sigma_t | t \in \mathbb{R}_+\}.$

A strongly continuous family of unitaries satisfying (i) will be called a $\tilde{\sigma}$-cocycle.

The general problem of conjugacy of two arbitrary $E_0$-semigroups can be reduced in the following way:

**Theorem 3.2.** Let $M$ be a von Neumann algebra and $\tilde{\rho} = \{\rho_t : t \in \mathbb{R}_+\}$, $\tilde{\sigma} = \{\sigma_t : t \in \mathbb{R}_+\}$ be two $E_0$-semigroups of $M$ decomposed as in Theorem 2.8:

$$\rho_t = \rho'_t \oplus \rho''_t, \ \sigma_t = \sigma'_t \oplus \sigma''_t, \ \forall t \in \mathbb{R}_+ .$$

Then $\tilde{\rho}$ and $\tilde{\sigma}$ are conjugate if and only if the following conditions are satisfied:

(i) The inner $E_0$-semigroups $\tilde{\rho}' = \{\rho'_t : t \in \mathbb{R}_+\}$ and $\tilde{\sigma}' = \{\sigma'_t : t \in \mathbb{R}_+\}$ are conjugate;
(ii) The properly outer $E_0$-semigroups $\tilde{\rho}'' = \{\rho''_t : t \in \mathbb{R}_+\}$ and $\tilde{\sigma}'' = \{\sigma''_t : t \in \mathbb{R}_+\}$ are conjugate.

**Proof.** ($\Rightarrow$) Assume that $\tilde{\rho}$ and $\tilde{\sigma}$ are conjugate $E_0$-semigroups, and let $\theta \in \text{Aut}(M)$ be such that $\theta \circ \rho_t = \sigma_t \circ \theta$, for all $t \in \mathbb{R}_+$. It follows that for any $t > 0$, we have

$$\theta^{-1} \circ \sigma_t \circ \theta |_{M \theta(p(\tilde{\rho}))} = \rho_t |_{M \theta(p(\tilde{\rho}))} = \rho'_t \in \text{End}_\infty(M \theta(p(\tilde{\rho}))).$$

In particular, this last relation implies that

$$\theta(p(\tilde{\rho})) \in M^\theta .$$

We claim that $\tilde{\sigma} |_{M \theta(p(\tilde{\rho}))}$ is an inner $E_0$-semigroup of the von Neumann algebra $M \theta(p(\tilde{\rho}))$. For proving this claim, let $t > 0$ be fixed, and let

$$\{u_k(t)\}_{k \in \mathbb{N}} \subset \text{Hom}_{M \theta(p(\tilde{\rho}))}(\text{Id}, \rho_t |_{M \theta(p(\tilde{\rho}))})$$

be an implementing set of the $\infty$-inner endomorphism $\rho_t |_{M \theta(p(\tilde{\rho}))}$. Then for any $x \in M \theta(p(\tilde{\rho}))$ and $i, j, k \in \mathbb{N}$, we have

$$\sigma_i(x) \theta(u_k(t)) = \theta(\rho_t(u_k(t))) \theta(\sigma_i(t)x) = \theta(\rho_t(u_k(t))(\theta(x)),$$

as well as

$$\theta(u_i(t)) \theta(u_j(t)) = \delta_{i,j} \cdot \theta(p(\tilde{\rho})) \text{ and } \sum_{k=1}^\infty \theta(u_k(t)) \theta(u_k(t))^* = \theta(p(\tilde{\rho})).$$

In conclusion, $\{\theta(u_k(t))\}_{k \in \mathbb{N}} \subset \text{Hom}_{M \theta(p(\tilde{\rho}))}(\text{Id}, \sigma_t |_{M \theta(p(\tilde{\rho}))})$ and the $\theta(u_k(t))$ satisfy the Cuntz relations (1.1). Hence

$$\sigma_t |_{M \theta(p(\tilde{\rho}))} \in \text{End}_\infty(M \theta(p(\tilde{\rho}))).$$
so \( \sigma|_{M(p(\tilde{\sigma}))} \) is an inner \( E_0 \)-semigroup.

It then follows from the definition of the projection \( p(\tilde{\sigma}) \) that \( \theta(p(\tilde{\rho})) \leq p(\tilde{\sigma}). \) Since \( \theta \) is an automorphism, we also have that
\[
\tilde{\rho}|_{M\theta^{-1}(p(\tilde{\sigma}))} \in \text{Einn}(M\theta^{-1}(p(\tilde{\sigma}))).
\]
Thus
\[
\theta(p(\tilde{\rho})) = p(\tilde{\sigma}).
\]

This relation implies that the mappings
\[
\theta|_{M(p(\tilde{\rho}))} : M(p(\tilde{\rho})) \to M(p(\tilde{\sigma})) \quad \text{and} \quad \theta|_{M(1-p(\tilde{\rho}))} : M(1-p(\tilde{\rho})) \to M(1-p(\tilde{\sigma}))
\]
are \(*\)-isomorphisms of von Neumann algebras. By construction, \( \theta|_{M(p(\tilde{\rho}))} \), respectively \( \theta|_{M(1-p(\tilde{\rho}))} \), implements the conjugacy between \( \tilde{\rho}' = \tilde{\rho}|_{M(p(\tilde{\rho}))} \) and \( \tilde{\sigma}' = \tilde{\sigma}|_{M(p(\tilde{\rho}))} \), respectively between \( \tilde{\rho}'' = \tilde{\rho}|_{M(1-p(\tilde{\rho}))} \) and \( \tilde{\sigma}'' = \tilde{\sigma}|_{M(1-p(\tilde{\rho}))} \).

