On Two Exponents of Approximation Related to a Real Number and Its Square

Damien Roy

Abstract. For each real number \( \xi \), let \( \hat{\lambda}_2(\xi) \) denote the supremum of all real numbers \( \lambda \) such that, for each sufficiently large \( X \), the inequalities \(|x_0| \leq X\), \(|x_0\xi - x_1| \leq X^{-\lambda}\) and \(|x_0\xi^2 - x_2| \leq X^{-\lambda}\) admit a solution in integers \( x_0, x_1 \) and \( x_2 \) not all zero, and let \( \hat{\omega}_2(\xi) \) denote the supremum of all real numbers \( \omega \) such that, for each sufficiently large \( X \), the dual inequalities \(|x_0 + x_1\xi + x_2\xi^2| \leq X^{-\omega}\), \(|x_1| \leq X\) and \(|x_2| \leq X\) admit a solution in integers \( x_0, x_1 \) and \( x_2 \) not all zero. Answering a question of Y. Bugeaud and M. Laurent, we show that the exponents \( \hat{\lambda}_2(\xi) \) where \( \xi \) ranges through all real numbers \( \left[ \Omega(\xi) : \Omega \right] > 2 \) form a dense subset of the interval \( \left[ \frac{1}{2}, \frac{\sqrt{5} - 1}{2} \right] \) while, for the same values of \( \xi \), the dual exponents \( \hat{\omega}_2(\xi) \) form a dense subset of \( \left[ 2, \frac{\sqrt{5} + 3}{2} \right] \). Part of the proof rests on a result of V. Jarník showing that \( \hat{\lambda}_2(\xi) = 1 - \frac{1}{\hat{\omega}_2(\xi)} \) for any real number \( \xi \) with \( \left[ \Omega(\xi) : \Omega \right] > 2 \).

1 Introduction

Let \( \xi \) and \( \eta \) be real numbers. Following the notation of Y. Bugeaud and M. Laurent [3], we define \( \hat{\lambda}(\xi, \eta) \) to be the supremum of all real numbers \( \lambda \) such that the inequalities

\[
|x_0| \leq X, \quad |x_0\xi - x_1| \leq X^{-\lambda} \quad \text{and} \quad |x_0\eta - x_2| \leq X^{-\lambda}
\]

admit a non-zero integer solution \((x_0, x_1, x_2) \in \mathbb{Z}^3\) for each sufficiently large value of \( X \). Similarly, we define \( \hat{\omega}(\xi, \eta) \) to be the supremum of all real numbers \( \omega \) such that the inequalities

\[
|x_0 + x_1\xi + x_2\eta| \leq X^{-\omega}, \quad |x_1| \leq X \quad \text{and} \quad |x_2| \leq X
\]

admit a non-zero solution \((x_0, x_1, x_2) \in \mathbb{Z}^3\) for each sufficiently large value of \( X \). An application of Dirichlet box principle shows that we have \( 1/2 \leq \hat{\lambda}(\xi, \eta) \) and \( 2 \leq \hat{\omega}(\xi, \eta) \). Moreover, in the (non-degenerate) case where \( 1, \xi \) and \( \eta \) are linearly independent over \( \mathbb{Q} \), a result of V. Jarník, kindly pointed out to the author by Yann Bugeaud, shows that these exponents are related by the formula

\[
\hat{\lambda}(\xi, \eta) = 1 - \frac{1}{\hat{\omega}(\xi, \eta)},
\]

with the convention that the right-hand side of this equality is 1 if \( \hat{\omega}(\xi, \eta) = \infty \) (see [7, Theorem 1]).

Received by the editors July 23, 2004; revised October 12, 2004.
Work partially supported by NSERC and CICMA
AMS subject classification: Primary: 11J13; secondary: 11J82.

211
In the case where \( \eta = \xi^2 \), we use the shorter notation \( \tilde{\lambda}_2(\xi) := \tilde{\lambda}(\xi, \xi^2) \) and \( \tilde{\omega}_2(\xi) := \tilde{\omega}(\xi, \xi^2) \) of [3]. The condition that \( 1, \xi \) and \( \xi^2 \) are linearly independent over \( \mathbb{Q} \) simply means that \( \xi \) is not an algebraic number of degree at most 2 over \( \mathbb{Q} \), a condition which we also write as \([\mathbb{Q}(\xi) : \mathbb{Q}] > 2\). Under this condition, it is known that these exponents satisfy

\[
2 \leq \tilde{\lambda}_2(\xi) \leq 1/\gamma = 0.618 \ldots \quad \text{and} \quad 2 \leq \tilde{\omega}_2(\xi) \leq \gamma^2 = 2.618 \ldots ,
\]

where \( \gamma = (1 + \sqrt{5})/2 \) denotes the golden ratio. By virtue of W. M. Schmidt's subspace theorem, the lower bounds in (2) are achieved by any algebraic number \( \xi \) of degree at least 3 (see [12, Ch. VI, Corollaries 1C, 1E]). They are also achieved by almost all real numbers \( \xi \), with respect to Lebesgue's measure (see [3, Theorem 2.3]). On the other hand, the upper bounds follow respectively from [5, Theorem 1a] and from [2]. They are achieved in particular by the so-called Fibonacci continued fractions (see [8, §2] or [9, §6]), a special case of the Sturmian continued fractions of [1].

Now, thanks to Jarník's formula (1), we recognize that each set of inequalities in (2) can be deduced from the other one.

Generalizing the approach of [8], Bugeaud and Laurent have computed the exponents \( \tilde{\lambda}_2(\xi) \) and \( \tilde{\omega}_2(\xi) \) for a general (characteristic) Sturmian continued fraction \( \xi \). They found that, after \( 1/\gamma \) and \( \gamma^2 \), the next largest values of \( \tilde{\lambda}_2(\xi) \) and \( \tilde{\omega}_2(\xi) \) for such numbers \( \xi \) are, respectively, \( 2 - \sqrt{2} \approx 0.586 \) and \( 1 + \sqrt{2} \approx 2.414 \), and they asked if there exists any transcendental real number \( \xi \) which satisfies either \( 2 - \sqrt{2} < \tilde{\lambda}_2(\xi) < 1/\gamma \) or \( 1 + \sqrt{2} < \tilde{\omega}_2(\xi) < \gamma^2 \) (see [3, §8]). Our main result below shows that such numbers exist.

