SETS OF POLYNOMIAL RINGS IN TWO VARIABLES AND FACTORIZATIONS OF POLYNOMIAL MORPHISMS

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ABSTRACT. Let $\mathbf{k}$ be a field. We study infinite strictly descending sequences $A_0 \supset A_1 \supset \cdots$ of rings where each A_i is a polynomial ring in two variables over $\mathbf{k}$, the aim being to describe those sequences satisfying $\bigcap_{i=0}^{\infty} A_i \neq \mathbf{k}$. We give a complete answer in characteristic zero, and partial results in arbitrary characteristic. We apply those results to the study of dominant morphisms $\mathbb{A}^2 \to \mathbb{A}^n$ and their factorizations, where $n \in \{1, 2\}$ and $\mathbb{A}^n$ is the affine n-space over $\mathbf{k}$.

1. Introduction

Throughout, $\mathbf{k}$ is an arbitrary field unless otherwise specified.

Let $\mathbb{A}^n$ denote the affine n-space over $\mathbf{k}$, i.e., $\mathbb{A}^n = \operatorname{Spec} A$ where A is a polynomial ring in n variables over $\mathbf{k}$. One of the aims of this paper is to study dominant morphisms $\mathbb{A}^2 \to \mathbb{A}^n$ (where $n \in \{1, 2\}$), and factorizations of such morphisms.

Section 2 is preparatory, Sections 3–5 constitute the algebraic core of the article, and Section 6 applies the theory of Sections 3–5 to factorizations of dominant morphisms $\mathbb{A}^2 \to \mathbb{A}^n$, $n \in \{1, 2\}$.

More precisely, Sections 3 and 4 study strictly descending infinite sequences $(A_i)_{i=0}^{\infty}$ where each A_i is a polynomial ring in two variables over $\mathbf{k}$ (by *strictly descending* we mean that $A_0 \supset A_1 \supset A_2 \supset \cdots$, where "$\supset$" denotes strict inclusion). Any such sequence satisfies $\mathbf{k} \subseteq \bigcap_{i=0}^{\infty} A_i$, and the aim of Sections 3 and 4 is to describe the sequences satisfying $\bigcap_{i=0}^{\infty} A_i \neq \mathbf{k}$. Section 3 is devoted to the proof of Thm 3.3, which settles the special case where $(A_i)_{i=0}^{\infty}$ is "birational" (we say that $(A_i)_{i=0}^{\infty}$ is *birational* if all A_i have the same field of fractions). Section 4 deals with the general case. Note that the proof of Thm 3.3 makes essential use of one of the main results of [CND14]. The following fact (Thm 1.1) is a consequence of the results of Sections 3 and 4 and is a good illustration of the type of result contained in those two sections. Recall that if A is a polynomial ring in two variables over $\mathbf{k}$ then by a *variable* of A we mean an element $F \in A$ for which there exists G satisfying $A = \mathbf{k}[F, G]$.

1.1. **Theorem.** *Let $\mathbf{k}$ be a field and $(A_i)_{i=0}^{\infty}$ a strictly descending infinite sequence where each A_i is a polynomial ring in two variables over $\mathbf{k}$ and $\bigcap_{i=0}^{\infty} A_i \neq \mathbf{k}$. Then*

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$\bigcap_{i=0}^{\infty} A_i = \mathbf{k}[F]$ *for some* F *transcendental over* $\mathbf{k}$*. Moreover, if we assume that at least one of the following conditions is satisfied:*

(i) *all* A_i *have the same field of fractions,*
(ii) $\operatorname{char} \mathbf{k} = 0$,

then F *is a variable of* A_i *for* $i \gg 0$.

Indeed, Prop. 4.5 implies that $\bigcap_{i=0}^{\infty} A_i = \mathbf{k}[F]$ for some F, and if (i) (resp. (ii)) is true then Thm 3.3 (resp. Thm 4.15) implies that F is a variable of A_i for $i \gg 0$. So Thm 1.1 follows from Prop. 4.5 and Thms 3.3 and 4.15.

Remark. We don't know if the last assertion of Thm 1.1 continues to be valid when both (i) and (ii) are false; note, however, the partial results Prop. 4.5 and Prop. 4.14.

Section 5 studies chain conditions in sets of polynomial rings. More precisely, let $\mathbf{k}$ be any field, and let us write "$A = \mathbf{k}^{[2]}$" as an abbreviation of "A is a polynomial ring in two variables over $\mathbf{k}$". Given rings $A \subseteq B$ such that $A = \mathbf{k}^{[2]}$ and $B = \mathbf{k}^{[2]}$, consider the set $\mathcal{R}(A, B) = \{ R \mid A \subseteq R \subseteq B \text{ and } R = \mathbf{k}^{[2]} \}$. Then Thm 5.6 states that the poset $(\mathcal{R}(A, B), \subseteq)$ satisfies ACC and DCC.

Let us say that an element F of $R = \mathbf{k}^{[2]}$ is *univariate in* R if there exists $(X, Y) \in R \times R$ such that $R = \mathbf{k}[X, Y]$ and $F \in \mathbf{k}[X]$. Given a pair (A, F) such that $A = \mathbf{k}^{[2]}$ and $F \in A \setminus \mathbf{k}$, define

$$\mathcal{U}(A, F) = \{ R \mid R = \mathbf{k}^{[2]},\ F \in R \subseteq A \text{ and } F \text{ is not univariate in } R \}$$
$$\mathcal{U}^*(A, F) = \{ R \in \mathcal{U}(A, F) \mid \operatorname{Frac} R = \operatorname{Frac} A \}.$$

Then Thms 5.11 and 5.12 state:

- $(\mathcal{U}(A, F), \subseteq)$ satisfies ACC, and if $\operatorname{char} \mathbf{k} = 0$ then it also satisfies DCC.
- $(\mathcal{U}^*(A, F), \subseteq)$ satisfies ACC and DCC.

We call the reader's attention on the fact that Theorems 5.6, 5.11 and 5.12 answer *very natural questions* about polynomial rings in two variables. Their proofs use much of the material developed in Sections 3 and 4 and some results on birational endomorphisms of the affine plane.

Section 6 applies the results of Sections 3–5 to answer some questions regarding factorizations of dominant morphisms $\mathbb{A}^2 \to \mathbb{A}^n$, $n \in \{1, 2\}$. Let us introduce the notations and definitions needed for discussing this.

Let $\mathbf{k}$ be a field and consider algebraic varieties and morphisms over $\mathbf{k}$. Recall that a morphism of varieties $f : X \to Y$ is *dominant* if $f(X)$ is Zariski-dense in Y, and that f is *birational* if there exist nonempty Zariski-open subsets U of X and V of Y such that f restricts to an isomorphism $U \to V$. Given varieties X, Y, we write $\operatorname{Mor}(X, Y)$ for the set of morphisms $X \to Y$. We consider the monoid $\operatorname{Dom}(X)$ of dominant morphisms $X \to X$ (the operation being the composition of morphisms), the monoid $\operatorname{Bir}(X)$ of birational morphisms $X \to X$, and the group $\operatorname{Aut}(X)$ of automorphisms of X. We have $\operatorname{Aut}(X) \subseteq \operatorname{Bir}(X) \subseteq \operatorname{Dom}(X)$.

The monoid $\operatorname{Dom}(\mathbb{A}^1)$ has been studied extensively. It is trivial to see—simply by considering degrees of polynomials—that $\operatorname{Dom}(\mathbb{A}^1)$ is an "atomic" monoid (a monoid M is *atomic* if every non invertible element of M is a finite composition of irreducible elements of M; see Def. 6.1). J.F. Ritt showed in 1922 that irreducible factorizations in $\mathrm{Dom}(\mathbb{A}^1)$ have certain uniqueness properties (cf. [Rit22]), and this gave rise to a research area that is still active today, with generalizations and applications in several fields. So the monoid $\mathrm{Dom}(\mathbb{A}^1)$, though much simpler than $\mathrm{Dom}(\mathbb{A}^2)$, is already an interesting object.

The first part of Section 6 is concerned with the monoid structure of $\mathrm{Dom}(\mathbb{A}^2)$ and in particular Cor. 6.2 states that $\mathrm{Dom}(\mathbb{A}^2)$ is atomic. This appears to be a new result, as far as we can see; in fact it is surprising to see how little is known about $\mathrm{Dom}(\mathbb{A}^2)$. The fact that $\mathrm{Dom}(\mathbb{A}^2)$ is atomic doesn't seem to be provable by simple minded considerations of degree, as in the case of $\mathrm{Dom}(\mathbb{A}^1)$. We obtain it as a trivial consequence of the fact (Thm 5.6) that $\mathcal{R}(A, B)$ satisfies ACC and DCC.

The second part of Section 6 is concerned with lean factorizations, which we now define. Given a morphism of varieties $f : X \to Y$, consider all factorizations of f of the following type:

$$(*) \qquad \begin{tikzcd} X \arrow[r, "\alpha"'] \arrow[rrr, bend left, "f=\beta\circ f'\circ\alpha"] & X \arrow[r, "f'"'] & Y \arrow[r, "\beta"'] & Y \end{tikzcd} \quad \text{with } \ \alpha \in \mathrm{Bir}(X),\ f' \in \mathrm{Mor}(X, Y),\ \beta \in \mathrm{Bir}(Y).$$

We say that f is a *lean morphism* if every factorization $(*)$ of f satisfies $\alpha \in \mathrm{Aut}(X)$ and $\beta \in \mathrm{Aut}(Y)$. By a *lean factorization* of f, we mean a factorization $(*)$ of f in which f' is lean.