(\( \Leftarrow \)) Let \( \theta' : M(p(\tilde{\rho})) \to M(p(\tilde{\sigma})) \) and \( \theta'' : M(1-p(\tilde{\rho})) \to M(1-p(\tilde{\sigma})) \) be \(*\)-isomorphisms of von Neumann algebras such that for any \( t \in \mathbb{R}_+ \),
\[
\theta' \circ \rho'_t = \sigma'_t \circ \theta', \quad \theta'' \circ \rho''_t = \sigma''_t \circ \theta''.
\]

If we define \( \theta = \theta' \oplus \theta'' \in \text{Aut}(M) \), then we have
\[
\theta \circ \rho_t = (\theta' \oplus \theta'') \circ (\rho'_t \oplus \rho''_t) = (\theta' \circ \rho'_t) \oplus (\theta'' \circ \rho''_t) = (\sigma'_t \circ \theta') \oplus (\sigma''_t \circ \theta'') = \sigma_t \circ \theta.
\]
This finishes the proof of the theorem. \( \square \)

A similar characterization can be established in the case of cocycle conjugacy:

**Corollary 3.3.** Let \( M \) be a von Neumann algebra and \( \tilde{\sigma} = \{\sigma_t : t \in \mathbb{R}_+\} \) be two \( E_0 \)-semigroups of endomorphisms of \( M \) decomposed as in Theorem 2.8:
\[
\rho_t = \rho'_t \oplus \rho''_t, \quad \sigma_t = \sigma'_t \oplus \sigma''_t, \quad \forall t \in \mathbb{R}_+.
\]

Then \( \tilde{\rho} \) and \( \tilde{\sigma} \) are cocycle conjugate if and only if the following conditions are satisfied:

1. The inner \( E_0 \)-semigroups \( \tilde{\rho}' = \{\rho'_t : t \in \mathbb{R}_+\} \) and \( \tilde{\sigma}' = \{\sigma'_t : t \in \mathbb{R}_+\} \) are cocycle conjugate;
2. The properly outer \( E_0 \)-semigroups \( \tilde{\rho}'' = \{\rho''_t : t \in \mathbb{R}_+\} \) and \( \tilde{\sigma}'' = \{\sigma''_t : t \in \mathbb{R}_+\} \) are cocycle conjugate.

**Proof.** (\( \Rightarrow \)) Let \( \{u_t\}_{t \in \mathbb{R}_+} \) be a \( \tilde{\sigma} \)-cocycle. For any \( t \in \mathbb{R}_+ \), we define
\[
u'_t := u_t p(\tilde{\sigma}) \in M(p(\tilde{\sigma})), \quad \nu''_t := u_t (1-p(\tilde{\sigma})) \in M(1-p(\tilde{\sigma})).
\]
It is easily seen that \( \{\nu'_t\}_{t \in \mathbb{R}_+} \) is a \( \tilde{\sigma}' \)-cocycle and \( \{\nu''_t\}_{t \in \mathbb{R}_+} \) is a \( \tilde{\sigma}'' \)-cocycle. On the other hand, the \( E_0 \)-semigroup \{Ad(\nu_t) \circ \sigma_t : t \in \mathbb{R}_+\} decomposes as
\[
\text{Ad}(\nu_t) \circ \sigma_t = (\text{Ad}(\nu'_t) \circ \sigma'_t) \oplus (\text{Ad}(\nu''_t) \circ \sigma''_t), \quad t \in \mathbb{R}_+,
\]
where
\[
\{\text{Ad}(\nu'_t) \circ \sigma'_t : t \in \mathbb{R}_+\} \in \text{Einn}(M(p(\tilde{\sigma})))
\]
and \( \{\text{Ad}(\nu''_t) \circ \sigma''_t : t \in \mathbb{R}_+\} \) is a properly outer \( E_0 \)-semigroup acting on the von Neumann algebra \( M(1-p(\tilde{\sigma})) \). The conclusion follows then from Theorem 3.2.
Let \( \{u_t\}_t \in \mathbb{R}_+ \) be a \( \tilde{\sigma}' \)-cocycle and \( \{u_t''\}_t \in \mathbb{R}_+ \) be a \( \tilde{\sigma}'' \)-cocycle. For any \( t \in \mathbb{R}_+ \), we define
\[
u_t := u'_t \oplus u''_t.
\]
Clearly, \( \{\nu_t\}_t \in \mathbb{R}_+ \) is a strongly continuous family of unitaries of \( M \). Moreover, for \( s, t \in \mathbb{R}_+ \), we have
\[
u_s\tilde{\sigma}_s(\nu_t) = (u'_s \oplus u''_s) \cdot (\sigma'_s \oplus \sigma''_s) (u'_t \oplus u''_t) = u'_s\tilde{\sigma}_s(\nu_t') \oplus u''_s\sigma''_s(\nu_t'') = u'_{s+t} \oplus u''_{s+t} = \nu_{s+t}.
\]
Therefore \( \{\nu_t\}_t \in \mathbb{R}_+ \) is a \( \tilde{\sigma} \)-cocycle. Then an argument similar to that one used in the proof of Theorem 3.2 shows that the \( E_0 \)-semigroups \( \tilde{\rho} \) and \( \{\text{Ad}(\nu_t)\circ \sigma_t : t \in \mathbb{R}_+\} \) are conjugate. The corollary is proved. □

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