**Theorem** The points \((\tilde{\lambda}_2(\xi), \tilde{\omega}_2(\xi))\) where \( \xi \) runs through all real numbers with \([\mathbb{Q}(\xi) : \mathbb{Q}] > 2\) form a dense subset of the curve \( C = \{ (1 - \omega^{-1}, \omega) ; 2 \leq \omega \leq \gamma^2 \} \).

Since \((\tilde{\lambda}_2(\xi), \tilde{\omega}_2(\xi)) = (1/2, 2)\) for any algebraic number \( \xi \) of degree at least 3, it follows in particular that \((1/\gamma, \gamma^2)\) is an accumulation point for the set of points \((\tilde{\lambda}_2(\xi), \tilde{\omega}_2(\xi))\) with \( \xi \) a transcendental real number. Because of Jarník’s formula (1), this theorem is equivalent to either one of the following two assertions.

**Corollary** The exponents \( \tilde{\lambda}_2(\xi) \) attached to transcendental real numbers \( \xi \) form a dense subset of the interval \([1/2, 1/\gamma]\). The corresponding dual exponents \( \tilde{\omega}_2(\xi) \) form a dense subset of \([2, \gamma^2]\).

The proof is inspired by the constructions of [9, §6] and [11, §5]. We produce countably many real numbers \( \xi \) of “Fibonacci type” (see §7 for a precise definition) for which we show that the exponents \( \tilde{\omega}_2(\xi) \) are dense in \([2, \gamma^2]\). By (1), this implies the theorem. One may then reformulate the question of Bugeaud and Laurent by asking if there exist transcendental real numbers \( \xi \) not of that type which satisfy \( \tilde{\omega}_2(\xi) > 1 + \sqrt{2} \). The work of S. Fischler announced in [6] should shed some light on this question.
2 Notation and Equivalent Definitions of the Exponents

We define the norm of a point \( x = (x_0, x_1, x_2) \in \mathbb{R}^3 \) as its maximum norm
\[
\|x\| = \max_{0 \leq i \leq 2} |x_i|.
\]

Given a second point \( y \in \mathbb{R}^3 \), we denote by \( x \wedge y \) the standard vector product of \( x \) and \( y \), and by \( \langle x, y \rangle \) their standard scalar product. Given a third point \( z \in \mathbb{R}^3 \), we also denote by \( \det(x, y, z) \) the determinant of the 3 \times 3 matrix whose rows are \( x, y \) and \( z \). Then we have the well-known relation
\[
\det(x, y, z) = \langle x, y \wedge z \rangle
\]
and we get the following alternative definition of the exponents \( \hat{\lambda}(\xi, \eta) \) and \( \hat{\omega}(\xi, \eta) \).

**Lemma 2.1** Let \( \xi, \eta \in \mathbb{R} \), and let \( y = (1, \xi, \eta) \). Then \( \hat{\lambda}(\xi, \eta) \) is the supremum of all real numbers \( \lambda \) such that, for each sufficiently large real number \( X \geq 1 \), there exists a point \( x \in \mathbb{Z}^3 \) with
\[
0 < \|x\| \leq X \quad \text{and} \quad \|x \wedge y\| \leq X^{-\lambda}.
\]
Similarly, \( \hat{\omega}(\xi, \eta) \) is the supremum of all real numbers \( \omega \) such that, for each sufficiently large real number \( X \geq 1 \), there exists a point \( x \in \mathbb{Z}^3 \) with
\[
0 < \|x\| \leq X \quad \text{and} \quad |\langle x, y \rangle| \leq X^{-\omega}.
\]

In the sequel, we will need the following inequalities.

**Lemma 2.2** For any \( x, y, z \in \mathbb{R}^3 \), we have
\[
\text{(3)} \quad \|\langle x, z \rangle y - \langle x, y \rangle z\| \leq 2\|x\|\|y \wedge z\|,
\]
\[
\text{(4)} \quad \|y\|\|x \wedge z\| \leq \|z\|\|x \wedge y\| + 2\|x\|\|y \wedge z\|.
\]

**Proof** Writing \( y = (y_0, y_1, y_2) \) and \( z = (z_0, z_1, z_2) \), we find
\[
\|\langle x, z \rangle y - \langle x, y \rangle z\| = \max_{i=0,1,2} |\langle x, y_i z - z_i y \rangle| \leq 2\|x\|\|y \wedge z\|,
\]
which proves (3). Similarly, one finds \( \|y_i x \wedge z - z_i x \wedge y\| \leq 2\|x\|\|y \wedge z\| \) for \( i = 0, 1, 2 \), and this implies (4).

For any non-zero point \( x \) of \( \mathbb{R}^3 \), let \( [x] \) denote the point of \( \mathbb{P}^2(\mathbb{R}) \) having \( x \) as a set of homogeneous coordinates. Then (4) has a useful interpretation in terms of the projective distance defined for non-zero points \( x \) and \( y \) of \( \mathbb{R}^3 \) by
\[
\text{dist}([x], [y]) = \text{dist}(x, y) = \frac{\|x \wedge y\|}{\|x\|\|y\|}.
\]

Indeed, for any triple of non-zero points \( x, y, z \in \mathbb{R}^3 \), it gives
\[
\text{(5)} \quad \text{dist}([x], [z]) \leq \text{dist}([x], [y]) + 2\text{dist}([y], [z]).
\]
3 Fibonacci Sequences in $GL_2(\mathbb{C})$

A Fibonacci sequence in a monoid is a sequence $(w_i)_{i \geq 0}$ of elements of that monoid such that $w_{i+2} = w_{i+1}w_i$ for each index $i \geq 0$. Clearly, such a sequence is entirely determined by its first two elements $w_0$ and $w_1$. We start with the following observation.

**Proposition 3.1** There exists a non-empty Zariski open subset $U$ of $GL_2(\mathbb{C})^2$ with the following property. For each Fibonacci sequence $(w_i)_{i \geq 0}$ with $(w_0, w_1) \in U$, there exists $N \in GL_2(\mathbb{C})$ such that the matrix

\[
y_i = \begin{cases} 
   w_iN & \text{if } i \text{ is even,} \\
   w_i^tN & \text{if } i \text{ is odd,}
\end{cases}
\]

is symmetric for each $i \geq 0$. Any matrix $N \in GL_2(\mathbb{C})$ such that $w_0N$, $w_1^tN$ and $w_1w_0N$ are symmetric satisfies this property. When $w_0$ and $w_1$ have integer coefficients, we may take $N$ with integer coefficients.