The second part of Section 6 presents two results on lean factorizations, namely, we show that every dominant morphism $\mathbb{A}^2 \to \mathbb{A}^2$ admits a lean factorization (Thm 6.5), and we determine which dominant morphisms $\mathbb{A}^2 \to \mathbb{A}^1$ admit a lean factorization (Thm 6.7). The insight provided by Thm 6.7 allows us, in [CND15b] and [CND15a], to make some progress in the open problem of classifying rational polynomials (let us say, briefly, that a *rational polynomial* is a dominant morphism $\mathbb{A}^2 \to \mathbb{A}^1$ whose general fibers are rational curves). The relation between lean factorizations and the classification of rational polynomials is explained in Section 1 of [CND15a].

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2. Preliminaries

Conventions. The symbol $\mathbf{k}$ denotes an arbitrary field, unless otherwise specified.

All algebraic varieties (in particular all curves and surfaces) are irreducible and reduced. Varieties and morphisms are over $\mathbf{k}$.

All rings are commutative and have a unity. The symbol A^* denotes the set of units of a ring A. If A is a subring of a ring B and $n \in \mathbb{N}$, the notation $B = A^{[n]}$ means that B is isomorphic (as an A-algebra) to the polynomial ring in n variables over A. If L/K is a field extension then "$L = K^{(n)}$" means that L is purely transcendental over K, of transcendence degree n; the transcendence degree of L over K is denoted $\mathrm{trdeg}_K L$. The field of fractions of an integral domain A is denoted $\mathrm{Frac}\, A$. If $A \subseteq B$ are domains, $\mathrm{trdeg}_A(B)$ is an abbreviation for $\mathrm{trdeg}_{\mathrm{Frac}\, A}(\mathrm{Frac}\, B)$. We adopt the conventions that $0 \in \mathbb{N}$, that "$\subset$" means strict inclusion and that "$\setminus$" denotes set difference.

2.1. **Variables and coordinate lines.** Let $\mathbf{k}$ be a field.

(i) Let $A = \mathbf{k}^{[2]}$. A *variable* of A is an element $F \in A$ for which there exists G satisfying $\mathbf{k}[F, G] = A$.

(ii) Consider the affine plane over $\mathbf{k}$, $\mathbb{A}^2 = \mathbb{A}^2_{\mathbf{k}} = \operatorname{Spec} A$ (where $A = \mathbf{k}^{[2]}$). A curve $C \subset \mathbb{A}^2$ is called a *coordinate line of* $\mathbb{A}^2$ if it is the zero-set of a variable of A.

Remark. It is clear that if $C \subset \mathbb{A}^2$ is a coordinate line of $\mathbb{A}^2$ then $C \cong \mathbb{A}^1$, and the Abhyankar-Moh-Suzuki Theorem ([AM75], [Suz74]) states that the converse is true if $\operatorname{char} \mathbf{k} = 0$. If $\operatorname{char} \mathbf{k} > 0$, there exist curves $C \subset \mathbb{A}^2$ satisfying $C \cong \mathbb{A}^1$ but which are not coordinate lines (for a survey of this topic, see [Gan11]).

2.2. **Birational morphisms.** Let $\mathbf{k}$ be an algebraically closed field and $\mathbb{A}^2 = \mathbb{A}^2_{\mathbf{k}}$ the affine plane over $\mathbf{k}$. Refer to Section 2 of [CND14] for background on birational endomorphisms of $\mathbb{A}^2$. Recall in particular that a *contracting curve* of a birational morphism $\Phi : \mathbb{A}^2 \to \mathbb{A}^2$ is a curve $C \subset \mathbb{A}^2$ such that $\Phi(C)$ is a point, and that a *missing curve* of Φ is a curve $C \subset \mathbb{A}^2$ such that $\operatorname{im}(\Phi) \cap C$ is a finite set of closed points of $\mathbb{A}^2$; Φ has finitely many contracting curves and missing curves. We write $\operatorname{Cont}(\Phi)$ (resp. $\operatorname{Miss}(\Phi)$) for the set of contracting (resp. missing) curves of Φ, and $c(\Phi) = |\operatorname{Cont}(\Phi)|$ and $q(\Phi) = |\operatorname{Miss}(\Phi)|$ for the cardinalities of these sets ($c(\Phi), q(\Phi) \in \mathbb{N}$).

2.2.1. **Notation.** Consider morphisms $\mathbb{A}^2 \xrightarrow{\Phi} \mathbb{A}^2 \xrightarrow{f} \mathbb{A}^1$ where Φ is birational and f is dominant. Then we write

$$\operatorname{Miss}_{\text{hor}}(\Phi, f) = \big\{\, C \in \operatorname{Miss}(\Phi) \ \mid\ f(C) \text{ is a dense subset of } \mathbb{A}^1 \,\big\}.$$

We refer to the elements of $\operatorname{Miss}_{\text{hor}}(\Phi, f)$ as the "f-horizontal" missing curves of Φ.

The following is used in the proof of Prop. 3.23:

2.2.2. **Lemma.** *Consider morphisms $\mathbb{A}^2 \xrightarrow{\Phi} \mathbb{A}^2 \xrightarrow{f} \mathbb{A}^1$ such that Φ is birational and f is dominant, and let $C \subset \mathbb{A}^2$ be a contracting curve of Φ. Suppose that $\operatorname{Miss}_{\text{hor}}(\Phi, f) = \varnothing$ and that $C = (f \circ \Phi)^{-1}(Q)$ for some $Q \in \mathbb{A}^1$. Then $f^{-1}(Q)$ is a coordinate line in $\mathbb{A}^2$.*

Proof. Define $\Gamma = f^{-1}(Q)$ and observe that $C = \Phi^{-1}(\Gamma)$.

Suppose that some irreducible component D of Γ is not a missing curve of Φ. Then there exists a curve $D' \subset \mathbb{A}^2$ such that $\Phi(D')$ is a dense subset of D. Then $\Phi(D') \subseteq D \subseteq \Gamma$, so $D' \subseteq \Phi^{-1}(\Gamma) = C$, so $D' = C$ and hence $\Phi(D') = \Phi(C)$ is a point, which is absurd.

This shows that Γ is a union of missing curves of Φ. As each missing curve of Φ is included in a fiber of f (because $\operatorname{Miss}_{\text{hor}}(\Phi, f) = \varnothing$), it follows that each missing curve of Φ is either included in Γ or disjoint from Γ; so [CND14, 2.17] implies:

- $\#\Gamma = \#\Phi^{-1}(\Gamma) = \#C = 1$ (where $\#M$ denotes the number of irreducible components of a closed set M), so Γ is irreducible;
- Φ factors as $\mathbb{A}^2 \xrightarrow{\Psi'} \mathbb{A}^2 \xrightarrow{\Psi} \mathbb{A}^2$, where Ψ and Ψ' are birational morphisms and where the irreducible curve Γ is the unique missing curve of Ψ.

Since $\operatorname{Miss}(\Psi) = \{\Gamma\}$ for some birational endomorphism Ψ of $\mathbb{A}^2$, [CND14, 3.4] implies that Γ is a coordinate line of $\mathbb{A}^2$. $\square$

2.3. **Dicriticals.** Let $\mathbf{k}$ be an algebraically closed field.[1]

Given a field extension $L \subseteq M$, let $\mathbb{V}(M/L)$ be the set of valuation rings R satisfying $L \subseteq R \subseteq M$, $\operatorname{Frac} R = M$ and $R \neq M$.

2.3.1. **Definition.** Consider a pair (F, A) such that $A = \mathbf{k}^{[2]}$ and $F \in A \setminus \mathbf{k}$.

(a) $\operatorname{dic}(F, A) =$ cardinality of the set $\left\{ R \in \mathbb{V}(\operatorname{Frac} A / \mathbf{k}(F)) \mid A \not\subseteq R \right\}$.

(b) If $f : \operatorname{Spec} A \to \operatorname{Spec} \mathbf{k}[F]$ is the dominant morphism determined by the inclusion homomorphism $\mathbf{k}[F] \to A$, we define $\operatorname{dic}(f) = \operatorname{dic}(F, A)$.

Note that (b) defines $\operatorname{dic}(f)$ for any dominant morphism $f : \mathbb{A}^2 \to \mathbb{A}^1$. We refer to $\operatorname{dic}(f)$ as the "number of dicriticals" of f; this use of terminology is justified in [CND15b, 2.3]. Note that $\operatorname{dic}(f)$ is a positive integer.

The following fact (needed for proving Prop. 3.23) is an immediate consequence of [CND15b, 2.9]:

2.3.2. **Corollary.** *Consider morphisms* $\mathbb{A}^2 \xrightarrow{\Phi} \mathbb{A}^2 \xrightarrow{f} \mathbb{A}^1$ *where* Φ *is birational and* f *is dominant. Then* $\operatorname{dic}(f \circ \Phi) = \operatorname{dic}(f) + |\operatorname{Miss}_{\mathrm{hor}}(\Phi, f)|$.

2.4. **Field generators.** Let $\mathbf{k}$ be a field.

2.4.1. **Notation.** Let us agree that the notation $A \preceq B$ means:

$$A = \mathbf{k}^{[2]}, \quad B = \mathbf{k}^{[2]}, \quad A \subseteq B \quad \text{and} \quad \operatorname{Frac} A = \operatorname{Frac} B.$$

2.4.2. **Definition.** Let $F \in A = \mathbf{k}^{[2]}$.

(i) F is a *field generator in* A if $\operatorname{Frac} A = \mathbf{k}(F, G)$ for some $G \in \operatorname{Frac} A$.

(ii) F is a *good field generator in* A if $\operatorname{Frac} A = \mathbf{k}(F, G)$ for some $G \in A$.

(iii) F is a *very good field generator in* A if it is a good field generator in each A' satisfying $F \in A' \preceq A$.

Refer to [CND15b] for details on these notions (see also [Rus77], [CN05]). "Good" and "very good" field generators appear in L. 3.2, Thm 3.3 and Thm 6.7, below.

3. Infinite chains of inclusions: the birational case

Throughout this section, we fix a field extension $K/\mathbf{k}$ where $\mathbf{k}$ is an arbitrary field and K is a purely transcendental extension of $\mathbf{k}$ of transcendence degree 2. We consider the set $\mathcal{A} = \mathcal{A}(K/\mathbf{k})$ whose elements are the rings A satisfying $\mathbf{k} \subset A \subset K$, $\operatorname{Frac} A = K$ and $A = \mathbf{k}^{[2]}$. By an *infinite descending chain in* $\mathcal{A}$, we mean an infinite sequence $\big(A_i\big)_{i \in \mathbb{N}}$ of elements A_i of $\mathcal{A}$ satisfying $A_0 \supset A_1 \supset A_2 \supset \dots$, where "$\supset$" denotes strict inclusion.

Note that $\mathbf{k} \subseteq \bigcap_{i=0}^{\infty} A_i$ holds for every infinite descending chain $\big(A_i\big)_{i \in \mathbb{N}}$ in $\mathcal{A}$. Our aim is to describe the chains $\big(A_i\big)_{i \in \mathbb{N}}$ satisfying $\bigcap_{i=0}^{\infty} A_i \neq \mathbf{k}$.

[1] It is shown in [CND15b] that all of paragraph 2.3 (including Cor. 2.3.2) remains valid for $\mathbf{k}$ an arbitrary field.

3.1. **Definition.** An infinite descending chain $\big(A_i\big)_{i\in\mathbb{N}}$ in $\mathcal{A}$ is said to be *simple* if there exist $N \in \mathbb{N}$ and $X, Y \in A_N$ satisfying $A_N = \mathbf{k}[X, Y]$ and:

$$\text{for each } i \geq N, \text{ there exists } \varphi_i(X) \in \mathbf{k}[X] \text{ such that } A_i = \mathbf{k}[X, \varphi_i(X)Y].$$

3.2. **Lemma.** *Let $\big(A_i\big)_{i\in\mathbb{N}}$ be an infinite descending chain in $\mathcal{A}$. Suppose that $\big(A_i\big)_{i\in\mathbb{N}}$ is simple, with notations N, X, Y, φ_i as in Def.* 3.1. *Then the following hold.*

(a) *If $N \leq i < j$ then $\varphi_i \mid \varphi_j$ in $\mathbf{k}[X]$ and $\deg_X \varphi_i < \deg_X \varphi_j$;*
(b) *$\bigcap_{i=0}^{\infty} A_i = \mathbf{k}[X]$;*
(c) *X is a good field generator of A_0.*

Proof. We leave (a) and (b) to the reader. Assertion (c) follows from $\mathbf{k}[X, Y] = A_N \subseteq A_0 \subset \mathbf{k}(X, Y)$. □

The main result of this section is:

3.3. **Theorem.** *For an infinite descending chain $\big(A_i\big)_{i\in\mathbb{N}}$ in $\mathcal{A}$, the following are equivalent:*

(a) *$\bigcap_{i=0}^{\infty} A_i \neq \mathbf{k}$;*
(b) *$\bigcap_{i=0}^{\infty} A_i = \mathbf{k}[F]$ for some good field generator F of A_0;*
(c) *$\big(A_i\big)_{i\in\mathbb{N}}$ is simple.*

Before starting the proof of the Theorem, note the following:

3.4. **Remark.** Given $A \in \mathcal{A}$ and $F \in A \setminus \mathbf{k}$, the following are equivalent:

(a) F is a good field generator of A;
(b) there exists an infinite descending chain $\big(A_i\big)_{i\in\mathbb{N}}$ in $\mathcal{A}$ satisfying $A_0 = A$ and $\bigcap_{i=0}^{\infty} A_i = \mathbf{k}[F]$.

Proof. If (a) holds then pick $G \in A$ satisfying $\mathbf{k}(F, G) = \operatorname{Frac} A$ and define $A_0 = A$ and $A_n = \mathbf{k}[F, F^n G]$ for $n \geq 1$; then (b) holds. The converse follows from Theorem 3.3. □

Let us observe right away that, in Thm 3.3, implications (c) ⇒ (b) ⇒ (a) are trivial (the first one follows from L. 3.2). So the proof of Thm 3.3 reduces to proving that

$$\bigcap_{i=0}^{\infty} A_i \neq \mathbf{k} \implies \big(A_i\big)_{i\in\mathbb{N}} \text{ is simple} \tag{1}$$

holds for every infinite descending chain $\big(A_i\big)_{i\in\mathbb{N}}$ in $\mathcal{A}$. Keep in mind that, from now-on, whenever we speak of the proof of Thm 3.3 we are really speaking of that of (1).

We now begin the proof of Thm 3.3 (i.e., of (1)). We first reduce the proof to the case where $\mathbf{k}$ is algebraically closed, and then prove that special case: Lemmas 3.13 and 3.24 constitute a proof of Theorem 3.3.

In paragraphs 3.5–3.13, $\mathbf{k}$ is any field.

3.5. **Lemma.** *Consider $\mathbf{k} \subset A \subseteq B$ where $A = \mathbf{k}^{[2]}$, $B = \mathbf{k}^{[2]}$ and $\operatorname{Frac} A = \operatorname{Frac} B$. Suppose that $X, Y \in B$ are such that $B = \mathbf{k}[X, Y]$ and $X \in A$. Then there exists $\varphi(X) \in \mathbf{k}[X] \setminus \{0\}$ such that $A = \mathbf{k}[X, \varphi(X)Y]$.*

Proof. This can be deduced from the much stronger result [Rus76, 1.3]. The present lemma (and its proof) being much easier, we include a proof for the reader's convenience.

Since $\mathbf{k}[X]$ and A are UFDs, $\mathbf{k}[X] \subset A \subseteq \mathbf{k}[X]^{[1]}$ and A has transcendence degree 1 over $\mathbf{k}[X]$, result [AEH72, 4.1] implies that $A = \mathbf{k}[X]^{[1]}$. So $A = \mathbf{k}[X, v]$ for some v. The conditions $\mathbf{k}(X)[v] \subseteq \mathbf{k}(X)[Y]$ and $\mathbf{k}(X, v) = \mathbf{k}(X, Y)$ imply that $\mathbf{k}(X)[v] = \mathbf{k}(X)[Y]$, which implies that $v = \varphi(X)Y + \psi(X)$ for some $\varphi(X), \psi(X) \in \mathbf{k}(X)$, $\varphi(X) \neq 0$. As v is a polynomial in X, Y, we have in fact $\varphi(X), \psi(X) \in \mathbf{k}[X]$. Then $A = \mathbf{k}[X, v] = \mathbf{k}[X, \varphi(X)Y + \psi(X)] = \mathbf{k}[X, \varphi(X)Y]$. $\square$

3.6. **Definition.** Suppose that $\mathbf{k} \subset A \subseteq B$, where $A = \mathbf{k}^{[2]}$, $B = \mathbf{k}^{[2]}$ and $\operatorname{Frac} A = \operatorname{Frac} B$. We say that the inclusion $A \subseteq B$ is *simple* if there exist X, Y such that $B = \mathbf{k}[X, Y]$ and $A = \mathbf{k}[X, \varphi(X)Y]$ for some $\varphi(X) \in \mathbf{k}[X]$.

3.7. **Lemma.** *Consider $\mathbf{k} \subset A \subseteq B \subset C$ where $A = \mathbf{k}^{[2]}$, $B = \mathbf{k}^{[2]}$, $C = \mathbf{k}^{[2]}$ and $\operatorname{Frac}(A) = \operatorname{Frac}(B) = \operatorname{Frac}(C)$. Suppose that $A \subset C$ and $B \subset C$ are simple (in the sense of Def.* 3.6*), and let $X, Y \in C$ be such that:*

$$C = \mathbf{k}[X, Y] \textit{ and } B = \mathbf{k}[X, \varphi(X)Y] \textit{ for some } \varphi(X) \in \mathbf{k}[X].$$

Then $A = \mathbf{k}[X, \psi(X)Y]$ for some $\psi(X) \in \mathbf{k}[X]$.