**Proof** Let $(w_i)_{i \geq 0}$ be a Fibonacci sequence in $GL_2(\mathbb{C})$ and let $N \in GL_2(\mathbb{C})$. Defining $y_i$ by (6) for each $i \geq 0$, we find $y_{i+3} = y_{i+1}^tSy_{i+2}$ with $S = N^{-1}$ if $i$ is even and $S = N^{-1}$ if $i$ is odd. Thus, $y_i$ is symmetric for each $i \geq 0$ if and only if it is so for $i = 0, 1, 2$.

Now, for any given point $(w_0, w_1) \in GL_2(\mathbb{C})^2$, the conditions that $w_0N$, $w_1^tN$ and $w_1w_0N$ are symmetric represent a system of three linear equations in the four unknown coefficients of $N$. Let $V$ be the Zariski open subset of $GL_2(\mathbb{C})^2$ consisting of all points $(w_0, w_1)$ for which this linear system has rank 3. Then, for each $(w_0, w_1) \in V$, the $3 \times 3$ minors of this linear system conveniently arranged into a $2 \times 2$ matrix provide a non-zero solution $N$ of the system, whose coefficients are polynomials in those of $w_0$ and $w_1$ with integer coefficients. Then the condition $\det(N) \neq 0$ in turn determines a Zariski open subset $U$ of $V$. To conclude, we note that $U$ is not empty as a short computation shows that it contains the point formed by $w_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $w_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. \(\blacksquare\)

**Definition 3.2** Let $M = Mat_{2 \times 2}(\mathbb{Z}) \cap GL_2(\mathbb{C})$ denote the monoid of $2 \times 2$ integer matrices with non-zero determinant. We say that a Fibonacci sequence $(w_i)_{i \geq 0}$ in $M$ is admissible if there exists a matrix $N \in M$ such that the sequence $(y_i)_{i \geq 0}$ given by (6) consists of symmetric matrices.

Since $M$ is Zariski dense in $GL_2(\mathbb{C})$, Proposition 3.1 shows that almost all Fibonacci sequences in $M$ are admissible. The following example is an illustration of this.

**Example 3.3** Fix integers $a, b, c$ with $a \geq 2$ and $c \geq b \geq 1$, and define

\[w_0 = \begin{pmatrix} 1 & b \\ a & a(b + 1) \end{pmatrix}, \quad w_1 = \begin{pmatrix} 1 & c \\ a & a(c + 1) \end{pmatrix}\]
and

\[ N = \begin{pmatrix} -1 + a(b + 1)(c + 1) & -a(b + 1) \\ -a(c + 1) & a \end{pmatrix}. \]

These matrices belong to \( M \) since \( \det(w_0) = \det(w_1) = a \) and \( \det(N) = -a \). Moreover, one finds that

\[ w_0N = \begin{pmatrix} -1 + a(c + 1) & -a \\ -a & 0 \end{pmatrix}, \quad w_1N = \begin{pmatrix} -1 + a(b + 1) & -a \\ -a & 0 \end{pmatrix} \]

and

\[ w_1w_0N = \begin{pmatrix} -1 + a & -a \\ -a & -a^2 \end{pmatrix} \]

are symmetric matrices. Therefore, the Fibonacci sequence \((w_i)_{i \geq 0}\) constructed on \( w_0 \) and \( w_1 \) is admissible with an associated sequence of symmetric matrices \((y_i)_{i \geq 0}\) given by (6), the first three matrices of this sequence being the above products \( y_0 = w_0N, \ y_1 = w_1N \) and \( y_2 = w_1w_0N \).

4 Fibonacci Sequences of 2 × 2 Integer Matrices

In the sequel, we identify \( \mathbb{R}^3 \) (resp., \( \mathbb{Z}^3 \)) with the space of 2 × 2 symmetric matrices with real (resp., integer) coefficients under the map

\[ x = (x_0, x_1, x_2) \mapsto \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix}. \]

Accordingly, it makes sense to define the determinant of a point \( x = (x_0, x_1, x_2) \) of \( \mathbb{R}^3 \) by \( \det(x) = x_0x_2 - x_1^2 \). Similarly, given symmetric matrices \( x, y \) and \( z \), we write \( x \wedge y \), \( (x, y) \) and \( \det(x, y, z) \) to denote respectively the vector product, scalar product and determinant of the corresponding points.

In this section we look at arithmetic properties of admissible Fibonacci sequences in the monoid \( M \) of Definition 3.2. For this purpose, we define the content of an integer matrix \( w \in \text{Mat}_{2 \times 2}(\mathbb{Z}) \) or of a point \( y \in \mathbb{Z}^3 \) as the greatest common divisor of their coefficients. We say that such a matrix or point is primitive if its content is 1.

**Proposition 4.1** Let \((w_i)_{i \geq 0}\) be an admissible Fibonacci sequence of matrices in \( M \) and let \((y_i)_{i \geq 0}\) be a corresponding sequence of symmetric matrices in \( M \). For each \( i \geq 0 \), define \( z_i = \det(w_i)^{-1}y_i \wedge y_{i+1} \). Then, for each \( i \geq 0 \), we have

(a) \( \tr(w_{i+3}) = \tr(w_{i+1})\tr(w_{i+2}) - \det(w_{i+1})\tr(w_i) \),

(b) \( y_{i+3} = \tr(w_{i+1})y_{i+2} - \det(w_{i+1})y_i \),

(c) \( z_{i+3} = \tr(w_{i+1})z_{i+2} + \det(w_i)z_i \),

(d) \( \det(y_i, y_{i+1}, y_{i+2}) = (-1)^i \det(y_0, y_1, y_2) \det(w_2)^{-1} \det(w_{i+2}) \),

(e) \( z_i \wedge z_{i+1} = (-1)^i \det(y_0, y_1, y_2) \det(w_2)^{-1}y_{i+1} \).