Proof. The assumption implies that there exist $X', Y' \in C$ such that $C = \mathbf{k}[X', Y']$ and $A = \mathbf{k}[X', \rho(X')Y']$ for some $\rho(X') \in \mathbf{k}[X']$. The elements $X, Y, X', Y' \in C$, $\varphi(X) \in \mathbf{k}[X]$ and $\rho(X') \in \mathbf{k}[X']$ are fixed from now-on. We claim:

$$\mathbf{k}[X] = \mathbf{k}[X']. \tag{2}$$

We first prove this under the assumption that $\mathbf{k}$ is algebraically closed. Consider the birational morphisms $\operatorname{Spec} C \xrightarrow{\Phi_{C,B}} \operatorname{Spec} B \xrightarrow{\Phi_{B,A}} \operatorname{Spec} A$ determined by the inclusion homomorphisms $A \hookrightarrow B \hookrightarrow C$. Also define $\Phi_{C,A} = \Phi_{B,A} \circ \Phi_{C,B} : \operatorname{Spec} C \to \operatorname{Spec} A$. Note that $\deg_X \varphi(X) \geq 1$, because $B \neq C$ (recall that "$\subset$" means strict inclusion). Choose a root $\lambda \in \mathbf{k}$ of $\varphi(X)$ and consider the curve $\Gamma \subset \operatorname{Spec} C$ defined by the prime ideal $\mathfrak{p} = (X - \lambda)$ of C. Then $\Gamma \in \operatorname{Cont}(\Phi_{C,B})$. As $\operatorname{Cont}(\Phi_{C,B}) \subseteq \operatorname{Cont}(\Phi_{C,A})$, we have $\Gamma \in \operatorname{Cont}(\Phi_{C,A})$; since $C = \mathbf{k}[X', Y']$ and $A = \mathbf{k}[X', \rho(X')Y']$, it follows that Γ is defined by the ideal $(X' - \mu)$ of C, for a suitable $\mu \in \mathbf{k}$. So $(X - \lambda) = \mathfrak{p} = (X' - \mu)$, which implies that (2) holds (when $\mathbf{k}$ is algebraically closed).

Next, we prove the general case of (2) (i.e., $\mathbf{k}$ is any field). Let $\bar{\mathbf{k}}$ be the algebraic closure of $\mathbf{k}$ and define $\bar{A} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} A$, $\bar{B} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} B$ and $\bar{C} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} C$. The elements $X, Y, X', Y' \in C$, $\varphi(X) \in \mathbf{k}[X]$ and $\rho(X') \in \mathbf{k}[X']$ satisfy:

- $\bar{C} = \bar{\mathbf{k}}[X, Y]$ and $\bar{B} = \bar{\mathbf{k}}[X, \varphi(X)Y]$,
- $\bar{C} = \bar{\mathbf{k}}[X', Y']$ and $\bar{A} = \bar{\mathbf{k}}[X', \rho(X')Y']$.

By the special case "$\mathbf{k} = \bar{\mathbf{k}}$" of (2), we obtain $\bar{\mathbf{k}}[X] = \bar{\mathbf{k}}[X']$. Thus X is integral over $\mathbf{k}[X']$ and X' is integral over $\mathbf{k}[X]$. As $X \in C = \mathbf{k}[X']^{[1]}$ is integral over $\mathbf{k}[X']$, we have $X \in \mathbf{k}[X']$; as $X' \in C = \mathbf{k}[X]^{[1]}$ is integral over $\mathbf{k}[X]$, we have $X' \in \mathbf{k}[X]$; so (2) is true in general.

By (2) we have $X \in \mathbf{k}[X']$, so $X \in A \subseteq C = \mathbf{k}[X, Y]$; then L. 3.5 implies that $A = \mathbf{k}[X, \psi(X)Y]$ for some $\psi(X) \in \mathbf{k}[X]$. $\square$

3.8. **Lemma.** *For an infinite descending chain* $\big(A_i\big)_{i\in\mathbb{N}}$ *in* $\mathcal{A}$*, the following conditions are equivalent:*

(a) $\big(A_i\big)_{i\in\mathbb{N}}$ *is simple (in the sense of Def.* 3.1*);*
(b) *there exists* $N \in \mathbb{N}$ *such that, for each* $i \geq N$*,* $A_i \subseteq A_N$ *is simple (in the sense of Def.* 3.6*).*

Proof. Suppose that (b) holds. Since $A_{N+1} \subset A_N$ is simple, there exist X, Y such that $A_N = \mathbf{k}[X,Y]$ and $A_{N+1} = \mathbf{k}[X, \varphi_{N+1}(X)Y]$ for some $\varphi_{N+1}(X) \in \mathbf{k}[X]$. The pair (X,Y) being fixed, we claim:

$$\text{for each } i \geq N \text{ there exists } \varphi_i(X) \in \mathbf{k}[X] \text{ such that } A_i = \mathbf{k}[X, \varphi_i(X)Y]. \tag{3}$$

Indeed, this is clear if $i = N, N+1$. If $i \geq N+2$ then applying L. 3.7 to $A_i \subset A_{N+1} \subset A_N$ shows that $A_i = \mathbf{k}[X, \varphi_i(X)Y]$ for some $\varphi_i(X) \in \mathbf{k}[X]$. So (3) is true and consequently (a) holds. The converse is trivial. □

We use some ideas found in the proof of Case 1 of [Rus76, 1.3] to prove Prop. 3.12, below.

3.9. **Notation.** Given $R = \mathbf{k}^{[2]}$, let $S(R)$ be the set of triples (x,y,f) satisfying:

(i) $R = \mathbf{k}[x,y]$ and $f \in R \setminus \mathbf{k}$,
(ii) the degree form of f is a monomial in x, y, i.e., $f = \lambda x^m y^n + f'$ for some $\lambda \in \mathbf{k}^*$, $m, n \in \mathbb{N}$ and $f' \in R$ such that $\deg(f') < m+n$,
(iii) $\deg_x(f) = m$ and $\deg_y(f) = n$,

where the degrees are the ones determined by (x,y).

3.10. **Lemma.** *If* $f \in R = \mathbf{k}^{[2]}$ *is a field generator of* R*, then there exists* (x,y) *such that* $(x,y,f) \in S(R)$*.*

Proof. If f is a variable of R then choose g such that $R = \mathbf{k}[f,g]$; then $(f,g,f) \in S(R)$. If f is not a variable of R then—as explained in the proof of [Rus76, 1.3]—the desired conclusion follows from [Rus75, 3.7 and 4.5]. □

3.11. **Lemma.** *Let* $R = \mathbf{k}[x,y] = \mathbf{k}^{[2]}$ *and* $f, g \in R \setminus \mathbf{k}$.

(a) $(x,y,fg) \in S(R) \Leftrightarrow (x,y,f), (x,y,g) \in S(R)$
(b) *If* $g \in \mathbf{k}[f] \setminus \mathbf{k}$ *then* $(x,y,f) \in S(R) \Leftrightarrow (x,y,g) \in S(R)$.

Proof. Throughout the proof of (a), we may assume that at least one of the conditions

$$\text{(i) } (x,y,fg) \in S(R), \qquad \text{(ii) } (x,y,f), (x,y,g) \in S(R)$$

holds. If (i) holds then the degree form of fg is a monomial in x, y, so the degree forms of f and g are monomials in x, y. This last condition also holds if (ii) holds, so we have:

$f = \lambda x^a y^b + f'$ and $g = \mu x^\alpha y^\beta + g'$, for some $\lambda, \mu \in \mathbf{k}^*$, $a, b, \alpha, \beta \in \mathbb{N}$ and $f', g' \in R$ satisfying $\deg(f') < a+b$ and $\deg(g') < \alpha + \beta$.

It follows that

$$\deg_x(f) \geq a, \quad \deg_y(f) \geq b, \quad \deg_x(g) \geq \alpha, \quad \deg_y(g) \geq \beta \tag{4}$$

and that

$$fg = \lambda\mu x^{a+\alpha} y^{b+\beta} + h', \quad \text{for some } h' \in R \text{ such that } \deg(h') < a+\alpha+b+\beta.$$

Now condition (i) holds iff $\deg_x(fg) = a+\alpha$ and $\deg_y(fg) = b+\beta$, iff the four equalities hold in (4), iff (ii) holds. This proves (a).

To prove (b), define $\bar{R} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} R$ where $\bar{\mathbf{k}}$ is the algebraic closure of $\mathbf{k}$, note that $\bar{R} = \bar{\mathbf{k}}[x,y]$, and observe that

$$(x,y,f) \in S(R) \Leftrightarrow (x,y,f) \in S(\bar{R}) \qquad \text{and} \qquad (x,y,g) \in S(R) \Leftrightarrow (x,y,g) \in S(\bar{R}).$$

In other words, we may assume that $\mathbf{k}$ is algebraically closed. Then $g = \lambda \prod_{i=1}^{n}(f-\lambda_i)$ for some $n \geq 1$, $\lambda \in \mathbf{k}^*$ and $\lambda_1, \dots, \lambda_n \in \mathbf{k}$, so part (a) implies that

$$(x,y,g) \in S(R) \;\Leftrightarrow\; \forall_i\, (x,y,f-\lambda_i) \in S(R).$$

Clearly, $(x,y,f-\lambda_i) \in S(R)$ is equivalent to $(x,y,f) \in S(R)$, so (b) is proved. □

3.12. **Proposition.** *Suppose that* $\mathbf{k} \subset A \subseteq B$, *where* $A = \mathbf{k}^{[2]}$, $B = \mathbf{k}^{[2]}$, *and* $\operatorname{Frac} A = \operatorname{Frac} B$. *Let* $\bar{\mathbf{k}}$ *be an algebraic extension of* $\mathbf{k}$, *define* $\bar{A} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} A$ *and* $\bar{B} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} B$. *Note that* $\bar{\mathbf{k}} \subset \bar{A} \subseteq \bar{B}$, $\bar{A} = \bar{\mathbf{k}}^{[2]}$, $\bar{B} = \bar{\mathbf{k}}^{[2]}$, *and* $\operatorname{Frac} \bar{A} = \operatorname{Frac} \bar{B}$. *If* $\bar{A} \subseteq \bar{B}$ *is simple (in the sense of Def.* 3.6*), then so is* $A \subseteq B$.