**Proof** For each index \( i \geq 0 \), let \( N_i \) denote the element of \( M \) for which \( y_i = w_iN_i \). According to (6), we have \( N_i = N \) if \( i \) is even and \( N_i = 'N \) if \( i \) is odd. We first prove
(b) following the argument of the proof of [10, Lemma 2.5(i)]. Multiplying both sides of the equality \( w_{i+2} = w_{i+1}w_i \) on the right by \( N_{i+2} = N_i \), we find

\[
y_{i+2} = w_{i+1}y_i,
\]

which can be rewritten as \( y_{i+2} = y_{i+1}N^{-1}_{i+1}y_i \). Taking the transpose of both sides, this gives \( y_{i+2} = y_iN^{-1}_i y_{i+1} = w_iy_{i+1} \). Replacing \( i \) by \( i+1 \) in the latter identity and combining it with (7), we get

\[
y_{i+3} = w_{i+1}y_{i+2} = w^2_{i+1}y_i.
\]

Then (b) follows from (7) and (8), using the fact that, by the Cayley–Hamilton theorem, we have \( w^2_{i+1} = \text{tr}(w_{i+1})w_{i+1} - \det(w_{i+1})I \). Multiplying both sides of (b) on the right by \( N^{-1}_i \) and taking the trace, we deduce that

\[
\text{tr}(y_{i+3}N^{-1}_i) = \text{tr}(w_{i+1})\text{tr}(w_{i+2}) - \det(w_{i+1})\text{tr}(w_i).
\]

This gives (a) because \( \text{tr}(y_{i+3}N^{-1}_i) = \text{tr}(y_{i+3}N^{-1}) = \text{tr}(w_{i+1}) \). Taking the exterior product of both sides of (b) with \( y_{i+1} \), we also find

\[
y_{i+1} \wedge y_{i+3} = \text{tr}(w_{i+1})\det(w_{i+1})z_{i+1} + \det(w_{i+1})\det(w_i)z_i.
\]

Similarly, replacing \( i \) by \( i+1 \) in (b) and taking the exterior product with \( y_{i+3} \) gives

\[
\det(w_{i+3})z_{i+3} = \det(w_{i+2})y_{i+1} \wedge y_{i+3}.
\]

Then (c) follows upon noting that \( \det(w_{i+3}) = \det(w_{i+2}) \det(w_{i+1}) \).

The formula (d) is clearly true for \( i = 0 \). If we assume that it holds for some integer \( i \geq 0 \), then using the formula for \( y_{i+3} \) given by (b) and taking into account the multilinearity of the determinant we find

\[
\det(y_{i+1}, y_{i+2}, y_{i+3}) = -\det(w_{i+1})\det(y_i, y_{i+1}, y_{i+2})
\]

\[
= (-1)^{i+1} \det(y_0, y_1, y_2) \frac{\det(w_{i+3})}{\det(w_2)}.
\]

This proves (d) by induction on \( i \). Then (e) follows since, for any \( x, y, z \in \mathbb{Z}^3 \), we have \( (x \wedge y) \wedge (y \wedge z) = \det(x, y, z) y \) which, in the present case, gives

\[
z_i \wedge z_{i+1} = \det(w_{i+2})^{-1}\det(y_i, y_{i+1}, y_{i+2})y_{i+1}.
\]

**Corollary 4.2** The notation being as in the proposition, assume that \( \text{tr}(w_i) \) and \( \det(w_i) \) are relatively prime for \( i = 0, 1, 2, 3 \) and that \( \det(y_0, y_1, y_2) \neq 0 \). Then for each \( i \geq 0 \),

(a) the points \( y_i, y_{i+1}, y_{i+2} \) are linearly independent,

(b) \( \text{tr}(w_i) \) and \( \det(w_i) \) are relatively prime,

(c) the matrix \( w_i \) is primitive.
(d) the content of \( y_i \) divides \( \det(y_2)/\det(w_2) \).

(e) the point \( \det(w_2)z_i \) belongs to \( \mathbb{Z}^3 \) and its content divides \( \det(y_2) \det(y_0, y_1, y_2) \).

**Proof** The assertion (a) follows from Proposition 4.1(d). Since (b) holds by hypothesis for \( i = 0, 1, 2, 3 \), and since \( \det(w_2) \) and \( \det(w_i) \) have the same prime factors for each \( i \geq 2 \), the assertion (b) follows, by induction on \( i \), from the fact that Proposition 4.1(a) gives \( \text{tr}(w_{i+1}) \equiv \text{tr}(w_i) \text{tr}(w_{i-1}) \mod \det(w_2) \) for each \( i \geq 3 \). Then (c) follows since the content of \( w_i \) divides both \( \text{tr}(w_i) \) and \( \det(w_i) \).

Let \( N \in \mathbb{M} \) such that \( y_2 = w_2N \). For each \( i \), we have \( y_i = w_iN_i \) where \( N_i = N \) if \( i \) is even and \( N_i = N \) if \( i \) is odd. This gives \( y_i \text{Adj}(N_i) = \det(N)w_i \) where \( \text{Adj}(N_i) \in \mathbb{M} \) denotes the adjoint of \( N_i \). Thus, by (c), the content of \( y_i \) divides \( \det(N) = \det(y_2)/\det(w_2) \), as claimed in (d).

The fact that \( \det(w_2)z_i \) belongs to \( \mathbb{Z}^3 \) is clear for \( i = 0, 1, 2 \) because \( \det(w_0) \) and \( \det(w_1) \) divide \( \det(w_2) \). Then Proposition 4.1(c) shows, by induction on \( i \), that \( \det(w_2)z_i \in \mathbb{Z}^3 \) for each \( i \geq 0 \). Moreover, the content of that point divides that of \( \det(w_2)^2z_i \cap z_{i+1} \) which, by (d) and Proposition 4.1(e), divides \( \det(y_0, y_1, y_2) \det(y_2) \). This proves (e).

**Example 4.3** Let \( (w_i)_{i \geq 0} \) and \( (y_i)_{i \geq 0} \) be as in Example 3.3. Since \( w_0, w_1 \) and \( N \) are congruent to matrices of the form \( \begin{pmatrix} 1 \pm 1 & * \\ 0 & 1 \end{pmatrix} \) modulo \( a \) and have determinant \( \pm a \), all matrices \( w_i \) and \( y_i \) are congruent to matrices of the same form modulo \( a \) and their determinant is, up to sign, a power of \( a \). Thus these matrices have relatively prime trace and determinant, and so are primitive for each \( i \geq 0 \). Since \( \det(y_0, y_1, y_2) = a^3(c - b) \), Proposition 4.1(e) shows that the points \( z_i = \det(w_i)^{-1}y_i \cap y_{i+1} \) satisfy \( z_i \cap z_{i+1} = (-1)^i a^2(c - b) y_{i+1} \) for each \( i \geq 0 \). Moreover, we find that \( a^{-1}z_0 = (0, 0, b - c), a^{-1}z_1 = (a, -1 + a(b + 1), -b) \) and \( a^{-1}z_2 = (a, -1 + a(c + 1), -c) \) are integer points. Then Proposition 4.1(c) shows, by induction on \( i \), that \( a^{-1}z_i \in \mathbb{Z}^3 \) for each \( i \geq 0 \). In particular, if \( c = b + 1 \), we deduce from the relation \( a^{-1}z_i \cap a^{-1}z_{i+1} = \pm y_{i+1} \) that \( a^{-1}z_i \) is a primitive integer point for each \( i \geq 0 \).

### 5 Growth Estimates

Define the norm of a \( 2 \times 2 \) matrix \( w = (w_{k,l}) \in \text{Mat}_{2 \times 2}(\mathbb{R}) \) as the largest absolute value of its coefficients \( \|w\| = \max_{1 \leq k,l \leq 2} |w_{k,l}| \), and define \( \gamma = (1 + \sqrt{5})/2 \) as in the introduction. In this section, we provide growth estimates for the norm and determinant of elements of certain Fibonacci sequences in \( \text{GL}_2(\mathbb{R}) \). We first establish two basic lemmas.

**Lemma 5.1** Let \( w_0, w_1 \in \text{GL}_2(\mathbb{R}) \). Suppose that, for \( i = 0, 1 \), the matrix \( w_i \) is of the form \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with \( 1 \leq a \leq \min\{b, c\} \) and \( \max\{b, c\} \leq d \). Then all matrices of the Fibonacci sequence \( (w_i)_{i \geq 0} \) constructed on \( w_0 \) and \( w_1 \) have this form and for each \( i \geq 0 \), they satisfy

\[
\|w_i\| \leq \|w_{i+2}\| \leq 2\|w_i\| \|w_{i+1}\|.
\]
Proof. The first assertion follows by recurrence on \( i \) and is left to the reader. It implies that \( ||w_i|| \) is equal to the element of index \((2, 2)\) of \( w_i \) for each \( i \geq 0 \). Then (9) follows by observing that, for any \( 2 \times 2 \) matrices \( w = (w_k) \) and \( w' = (w'_k) \) with positive real coefficients, the product \( w'w = (w''_k) \) satisfies \( w_{2,2}w'_{2,2} < w''_{2,2} \leq 2||w||||w'||. \)

Lemma 5.2 Let \( (r_i)_{i \geq 0} \) be a sequence of positive real numbers. Assume that there exist constants \( c_1, c_2 > 0 \) such that \( c_1 r_i r_{i+1} \leq r_{i+2} \leq c_2 r_i r_{i+1} \) for each \( i \geq 0 \). Then there also exist constants \( c_3, c_4 > 0 \) such that \( c_3 r_i^2 \leq r_{i+1} \leq c_4 r_i^2 \) for each \( i \geq 0 \).

Proof. Define \( c_3 = \frac{c_1^2}{c_2} \) and \( c_4 = cc_2^2/c_1 \), where \( c \geq 1 \) is chosen so that the condition \( c_3 \leq r_{i+1}/r_i^2 \leq c_4 \) holds for \( i = 0 \). Assuming that the same condition holds for some index \( i \geq 0 \), we find

\[
\frac{r_{i+2}}{r_{i+1}^2} \geq \frac{c_1}{c_2} \frac{r_i}{r_i^{1/2}} \geq c_1 c_2^{1/2} c_3 \geq c_3,
\]

and similarly \( r_{i+2}/r_{i+1}^2 \leq c_4 \). This proves the lemma by recurrence on \( i \).

Proposition 5.3 Let \( (w_i)_{i \geq 0} \) be a Fibonacci sequence in \( GL_2(\mathbb{R}) \). Suppose that there exist real numbers \( c_1, c_2 > 0 \) such that

\[
(10) \quad c_1 ||w_i|| ||w_{i+1}|| \leq ||w_{i+2}|| \leq c_2 ||w_i|| ||w_{i+1}||
\]

for each \( i \geq 0 \). Then there exist constants \( c_3, c_4 > 0 \) for which the inequalities

\[
(11) \quad c_3 ||w_i||^\gamma \leq ||w_{i+1}|| \leq c_4 ||w_i||^\gamma, \quad c_1 |\det(w_i)|^\gamma \leq |\det(w_{i+1})| \leq c_4 |\det(w_i)|^\gamma
\]

hold for each \( i \geq 0 \). Moreover, if there exist \( \alpha, \beta \geq 0 \) such that

\[
(12) \quad (c_2 ||w_i||)^\alpha \leq |\det(w_i)| \leq (c_1 ||w_i||)^\beta
\]

holds for \( i = 0, 1 \), then this relation extends to each \( i \geq 0 \).

Proof. The first assertion of the proposition follows from Lemma 5.2 applied once with \( r_i = ||w_i|| \) and once with \( r_i = |\det(w_i)| \). To prove the second assertion, assume that for some index \( j \geq 0 \) the condition (12) holds both with \( i = j \) and \( i = j + 1 \). We find

\[
|\det(w_{j+2})| = |\det(w_{j+1})| |\det(w_j)| \geq (c_2 ||w_{j+1}||)^\alpha (c_2 ||w_j||)^\alpha \geq (c_2 ||w_{j+2}||)^\alpha
\]

and similarly \( |\det(w_{j+2})| \leq (c_1 ||w_{j+2}||)^\beta \). Therefore, (12) holds with \( i = j + 2 \). By recurrence on \( i \), this shows that (12) holds for each \( i \geq 0 \) if it holds for \( i = 0, 1 \).
On Two Exponents of Approximation

Example 5.4 Let the notation be as in Example 3.3. Since $w_0$ and $w_i$ satisfy the hypotheses of Lemma 5.1, the Fibonacci sequence $(w_i)_{i \geq 0}$ that they generate fulfills for each $i \geq 0$ the condition (10) of Proposition 5.3 with $c_1 = 1$ and $c_2 = 2$. As $\det(w_0) = \det(w_i) = a$, we also note that for this choice of $c_1$ and $c_2$ the condition (12) holds for $i = 0, 1$ with

$$\alpha = \frac{\log a}{\log(2a(c + 1))} \quad \text{and} \quad \beta = \frac{\log a}{\log(a(b + 1))}.$$

Then, for an appropriate choice of $c_1, c_4 > 0$, both (11) and (12) hold for each $i \geq 0$. Moreover, the estimates (9) of Lemma 5.1 imply that the sequence $(w_i)_{i \geq 0}$ is unbounded.

6 Construction of a Real Number

Given sequences of non-negative real numbers with general terms $a_i$ and $b_i$, we write $a_i \ll b_i$ or $b_i \gg a_i$ if there exists a real number $c > 0$ such that $a_i \leq c b_i$ for all sufficiently large values of $i$. We write $a_i \sim b_i$ when $a_i \ll b_i$ and $b_i \ll a_i$. With this notation, we now prove the following result (cf. [11, §5]).