Proof. Consider the inclusions:

$$\begin{tikzcd} B \arrow[r, hook] & \bar{B} \\ A \arrow[u, hook] \arrow[r, hook] & \bar{A} \arrow[u, hook] \end{tikzcd}$$

We may assume that $\bar{\mathbf{k}}$ is the algebraic closure of $\mathbf{k}$ and that $A \neq B$. Then $\bar{A} \neq \bar{B}$, so the corresponding birational morphism $f : \mathbb{A}^2_{\bar{\mathbf{k}}} = \operatorname{Spec} \bar{B} \to \mathbb{A}^2_{\bar{\mathbf{k}}} = \operatorname{Spec} \bar{A}$ is not an isomorphism and hence (by [CND14, 2.6(b)]) has a contracting curve; consequently, there exists a height 1 prime ideal $\mathfrak{q}$ of $\bar{B}$ such that $\mathfrak{q} \cap \bar{A}$ is a maximal ideal of $\bar{A}$. Note that $\bar{A}$ and $\bar{B}$ are integral extensions of A and B respectively, and that the Going-Down Theorem holds for $A \subseteq \bar{A}$ and for $B \subseteq \bar{B}$; so $\mathfrak{p} = \mathfrak{q} \cap B$ is a height 1 prime ideal of B and $\mathfrak{m} = \mathfrak{p} \cap A$ is a maximal ideal of A.

The assumption that $\bar{A} \subseteq \bar{B}$ is simple means that there exist x', y' such that $\bar{B} = \bar{\mathbf{k}}[x', y']$ and, for some $P(x') \in \bar{\mathbf{k}}[x']$, $\bar{A} = \bar{\mathbf{k}}[x', P(x')y']$. There follows:

(5) each contracting curve of f is the zero-set of a polynomial (of degree 1) belonging to $\bar{\mathbf{k}}[x']$.

Let b be an irreducible element of B such that $\mathfrak{p} = bB$, let $\beta_1, \dots, \beta_n$ be prime elements of $\bar{B}$ satisfying $b = \prod_{i=1}^{n} \beta_i$ and let $\mathfrak{q}_i = \beta_i \bar{B}$ $(1 \leq i \leq n)$. Then, for each i, $\mathfrak{q}_i \cap B = \mathfrak{p}$, so $(\mathfrak{q}_i \cap \bar{A}) \cap A = \mathfrak{m}$ and consequently $\mathfrak{q}_i \cap \bar{A}$ is a maximal ideal of $\bar{A}$. By (5), it follows that $\beta_i \in \bar{\mathbf{k}}[x']$ for all i, so $b \in \bar{\mathbf{k}}[x']$.

Pick v, w such that $A = \mathbf{k}[v, w]$. Then $v \in B$ and $\operatorname{Frac} B = \mathbf{k}(v, w)$, so v is a field generator of B. By L. 3.10, there exist x, y such that $B = \mathbf{k}[x, y]$ and $(x, y, v) \in S(B)$.

Let $H(T) \in \mathbf{k}[T] \setminus \{0\}$ be the minimal polynomial of $v + \mathfrak{m} \in A/\mathfrak{m}$ over $\mathbf{k}$; then $H(v) \notin \mathbf{k}$ and $H(v) \in \mathfrak{m} \subseteq bB$, i.e., $b \mid H(v)$ in B. As $(x, y, v) \in S(B)$ and $H(v) \in \mathbf{k}[v] \setminus \mathbf{k}$, L. 3.11 gives $(x, y, H(v)) \in S(B)$ and then $(x, y, b) \in S(B)$ because $b \mid H(v)$.

As $\bar{B} = \bar{\mathbf{k}}[x,y]$, we also have $(x,y,b) \in S(\bar{B})$; as $b \in \bar{\mathbf{k}}[x']$, using L. 3.11 again gives $(x,y,x') \in S(\bar{B})$. So $x' = \lambda x^m y^n + g(x,y)$ for some $\lambda \in \bar{\mathbf{k}}^*$, $m,n \in \mathbb{N}$ and $g(x,y) \in \bar{\mathbf{k}}[x,y]$ such that $\deg_{(x,y)}(g) < m+n$, $\deg_x(x') = m$, and $\deg_y(x') = n$. As x' is a variable of $\bar{\mathbf{k}}[x,y]$, these conditions imply that (m,n) is either $(1,0)$ or $(0,1)$, so $x' = \lambda u + \mu$ for some $u \in \{x,y\}$, $\lambda \in \bar{\mathbf{k}}^*$ and $\mu \in \bar{\mathbf{k}}$. As $x' \in \bar{A}$ we then have $u \in \bar{A}$, so u is integral over A and $u \in B \subset \operatorname{Frac} A$, so $u \in A$. We showed that some variable of B belongs to A; by L. 3.5, $A \subseteq B$ is simple. $\square$

3.13. **Lemma.** *If Theorem* 3.3 *is valid in the special case where* $\mathbf{k}$ *is algebraically closed, then it is valid in general.*

Proof. Assume that Theorem 3.3 is valid in the algebraically closed case. Let $K/\mathbf{k}$ be as in the introduction of Section 3 (with $\mathbf{k}$ an arbitrary field) and consider an infinite descending chain $\big(A_i\big)_{i\in\mathbb{N}}$ in $\mathcal{A}(K/\mathbf{k})$ such that $\bigcap_{i=0}^{\infty} A_i \neq \mathbf{k}$. It has to be shown that $\big(A_i\big)_{i\in\mathbb{N}}$ is simple, in the sense of Def. 3.1.

Let $\bar{\mathbf{k}}$ be the algebraic closure of $\mathbf{k}$ and, for each $i \in \mathbb{N}$, $\bar{A}_i = \bar{\mathbf{k}} \otimes_{\mathbf{k}} A_i$. Then all $\bar{A}_i$ have the same field of fractions L, and $\big(\bar{A}_i\big)_{i\in\mathbb{N}}$ is an infinite descending chain in $\mathcal{A}(L/\bar{\mathbf{k}})$ such that $\bigcap_{i=0}^{\infty} \bar{A}_i \neq \bar{\mathbf{k}}$. Since we are assuming that the algebraically closed case of Theorem 3.3 is valid, it follows that $\big(\bar{A}_i\big)_{i\in\mathbb{N}}$ is simple, in the sense of Def. 3.1; by L. 3.8, there exists $N \in \mathbb{N}$ such that

for each $i \geq N$, the inclusion $\bar{A}_i \subseteq \bar{A}_N$ is simple, in the sense of Def. 3.6.

By Prop. 3.12, it follows that the same N satisfies:

for each $i \geq N$, the inclusion $A_i \subseteq A_N$ is simple, in the sense of Def. 3.6.

Then L. 3.8 implies that $\big(A_i\big)_{i\in\mathbb{N}}$ is simple, in the sense of Def. 3.1. $\square$

The case where $\mathbf{k}$ is algebraically closed

Our next aim is to prove L. 3.24, which asserts that Theorem 3.3 is valid under the assumption that $\mathbf{k}$ is algebraically closed. The proof of L. 3.24 is in several parts, and makes heavy use of the results of [CND14]. In particular the proof of L. 3.18 makes essential use of [CND14, Th. 3.15(c)]. Also note that L. 2.2.2 and Cor. 2.3.2 (of the present paper) are used in the proof of Prop. 3.23.

Until the end of Section 3, we assume that $\mathbf{k}$ is algebraically closed.

3.14. **Definition.** Let A, B be $\mathbf{k}$-algebras such that

$$A \subseteq B, \quad \operatorname{Frac} A = \operatorname{Frac} B, \quad A = \mathbf{k}^{[2]} \quad \text{and} \quad B = \mathbf{k}^{[2]}$$

and let $\Phi : \operatorname{Spec} B \to \operatorname{Spec} A$ be the birational morphism determined by $A \hookrightarrow B$.

(1) We say that Φ *is of type* $(\boldsymbol{\mathcal{V}})$ if there exist X, Y satisfying:

$$B = \mathbf{k}[X,Y] \text{ and } A = \mathbf{k}[X, \varphi(X)Y], \text{ for some } \varphi(X) \in \mathbf{k}[X] \setminus \{0\}.$$

(2) We say that Φ *is of type* $(\mathfrak{G})$ if there exist X, Y satisfying:

$$B = \mathbf{k}[X,Y] \text{ and } A = \mathbf{k}[X^aY^b\,,\, X^cY^d], \text{ for some } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{M},$$

where $\mathbf{M} = \left\{ \left(\begin{smallmatrix} i & j \\ k & \ell \end{smallmatrix}\right) \;\middle|\; i,j,k,\ell \in \mathbb{N} \text{ and } i\ell - jk = \pm 1 \right\}$.

3.15. **Definition.** Let $\big(A_i\big)_{i\in\mathbb{N}}$ be an infinite descending chain in $\mathcal{A}$.

(a) Given natural numbers $i \le j$, we write $\Phi_{ij} : \operatorname{Spec} A_i \to \operatorname{Spec} A_j$ for the morphism determined by the inclusion $A_i \hookleftarrow A_j$. Thus Φ_{ij} is a birational morphism from $\operatorname{Spec} A_i \cong \mathbb{A}^2$ to $\operatorname{Spec} A_j \cong \mathbb{A}^2$. Note that $\Phi_{jk} \circ \Phi_{ij} = \Phi_{ik}$ whenever $i \le j \le k$.