Proposition 6.1 Let $(w_i)_{i \geq 0}$ be an admissible Fibonacci sequence in $\mathbb{M}$ and let $(y_i)_{i \geq 0}$ be a corresponding sequence of symmetric matrices in $\mathbb{M}$. Assume that $(w_i)_{i \geq 0}$ is unbounded and satisfies the conditions

\begin{equation}
\|w_{i+1}\| \sim \|w_i\|^\gamma, \quad |\det(w_{i+1})| \sim |\det(w_i)|^\gamma \quad \text{and} \quad |\det(w_i)| \ll \|w_i\|^\beta.
\end{equation}

for a real number $\beta$ with $0 < \beta < 2$. Viewing each $y_i$ as a point in $\mathbb{Z}^3$, assume that $\det(y_0, y_1, y_2) \neq 0$ and define $z_i = (\det(w_i))^{-1}y_i \wedge y_{i+1}$ for each $i \geq 0$. Then we have

\begin{equation}
\|y_i\| \sim \|w_i\|, \quad |\det(y_i)| \sim |\det(w_i)|, \quad \|z_i\| \sim \|w_{i-1}\|,
\end{equation}

and there exists a non-zero point $y$ of $\mathbb{R}^3$ with $\det(y) = 0$ such that

\begin{equation}
\|y_i \wedge y\| \sim \frac{|\det(w_i)|}{\|w_i\|} \quad \text{and} \quad |z_i \wedge y| \sim \frac{|\det(w_{i+1})|}{\|w_{i+2}\|}.
\end{equation}

If $\beta < 1$, the coordinates of such a point $y$ are linearly independent over $\mathbb{Q}$ and we may assume that $y = (1, \xi, \xi^2)$ for some real number $\xi$ with $[\mathbb{Q}(\xi): \mathbb{Q}] > 2$.

Proof For each $i \geq 0$, let $N_i$ denote the element of $\mathbb{M}$ for which $y_i = w_i N_i$. Putting $N = N_0$, we have by hypothesis $N_i = N$ when $i$ is even and $N_i = iN$ otherwise. This implies that $\|y_i\| \sim \|w_i\|$ and $|\det(y_i)| \sim |\det(w_i)|$. In the sequel, we will repeatedly use these relations as well as the hypothesis (13).

We claim that we have

\begin{equation}
\|y_i \wedge y_{i+1}\| \ll |\det(w_i)| \|w_{i-1}\|.
\end{equation}
To prove this, we define \( J = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \) and note that for each \( i \geq 0 \) the coefficients of the diagonal of \( y_i J y_{i+1} \) coincide with the first and third coefficients of \( y_i \wedge y_{i+1} \) while the sum of the coefficients of \( y_i J y_{i+1} \) outside of the diagonal is the middle coefficient of \( y_i \wedge y_{i+1} \) multiplied by \(-1\). This gives

\[
(17) \quad \| y_i \wedge y_{i+1} \| \leq 2 \| y_i J y_{i+1} \|.
\]

Since \( y_{i+1} = y_i N_i^{-1} y_{i-1} \) and since \( \mathbf{x} J \mathbf{x} = \det(\mathbf{x}) J \) for any symmetric matrix \( \mathbf{x} \), we also find that \( y_i J y_{i+1} = \det(y_i) J N_i^{-1} y_{i-1} \) and therefore \( \| y_i J y_{i+1} \| \ll \| \det(w_i) \| \| w_{i-1} \| \). Combining this with \((17)\) proves our claim \((16)\), which can also be written in the form

\[
(18) \quad \| z_i \| \ll \| w_{i-1} \|.
\]

As \( \| y_i \| \sim \| w_i \| \) and \( \| y_{i+1} \| \sim \| w_i \| \), the estimate \((16)\) shows, in the notation of \$2\$, that

\[
(19) \quad \text{dist}(\{y_i\}, \{y_{i+1}\}) \leq c \delta_i, \quad \text{where} \quad \delta_i = \frac{|\det(w_i)|}{\| w_i \|^2}
\]

and where \( c \) is some positive constant which does not depend on \( i \). Since by hypothesis we have \( |\det(w_i)| \ll \| w_i \|^\beta \) with \( \beta < 2 \), we find that \( \lim_{i \to \infty} \delta_i = 0 \). Since moreover, we have \( \delta_{i+1} \ll \delta_i \), we deduce that there exists an index \( i_0 \geq 1 \) such that \( \delta_{i+1} \leq \delta_i / 4 \) for each \( i \geq i_0 \). Then, using \((5)\), we deduce that

\[
(20) \quad \text{dist}(\{y_i\}, \{y_j\}) \leq \sum_{k=i}^{j-1} 2^{k-i} \text{dist}(\{y_k\}, \{y_{k+1}\}) \leq c \sum_{k=i}^{j-1} 2^{k-i} \delta_k \leq 2c \delta_i
\]

for each choice of \( i \) and \( j \) with \( i_0 \leq i < j \). Thus the sequence \((\{y_i\})_{i \geq 0} \) converges in \( L^2(\mathbb{R}) \) to a point \( \{y\} \) for some non-zero \( y \in \mathbb{R}^3 \). Since the ratio \( |\det(y_i)| / \| y_i \|^2 \) depends only on the class \( \{y_i\} \) of \( y_i \) in \( L^2(\mathbb{R}) \) and tends to 0 like \( \delta_i \) as \( i \to \infty \), we deduce by continuity that \( |\det(y)| / \| y \|^2 = 0 \) and thus that \( \det(y) = 0 \). By continuity, \((20)\) also leads to \( \text{dist}(\{y_i\}, \{y\}) \leq 2c \delta_i \) for each \( i \geq i_0 \), and so

\[
(21) \quad \| y_i \wedge y \| \ll \frac{|\det(w_i)|}{\| w_i \|}.
\]

Applying \((3)\) together with the above estimates \((18)\) and \((21)\), we find

\[
\| \langle z_i, y \rangle y_{i+2} - \langle z_i, y_{i+2} \rangle y \| \leq 2 \| z_i \| \| y \| \ll \| w_{i-1} \| \frac{|\det(w_{i+2})|}{\| w_{i+2} \|} \ll |\det(w_{i+1})| \delta_i.
\]

Using Proposition 4.1(d), we also get

\[
(22) \quad \| \langle z_i, y_{i+2} \rangle y \| = \frac{|\det(y_i, y_{i+1}, y_{i+2})|}{|\det(w_i)|} \| y \| \sim |\det(w_{i+1})|.
\]
Combining the above two estimates, we deduce that \( \| (z_i, y) y_{i+2} \| \sim | \det(w_{i+1}) | \) and therefore that \( |z_i, y| \sim | \det(w_{i+1}) |/\| w_{i+2} \| \). The latter estimate is the second half of (15). It implies

\[
\| (z_{i+1}, y) y_i \| \sim \frac{|\det(w_{i+2})|}{\| w_{i+3} \|} \| w_i \| \sim |\det(w_i)| \delta_{i+1}.
\]

Since \( (z_{i+1}, y_i) = \det(w_{i-1})^{-1} (z_i, y_{i+2}) \), the estimate (22) can also be written in the form \( \| (z_{i+1}, y_i) y \| \sim | \det(w_i) | \). Then, applying (3) once again, we find

\[
2\| z_{i+1} \| \| y_i \| \geq \| (z_{i+1}, y_i) y_i - (z_{i+1}, y_i) y \| \geq | \det(w_i) |.
\]

Since, by (18) and (21), we have \( \| z_{i+1} \| \ll \| w_i \| \) and \( \| y_i \| \ll | \det(w_i) |/\| w_i \| \), we conclude from this that \( \| z_{i+1} \| \sim \| w_i \| \) and \( \| y_i \| \sim | \det(w_i) |/\| w_i \| \), which completes the proof of (14) and (15).

Now, assume that \( \beta < 1 \), and let \( u \in \mathbb{Z}^3 \) such that \( \langle u, y \rangle = 0 \). By (3), we have

\[
(23) \quad 2\| u \| \| y_i \| \geq \| (u, y_i) y_i - (u, y_i) y \| = | (u, y_i) | \| y \|.
\]

for each \( i \geq 0 \). Since \( \| y_i \| \sim | \det(w_i) |/\| w_i \| \ll \| w_i \| \beta^{-1} \) tends to 0 as \( i \to \infty \), we deduce from (23) that the integer \( \langle u, y_i \rangle \) must vanish for all sufficiently large values of \( i \). This implies that \( u = 0 \) because it follows from the hypothesis \( \det(y_0, y_1, y_2) \neq 0 \) and the formula in Proposition 4.1(d) that any three consecutive points of the sequence \( (y_i)_{i=0}^\infty \) are linearly independent. Thus the coordinates of \( y \) must be linearly independent over \( \mathbb{Q} \). In particular, the first coordinate of \( y \) is non-zero and, dividing \( y \) by this coordinate, we may assume that it is equal to 1. Then, upon denoting by \( \xi \) the second coordinate of \( y \), the condition \( \det(y) = 0 \) implies that \( y = (1, \xi, \xi^2) \) and thus \( |\mathbb{Q}(\xi):\mathbb{Q}| > 2 \).

7 Estimates for the Exponent \( \hat{\omega}_2 \)

We first prove the following result and then deduce from it our main theorem in §1.

**Proposition 7.1** Let \( (w_i)_{i \geq 0} \) be an admissible Fibonacci sequence in \( M \), and let \( (y_i)_{i \geq 0} \) be a corresponding sequence of symmetric matrices in \( M \). Assume that \( (w_i)_{i \geq 0} \) is unbounded and satisfies

\[
(24) \quad \| w_{i+1} \| \sim \| w_i \|^{\gamma}, \quad | \det(w_{i+1}) | \sim | \det(w_i) |^{\gamma}, \quad \| w_i \|^{\alpha} \ll | \det(w_i) | \ll \| w_i \|^{\beta}
\]

for real numbers \( \alpha \) and \( \beta \) with \( 0 \leq \alpha \leq \beta < \gamma^{-2} \). Assume moreover that \( \text{tr}(w_i) \) and \( \det(w_i) \) are relatively prime for \( i = 0, 1, 2, 3 \) and that \( \det(y_0, y_1, y_2) \neq 0 \). Then the real number \( \xi \) which comes out from the last assertion of Proposition 6.1 satisfies

\[
\gamma^2 - \beta \gamma \leq \hat{\omega}_2(\xi) \leq \gamma^2 - \alpha \gamma.
\]
Proof. Put \( y = (1, \xi, \xi^2) \) and define the sequence \((z_i)_{i \geq 0}\) as in Proposition 4.1. Since \( \|y\| \geq 1 \), the inequality (3) combined with the estimates of Proposition 6.1 shows that, for any point \( z \in \mathbb{Z}^3 \) and any index \( i \geq 1 \), we have

\[
(25) \quad |\langle z, y_i \rangle| \leq \|y_i\| |\langle z, y \rangle| + 2 \|z\| \|y_i \wedge y\| < c_5 \max \left\{ \|w_i\| |\langle z, y \rangle|, \|z\| \frac{|\det(w_i)|}{\|w_i\|} \right\},
\]

with a constant \( c_5 > 0 \) which is independent of \( z \) and \( i \). Suppose that a point \( z \in \mathbb{Z}^3 \) satisfies

\[
(26) \quad 0 < \|z\| \leq Z_i := c_6 \|w_i\| \quad \text{and} \quad |\langle z, y \rangle| \leq \frac{|\det(w_{i+1})|}{\|w_{i+2}\|},
\]

where \( c_6 = c_5^{-1} |\det(y_2)|^{-1} \). Using (25) with \( i \) replaced by \( i + 1 \), we find

\[
|\langle z, y_{i+1} \rangle| \ll |\det(w_i)|^\gamma \|w_i\|^{-1/\gamma}.
\]

Since \( |\det(w_i)| \ll \|w_i\|^\beta \) with \( \beta < \gamma^{-2} \), this gives \( |\langle z, y_{i+1} \rangle| < 1 \) provided that \( i \) is sufficiently large. Then the integer \((z, y_{i+1})\) must be zero and, by Proposition 4.1(e), we deduce that \( z = az_i + bz_{i+1} \) for some \( a, b \in \mathbb{Q} \) where \( b \) is given by

\[
z_i \wedge z = bz_i \wedge z_{i+1} = (-1)^i b \det(y_0, y_1, y_2) \det(w_2)^{-1} y_{i+1}.
\]

Since \( \det(w_2)z_i \wedge z \in \mathbb{Z}^3 \) and since, by Corollary 4.2(d), the content of \( y_{i+1} \) divides \( \det(y_2)/\det(w_2) \), this implies that \( b \det(y_0, y_1, y_2) \det(w_2)^{-1} \det(y_2) / \det(w_2) \) is an integer. So, if \( b \) is non-zero, it satisfies the lower bound

\[
|b| \geq \left| \frac{\det(y_2)}{\det(w_2)} \right|.
\]