(b) Let τ be one of the symbols $(\boldsymbol{\mathcal{V}})$, $(\boldsymbol{\mathcal{G}})$. We say that the chain $\big(A_i\big)_{i\in\mathbb{N}}$ is *of type* τ if there exists $N \in \mathbb{N}$ such that, for all $i, j \in \mathbb{N}$ such that $N \le i \le j$, Φ_{ij} is of type τ in the sense of Def. 3.14.

(c) We say that the chain $\big(A_i\big)_{i\in\mathbb{N}}$ *is of bounded type* if $\big\{\, c(\Phi_{0j}) \mid j \in \mathbb{N} \,\big\}$ is a finite subset of $\mathbb{N}$, or equivalently, if there exists $k \in \mathbb{N}$ such that $c(\Phi_{0j}) \le c(\Phi_{0k})$ for all $j \in \mathbb{N}$. (Here, $c(\Phi_{ij})$ denotes the number of contracting curves of Φ_{ij} (see paragraph 2.2); recall that $c(\Phi_{ij}) = q(\Phi_{ij})$, cf. [CND14, 2.2 and 2.6].)

3.16. **Remark.** We have $\operatorname{Cont}(\Phi_{0j}) \subseteq \operatorname{Cont}(\Phi_{0,j+1})$ and hence $c(\Phi_{0j}) \le c(\Phi_{0,j+1})$ for all $j \in \mathbb{N}$. It follows that $\big(A_i\big)_{i\in\mathbb{N}}$ is of bounded type if and only if the sequence $\big(c(\Phi_{0j})\big)_{j\in\mathbb{N}}$ eventually stabilizes.

3.17. **Lemma.** *Let $\big(A_i\big)_{i\in\mathbb{N}}$ be an infinite descending chain in $\mathcal{A}$, of type $(\boldsymbol{\mathcal{V}})$. Then $\big(A_i\big)_{i\in\mathbb{N}}$ is simple.*

Proof. The assumption implies that $\big(A_i\big)_{i\in\mathbb{N}}$ satisfies condition (b) of L. 3.8. □

3.18. **Lemma.** *Let $\big(A_i\big)_{i\in\mathbb{N}}$ be an infinite descending chain in $\mathcal{A}$, of bounded type. Then there exists $N \in \mathbb{N}$ such that, for each choice of natural numbers i, j such that $N \le i \le j$, Φ_{ij} is of type $(\boldsymbol{\mathcal{V}})$ or $(\boldsymbol{\mathcal{G}})$.*

Proof. Because $\big(A_i\big)_{i\in\mathbb{N}}$ is of bounded type, there exists $t \in \mathbb{N}$ such that $\forall_{j\in\mathbb{N}}\ c(\Phi_{0j}) \le c(\Phi_{0t})$. Consider natural numbers $i \le j$ such that $\operatorname{Cont}(\Phi_{ij}) \nsubseteq \operatorname{Miss}(\Phi_{0i})$; then [CND14, 2.16] implies that $c(\Phi_{0j}) > c(\Phi_{0i})$, so $c(\Phi_{0i}) < c(\Phi_{0t})$ and hence $i < t$. This shows that

$$\text{(6)} \qquad \textit{for all } i, j \in \mathbb{N} \textit{ satisfying } t \le i \le j,\ \operatorname{Cont}(\Phi_{ij}) \subseteq \operatorname{Miss}(\Phi_{0i}).$$

Next, define $\Omega = \big\{\, (i,j) \in \mathbb{N}^2 \mid i \le j \text{ and } \exists_{k\ge j}\ \operatorname{Miss}(\Phi_{ij}) \subseteq \operatorname{Cont}(\Phi_{jk}) \,\big\}$. Let us show that there exists $\ell \in \mathbb{N}$ satisfying:

$$\text{(7)} \qquad \forall_{i,j\in\mathbb{N}}\ \big(\ell \le i \le j \implies (i,j) \in \Omega\big).$$

By contradiction, suppose that ℓ does not exist. Then there exists an infinite sequence $(i_1, j_1), (i_2, j_2), \dots$ of elements of $\mathbb{N}^2 \setminus \Omega$ satisfying $i_1 < j_1 \le i_2 < j_2 \le i_3 < j_3 \le \dots$. Define $j_0 = 0$ and note that the sequence $j_0 < j_1 < j_2 < \dots$ satisfies

$$\text{(8)} \qquad (j_\nu, j_{\nu+1}) \notin \Omega \textit{ for all } \nu \in \mathbb{N}.$$

Indeed, if $(j_\nu, j_{\nu+1}) \in \Omega$ then there exists $k \ge j_{\nu+1}$ such that

$$\operatorname{Miss}(\Phi_{i_{\nu+1}j_{\nu+1}}) \subseteq \operatorname{Miss}(\Phi_{j_\nu j_{\nu+1}}) \subseteq \operatorname{Cont}(\Phi_{j_{\nu+1}k})$$

(where the first inclusion is a simple consequence of $j_\nu \le i_{\nu+1} < j_{\nu+1}$ and of [CND14, 2.12]), and this implies that $(i_{\nu+1}, j_{\nu+1}) \in \Omega$, a contradiction. So (8) is proved.

Now let s be a positive integer; we claim:

$$\text{(9)} \qquad q(\Phi_{j_{\nu-1}j_s}) > q(\Phi_{j_\nu j_s}), \quad \text{for all } \nu \in \{1, \dots, s\}.$$

Indeed, let $\nu \in \{1, \dots, s\}$ and consider:

$$\begin{tikzcd} \mathbb{A}^2 \arrow[r, "\Phi_{j_{\nu-1}j_\nu}"'] \arrow[rr, bend left, "\Phi_{j_{\nu-1}j_s}"] & \mathbb{A}^2 \arrow[r, "\Phi_{j_\nu j_s}"'] & \mathbb{A}^2 \end{tikzcd}$$

We have $\operatorname{Miss}(\Phi_{j_{\nu-1}j_\nu}) \nsubseteq \operatorname{Cont}(\Phi_{j_\nu j_s})$ by (8), then $q(\Phi_{j_{\nu-1}j_s}) > q(\Phi_{j_\nu j_s})$ by [CND14, 2.16], and this proves (9). Applying (9) repeatedly gives

$$q(\Phi_{j_0 j_s}) > q(\Phi_{j_1 j_s}) > q(\Phi_{j_2 j_s}) > \cdots > q(\Phi_{j_{s-1} j_s}) > q(\Phi_{j_s j_s}) = 0,$$

so $c(\Phi_{0 j_s}) = q(\Phi_{0 j_s}) = q(\Phi_{j_0 j_s}) \geq s$. Since this holds for arbitrary $s \geq 1$, we obtain that $\big\{ c(\Phi_{0j}) \mid j \in \mathbb{N} \big\}$ is not a finite set, which contradicts the hypothesis that $\big(A_i\big)_{i\in\mathbb{N}}$ is of bounded type. This contradiction shows that there exists $\ell \in \mathbb{N}$ satisfying (7).

Choose $\ell \in \mathbb{N}$ satisfying (7) and define $N = \max(t, \ell)$. Consider any $i, j \in \mathbb{N}$ such that $N \leq i \leq j$. Then (6) and (7) imply that

(10) $$\operatorname{Cont}(\Phi_{ij}) \subseteq \operatorname{Miss}(\Phi_{0i}) \quad \text{and} \quad \exists_{k \geq j}\ \operatorname{Miss}(\Phi_{ij}) \subseteq \operatorname{Cont}(\Phi_{jk}).$$

By [CND14, 2.15] and (10),

(11) $\operatorname{Cont}(\Phi_{ij})$ and $\operatorname{Miss}(\Phi_{ij})$ are admissible.

Then [CND14, Th. 3.15(c)] and (11) imply that Φ_{ij} is of type $(\mathcal{V})$ or $(\mathcal{G})$. $\square$

3.19. **Lemma.** *Let $\big(A_i\big)_{i\in\mathbb{N}}$ be an infinite descending chain in $\mathcal{A}$. Let $i \leq j$ be natural numbers and suppose that Φ_{ij} is of type $(\mathcal{G})$ but not of type $(\mathcal{V})$. Then Φ_{ij} has exactly 2 contracting curves and 2 missing curves. Let X, Y be irreducible elements of A_i whose zero-sets in $\operatorname{Spec} A_i$ are the contracting curves of Φ_{ij}. Then*

1. $A_i = \mathbf{k}[X, Y]$
2. $A_j = \mathbf{k}[X^aY^b\,,\, X^cY^d]$, *for some* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{M}$ *such that* $\min(a, b, c, d) \geq 1$.
3. X^aY^b *and* X^cY^d *are irreducible elements of A_j and their zero-sets in* $\operatorname{Spec} A_j$ *are precisely the two missing curves of* Φ_{ij}.

Proof. By [CND14, 2.6(a)], if Φ_{ij} has two contracting curves then it also has two missing curves. Since Φ_{ij} is of type $(\mathcal{G})$, there exists (X_0, Y_0) satisfying

$$A_i = \mathbf{k}[X_0, Y_0] \text{ and } A_j = \mathbf{k}[X_0^{a_0}Y_0^{b_0}\,,\, X_0^{c_0}Y_0^{d_0}] \text{ for some } \left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right) \in \mathbf{M}.$$

We have $\min(a_0, b_0, c_0, d_0) \geq 1$, otherwise Φ_{ij} would be of type $(\mathcal{V})$. It follows that Φ_{ij} has exactly two contracting curves, namely, the zero-set of X_0 and the zero-set of Y_0 in $\operatorname{Spec} A_i$. Let X, Y be as in the statement of the lemma. Then the principal ideals (X_0), (Y_0), (X) and (Y) of A_i satisfy either (i) $(X_0) = (X)$ and $(Y_0) = (Y)$, or (ii) $(X_0) = (Y)$ and $(Y_0) = (X)$. Note that assertion (1) is true in both cases. If (i) holds (resp. (ii) holds) then $A_j = \mathbf{k}[X^aY^b\,,\, X^cY^d]$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} b_0 & a_0 \\ d_0 & c_0 \end{smallmatrix}\right)$). In both cases we have $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{M}$ and $\min(a, b, c, d) \geq 1$, so assertion (2) holds. Assertion (3) is an easy consequence of (1) and (2). $\square$

3.20. **Proposition.** *If $\big(A_i\big)_{i\in\mathbb{N}}$ is of bounded type and not of type $(\mathcal{V})$, then $\bigcap_{i=0}^{\infty} A_i = \mathbf{k}$.*

Proof. For each $\tau \in \{(\mathcal{V}), (\mathcal{G})\}$, define $S_\tau = \{ (i,j) \in \mathbb{N}^2 \mid i \le j \text{ and } \Phi_{ij} \text{ is of type } \tau \}$. As $(A_i)_{i\in\mathbb{N}}$ is of bounded type, L. 3.18 implies that there is a natural number N satisfying:

(12) *for all $i,j \in \mathbb{N}$ such that $N \le i \le j$, $(i,j) \in S_{(\mathcal{V})} \cup S_{(\mathcal{G})}$.*

As $(A_i)_{i\in\mathbb{N}}$ is not of type $(\mathcal{V})$, there exist $j_0, j_1 \in \mathbb{N}$ such that $N \le j_0 < j_1$ and $(j_0, j_1) \in S_{(\mathcal{G})} \setminus S_{(\mathcal{V})}$. Suppose that $N \le j_0 < j_1 < \dots < j_k$ (where $k \ge 1$) are natural numbers such that $(j_\nu, j_{\nu+1}) \in S_{(\mathcal{G})} \setminus S_{(\mathcal{V})}$ for all $\nu \in \{0, \dots, k-1\}$. As $(A_i)_{i\in\mathbb{N}}$ is not of type $(\mathcal{V})$, there exist $i, j_{k+1} \in \mathbb{N}$ such that $j_k \le i < j_{k+1}$ and $(i, j_{k+1}) \in S_{(\mathcal{G})} \setminus S_{(\mathcal{V})}$. Because $\Phi_{ij_{k+1}}$ is of type $(\mathcal{G})$ but not of type $(\mathcal{V})$, L. 3.19 implies that there exist missing curves $D \neq D'$ of $\Phi_{ij_{k+1}}$ such that $D \cap D' \neq \varnothing$; as D, D' are missing curves of $\Phi_{j_k j_{k+1}}$, it follows that $\Phi_{j_k j_{k+1}}$ is not of type $(\mathcal{V})$, so $(j_k, j_{k+1}) \in S_{(\mathcal{G})} \setminus S_{(\mathcal{V})}$. By induction, this constructs an infinite sequence $N \le j_0 < j_1 < j_2 < \dots$ of natural numbers satisfying:

(13) *for each $\nu \in \mathbb{N}$, $\Phi_{j_\nu j_{\nu+1}}$ is of type $(\mathcal{G})$ but not of type $(\mathcal{V})$.*

We claim that

(14) *for each $\nu \in \mathbb{N}$,* $\operatorname{Miss}(\Phi_{j_\nu j_{\nu+1}}) = \operatorname{Cont}(\Phi_{j_{\nu+1}, j_{\nu+2}})$.

To see this, first note that (by L. 3.19) $\Phi_{j_\nu j_{\nu+1}}$ has two contracting curves which have a common point; thus $\Phi_{j_\nu j_{\nu+2}}$ has the same property, and hence is not of type $(\mathcal{V})$; in view of (12), we get that $\Phi_{j_\nu j_{\nu+2}}$ is of type $(\mathcal{G})$ but not of type $(\mathcal{V})$. So each element Φ of $\{\Phi_{j_\nu j_{\nu+1}}, \Phi_{j_{\nu+1} j_{\nu+2}}, \Phi_{j_\nu j_{\nu+2}}\}$ is of type $(\mathcal{G})$ but not of type $(\mathcal{V})$ and hence satisfies $c(\Phi) = 2 = q(\Phi)$. Then [CND14, 2.16(a)] implies that $\operatorname{Miss}(\Phi_{j_\nu j_{\nu+1}}) \subseteq \operatorname{Cont}(\Phi_{j_{\nu+1}, j_{\nu+2}})$, and [CND14, 2.16(b)] implies that $\operatorname{Miss}(\Phi_{j_\nu j_{\nu+1}}) \supseteq \operatorname{Cont}(\Phi_{j_{\nu+1}, j_{\nu+2}})$; so (14) is true.

Let us now explain how to construct a sequence of pairs (X_ν, Y_ν) satisfying the following conditions for all $\nu \in \mathbb{N}$:

(i) $A_{j_\nu} = \mathbf{k}[X_\nu, Y_\nu]$

(ii) $(X_{\nu+1}, Y_{\nu+1}) = (X_\nu^{a_{\nu+1}} Y_\nu^{b_{\nu+1}}, X_\nu^{c_{\nu+1}} Y_\nu^{d_{\nu+1}})$ for some $\left(\begin{smallmatrix} a_{\nu+1} & b_{\nu+1} \\ c_{\nu+1} & d_{\nu+1} \end{smallmatrix}\right) \in \mathbf{M}$ satisfying $\min(a_{\nu+1}, b_{\nu+1}, c_{\nu+1}, d_{\nu+1}) \ge 1$.

By (13), $\Phi_{j_0 j_1}$ is of type $(\mathcal{G})$ but not of type $(\mathcal{V})$. So $\Phi_{j_0 j_1}$ has exactly 2 contracting curves; we begin the construction by choosing irreducible elements X_0, Y_0 of A_{j_0} whose zero-sets in $\operatorname{Spec} A_{j_0}$ are the contracting curves of $\Phi_{j_0 j_1}$. Then L. 3.19 implies that $A_{j_0} = \mathbf{k}[X_0, Y_0]$ and $A_{j_1} = \mathbf{k}[X_0^{a_1} Y_0^{b_1}, X_0^{c_1} Y_0^{d_1}]$ for some $\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right) \in \mathbf{M}$ satisfying $\min(a_1, b_1, c_1, d_1) \ge 1$. Define $(X_1, Y_1) = (X_0^{a_1} Y_0^{b_1}, X_0^{c_1} Y_0^{d_1})$; then, again by L. 3.19, the zero-sets of X_1 and Y_1 in $\operatorname{Spec} A_{j_1}$ are precisely the missing curves of $\Phi_{j_0 j_1}$; so, by (14), they are the contracting curves of $\Phi_{j_1 j_2}$. It is now clear that, using induction together with L. 3.19, (13) and (14), we obtain an infinite sequence of pairs (X_ν, Y_ν) with the desired properties.

The above statements (i) and (ii) imply that, for all $\nu \in \mathbb{N} \setminus \{0\}$,

$$(X_\nu, Y_\nu) = (X_0^{\alpha_\nu} Y_0^{\beta_\nu}, X_0^{\gamma_\nu} Y_0^{\delta_\nu}), \text{ where } \left(\begin{smallmatrix} \alpha_\nu & \beta_\nu \\ \gamma_\nu & \delta_\nu \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_\nu & b_\nu \\ c_\nu & d_\nu \end{smallmatrix}\right) \cdots \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right) \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right).$$

It also follows that $\left(\begin{smallmatrix} \alpha_\nu & \beta_\nu \\ \gamma_\nu & \delta_\nu \end{smallmatrix}\right) \in \mathbf{M}$ and $\min(\alpha_\nu, \beta_\nu, \gamma_\nu, \delta_\nu) \ge 2^{\nu-1}$ for all $\nu \in \mathbb{N} \setminus \{0\}$.