We note that \( \langle z_i, y_i \rangle = 0 \) and by Proposition 4.1(d) that

\[
\langle z_{i+1}, y_i \rangle = \frac{\det(y_i, y_{i+1}, y_{i+2})}{\det(w_{i+1})} = (-1)^i \frac{\det(y_0, y_1, y_2)}{\det(w_2)} \frac{\det(w_i)}{\det(w_{i+1})}.
\]

Therefore, if \( b \neq 0 \), the point \( z = az_i + bz_{i+1} \) satisfies

\[
|\langle z, y_i \rangle| = |b| \left| \langle z_{i+1}, y_i \rangle \right| \geq \left| \det(y_2) \right|^{-1} \left| \det(w_i) \right| = c_5 c_6 |\det(w_i)|.
\]

However, (25) and (26) give

\[
|\langle z, y_i \rangle| \ll c_5 \max \left\{ \frac{|\det(w_{i+1})| \|w_i\|}{\|w_{i+2}\|}, c_6 \left| \det(w_i) \right| \right\} = c_5 c_6 |\det(w_i)|
\]

if \( i \) is sufficiently large, because the ratio \( |\det(w_{i+1})| \|w_i\| / \|w_{i+2}\| \ll \|w_i\|^\beta \gamma^{-2} \) tends to 0 as \( i \rightarrow \infty \). Comparison with the previous inequality then forces \( b = 0 \), and so we get \( z = az_i \) with \( a \neq 0 \). Since \( \det(w_2)z_i \) is, by Corollary 4.2(e), an integer point whose content divides \( \det(y_2) \det(y_0, y_1, y_2) \), we deduce that

\[
a \det(y_2) \det(y_0, y_1, y_2) / \det(w_2)
\]
On Two Exponents of Approximation

is a non-zero integer and therefore, using the second part of (15) in Proposition 6.1, we find that

\[ |(z, y)| = |a| |(z_i, y)| \geq \frac{|\det(w_2)|}{|\det(y_2)\det(y_0, y_1, y_2)|} |(z_i, y)| \gg \frac{|\det(w_{i+1})|}{\|w_{i+2}\|}. \]

Since this holds for any point \( z \) satisfying (26) with \( i \) sufficiently large, we deduce that for any index \( i \geq 0 \) and any point \( z \in \mathbb{Z}^1 \) with \( 0 < \|z\| \leq Z \), we have

\[ |(z, y)| \gg \frac{|\det(w_{i+1})|}{\|w_{i+2}\|} \gg \|w_i\|^{\gamma_0 - \gamma^2} \gg Z_1^{\gamma_0 - \gamma^2}. \]

This shows that \( \hat{\omega}_2(\xi) \leq \gamma^2 - \gamma \alpha \).

Finally, for any real number \( Z \geq \|z_0\| \), there exists an index \( i \geq 0 \) such that \( \|z_i\| \leq Z < \|z_{i+1}\| \) and, for such choice of \( i \), we find by Proposition 6.1 that

\[ |(z_i, y)| \ll \frac{|\det(w_{i+1})|}{\|w_{i+2}\|} \ll \|w_i\|^{\beta_0 - \gamma^2} \sim \|z_{i+1}\|^{\beta_1 - \gamma^2} \ll Z^{\beta_0 - \gamma^2}, \]

showing that \( \hat{\omega}_2(\xi) \geq \gamma^2 - \gamma \beta \).

Let us say that a real number \( \xi \) is of “Fibonacci type” if there exist an unbounded Fibonacci sequence \( (w_i)_{i \geq 0} \) in \( \mathbb{M} \) and a real number \( \theta \) with \( \theta > 1/\gamma \) such that \( \|(\xi, -1)w_i\| \leq \|w_i\|^{-\theta} \) for each sufficiently large index \( i \). There are countably many such numbers, and any real number \( \xi \) obtained from Proposition 6.1 with \( \beta < \gamma^{-2} \) is of this type. The following corollary shows that the exponents \( \hat{\omega}_2(\xi) \) attached to transcendental numbers of Fibonacci type are dense in the interval \([2, \gamma^2]\). By Jarník’s formula (1), this implies our main theorem in §1.

**Corollary 7.2** Let \( t \) and \( \epsilon \) be real numbers with \( 0 < t < \gamma^{-2} \) and \( \epsilon > 0 \). Then there exist a transcendental real number \( \xi \) and an unbounded Fibonacci sequence \( (w_i)_{i \geq 0} \) in \( \mathbb{M} \) which satisfy

(a) \( \|(\xi, -1)w_i\| \leq \|w_i\|^{-1 + t} \) for each sufficiently large \( i \),

(b) \( \gamma^2 - \gamma \epsilon \leq \hat{\omega}_2(\xi) \leq \gamma^2 - (t - \epsilon) \gamma \).

**Proof** Since \( t < 1 \), there exist integers \( k \) and \( \ell \) with \( 0 < \ell < k \) and \( t - \epsilon \leq \ell/(k+2) \leq \ell/k < t \). For such a choice of \( k \) and \( \ell \), consider the Fibonacci sequence \( (w_i)_{i \geq 0} \) of Example 3.3 with parameters \( a = 2^\ell, b = 2^{k-\ell} - 1 \) and \( c = 2^{k-\ell} \). According to Example 4.3, \( w_i \) has relatively prime trace and determinant for each \( i \geq 0 \) and the corresponding sequence of symmetric matrices \( (y_i)_{i \geq 0} \) satisfies \( \det(y_0, y_1, y_2) = 2^{k\ell} \neq 0 \). Moreover, Example 5.4 shows that \( (w_i)_{i \geq 0} \) is unbounded and satisfies the estimates (24) of Proposition 7.1 with \( \alpha = \ell/(k+2) \) and \( \beta = \ell/k \) (note that the example provides a slightly larger value for \( \alpha \)). So, Proposition 7.1 applies and shows that the corresponding real number \( \xi \) constructed by Proposition 6.1 satisfies the above condition (b). In particular, \( \xi \) is transcendental since \( \hat{\omega}_2(\xi) > 2 \). Moreover, since \( \|(\xi, -1)w_i\| \sim \|(\xi, -1)y_i\| \sim \|y_i \wedge y\| \), the first estimate in (15) leads to (a).
Acknowledgments  The author warmly thanks Yann Bugeaud for pointing out the results of Jarník in [7] which brought a notable simplification to the present paper.

References


Département de Mathématiques
Université d’Ottawa
585 King Edward
Ottawa, ON
K1N 6N5

e-mail: droy@uottawa.ca