Now let $\deg_0 : A_{j_0} \setminus \{0\} = \mathbf{k}[X_0, Y_0] \setminus \{0\} \to \mathbb{N}$ be the total degree of polynomials in X_0, Y_0. We claim that, given any $\nu \in \mathbb{N}$,

$$\text{if } f \in A_{j_\nu} = \mathbf{k}[X_\nu, Y_\nu] \text{ and } f \notin \mathbf{k} \text{ then } \deg_0(f) \geq 2^\nu. \tag{15}$$

This is obvious if $\nu = 0$, so let $\nu \in \mathbb{N} \setminus \{0\}$ and $f = \sum_{i,j} a_{ij} X_\nu^i Y_\nu^j \in A_{j_\nu} \setminus \mathbf{k}$ (where $a_{ij} \in \mathbf{k}$). Then

$$f = \textstyle\sum_{i,j} a_{ij} X_0^{\alpha_\nu i + \gamma_\nu j} Y_0^{\beta_\nu i + \delta_\nu j}, \tag{16}$$

where for each $(i,j) \neq (0,0)$ we have $\alpha_\nu i + \gamma_\nu j + \beta_\nu i + \delta_\nu j \geq \min(\alpha_\nu + \beta_\nu, \gamma_\nu + \delta_\nu) \geq 2^\nu$. We have $\det\left(\begin{smallmatrix} \alpha_\nu & \beta_\nu \\ \gamma_\nu & \delta_\nu \end{smallmatrix}\right) = \pm 1$ by definition of $\mathbf{M}$, so

$$\text{the map } \mathbb{N}^2 \to \mathbb{N}^2,\ (i,j) \mapsto (\alpha_\nu i + \gamma_\nu j, \beta_\nu i + \delta_\nu j), \text{ is injective.}$$

It follows that there are no cancellations in the sum (16). So $\deg_0(f) \geq 2^\nu$ and (15) is proved. Clearly, (15) implies that $\bigcap_{i=1}^\infty A_i = \mathbf{k}$, and this completes the proof. $\square$

3.21. **Lemma.** *Let $\big(A_i\big)_{i\in\mathbb{N}}$ be an infinite descending chain in $\mathcal{A}$ and let $R = \bigcap_{i=0}^\infty A_i$. Then for each $F' \in R \setminus \mathbf{k}$ there exists $F \in R \setminus \mathbf{k}$ such that $\mathbf{k}[F'] \subseteq \mathbf{k}[F]$ and such that $\mathbf{k}[F]$ is a maximal element of the set $\Sigma = \big\{ \mathbf{k}[G] \mid G \in K \setminus \mathbf{k} \big\}$.*

Proof. Recall that K is defined in the introduction of Section 3. If $\mathbf{k}[G_0] \subseteq \mathbf{k}[G_1] \subseteq \dots$ is an ascending sequence of elements of Σ and if Ω denotes the algebraic closure of $\mathbf{k}(G_0)$ in K then $\Omega/\mathbf{k}$ is a finitely generated extension, so $\Omega/\mathbf{k}(G_0)$ is a finite extension. As $\mathbf{k}(G_i) \subseteq \Omega$ for all i, the sequence of fields $\big(\mathbf{k}(G_i)\big)_{i\in\mathbb{N}}$ stabilizes and, consequently, the sequence of rings $\big(\mathbf{k}[G_i]\big)_{i\in\mathbb{N}}$ stabilizes. This shows that Σ satisfies ACC. So, given $F' \in R \setminus \mathbf{k}$, we may choose $F \in K$ such that $\mathbf{k}[F'] \subseteq \mathbf{k}[F]$ and such that $\mathbf{k}[F]$ is a maximal element of Σ. For each $i \in \mathbb{N}$, A_i is a normal domain, $F \in \operatorname{Frac} A_i$ and F is integral over $\mathbf{k}[F'] \subset A_i$; so $F \in A_i$ and hence $F \in R$. $\square$

The following well-known fact is needed in the next proof.

3.22. **Lemma.** *Let $\mathbf{k}$ be an algebraically closed field, $n \geq 1$, $A = \mathbf{k}^{[n]}$ and $F \in A \setminus \mathbf{k}$. The set $\big\{ \lambda \in \mathbf{k} \mid F - \lambda \text{ is not irreducible in } A \big\}$ is infinite if and only if $F = P(G)$ for some $G \in A$ and $P(T) \in \mathbf{k}[T]$ such that $\deg_T P(T) > 1$.*

Proof. This can be derived from a general Theorem on linear systems proved by Bertini (and reproved by Zariski) in characteristic zero, then generalized to all characteristics by Matsusaka [Mat50]. For the result as stated here, see [Sch00], Chap. 3, § 3, Cor. 1. $\square$

3.23. **Proposition.** *Let $\big(A_i\big)_{i\in\mathbb{N}}$ be an infinite descending chain in $\mathcal{A}$ which is not of bounded type. If $\bigcap_{i=0}^\infty A_i \neq \mathbf{k}$, then $\big(A_i\big)_{i\in\mathbb{N}}$ is simple.*

Proof. Pick $F \in \bigcap_{i=0}^\infty A_i$, $F \notin \mathbf{k}$, such that $\mathbf{k}[F]$ is a maximal element of the set Σ defined in L. 3.21. For each $i \in \mathbb{N}$, let $\mathbb{A}^2 \xrightarrow{f_i} \mathbb{A}^1$ be the morphism determined by the inclusion map $A_i \hookleftarrow \mathbf{k}[F]$. For each $i \in \mathbb{N}$, Cor. 2.3.2 gives

$$\begin{tikzcd} \mathbb{A}^2 \arrow[r, "\Phi_{i,i+1}"'] \arrow[rr, bend left, "f_i"] & \mathbb{A}^2 \arrow[r, "f_{i+1}"'] & \mathbb{A}^1 \end{tikzcd} \qquad \operatorname{dic}(f_i) = \operatorname{dic}(f_{i+1}) + |\operatorname{Miss}_{\mathrm{hor}}(\Phi_{i,i+1}, f_{i+1})|,$$

which shows that $\operatorname{dic}(f_i) \geq \operatorname{dic}(f_{i+1})$ for all $i \in \mathbb{N}$; so there exists $N \in \mathbb{N}$ such that $\big(\operatorname{dic}(f_i)\big)_{i=N}^{\infty}$ is a constant sequence. Then, given any $i, j \in \mathbb{N}$ such that $N \leq i \leq j$, we get $\operatorname{Miss}_{\mathrm{hor}}(\Phi_{ij}, f_j) = \varnothing$ by applying Cor. 2.3.2 again:

$$\begin{tikzcd} \mathbb{A}^2 \arrow[r, "\Phi_{i,j}"'] \arrow[rr, bend left, "f_i"] & \mathbb{A}^2 \arrow[r, "f_j"'] & \mathbb{A}^1 \end{tikzcd} \qquad \operatorname{dic}(f_i) = \operatorname{dic}(f_j) + |\operatorname{Miss}_{\mathrm{hor}}(\Phi_{i,j}, f_j)|.$$

To prove the Lemma it suffices to show that $\big(A_i\big)_{i=N}^{\infty}$ is simple. The reader may verify that $\big(A_i\big)_{i=N}^{\infty}$ is not of bounded type, and it is clear that $\bigcap_{i=N}^{\infty} A_i \neq \mathbf{k}$; so we may as well replace $\big(A_i\big)_{i \in \mathbb{N}}$ by $\big(A_i\big)_{i=N}^{\infty}$ throughout, i.e., we may assume that

(17) $\operatorname{Miss}_{\mathrm{hor}}(\Phi_{ij}, f_j) = \varnothing$ *for all natural numbers* $i \leq j$.

By maximality of $\mathbf{k}[F]$ in Σ, it follows that $\big\{\, \lambda \in \mathbf{k} \mid F - \lambda \text{ is not irreducible in } A_0 \,\big\}$ is a finite set (see L. 3.22); so there exists a finite set S of closed points of $\mathbb{A}^1$ such that:

(18) *For each closed point* $y \in \mathbb{A}^1 \setminus S$, $f_0^{-1}(y)$ *is an irreducible curve.*

Let M_S be the number of irreducible curves included in the set $f_0^{-1}(S)$. As $\big(A_i\big)_{i \in \mathbb{N}}$ is not of bounded type, we can choose $j \in \mathbb{N}$ such that $c(\Phi_{0j}) > M_S$. In fact, there exists $N_S \in \mathbb{N}$ such that $c(\Phi_{0j}) > M_S$ holds for all $j \geq N_S$. We claim:

(19) *For each* $j \geq N_S$, F *is a variable of* A_j.

Indeed, consider $j \geq N_S$. Then $c(\Phi_{0j}) > M_S$ and hence there exists a contracting curve C of Φ_{0j} such that $C \nsubseteq f_0^{-1}(S)$. Since

$$\begin{tikzcd} \mathbb{A}^2 \arrow[r, "\Phi_{0j}"'] \arrow[rr, bend left, "f_0"] & \mathbb{A}^2 \arrow[r, "f_j"'] & \mathbb{A}^1 \end{tikzcd}$$

commutes, $f_0(C)$ is a point, and hence is a point $Q \in \mathbb{A}^1 \setminus S$. By (18), $C = f_0^{-1}(Q)$. As $\operatorname{Miss}_{\mathrm{hor}}(\Phi_{0j}, f_j) = \varnothing$ by (17), $\mathbb{A}^2 \xrightarrow{\Phi_{0j}} \mathbb{A}^2 \xrightarrow{f_j} \mathbb{A}^1$ satisfies the hypothesis of L. 2.2.2 and consequently $f_j^{-1}(Q)$ is a coordinate line in $\mathbb{A}^2 = \operatorname{Spec} A_j$. For a suitable choice of $\lambda \in \mathbf{k}$, $f_j^{-1}(Q)$ is the zero-set of the ideal $(F - \lambda)A_j$; so $F - \lambda$ is a power of a variable of A_j; by maximality of $\mathbf{k}[F]$ in Σ, $F - \lambda$ is in fact a variable of A_j. So F is a variable of A_j, i.e., (19) is proved.

By (19) there exists G such that $A_{N_S} = \mathbf{k}[F, G]$ and, given any $j \geq N_S$,

$$F \in A_j \subseteq A_{N_S} = \mathbf{k}[F, G];$$

then L. 3.5 implies that $A_j = \mathbf{k}[F, \varphi_j(F)G]$ for some $\varphi_j(F) \in \mathbf{k}[F]$, showing that $\big(A_i\big)_{i \in \mathbb{N}}$ is simple. □

3.24. **Lemma.** *Theorem* 3.3 *is valid if* $\mathbf{k}$ *is algebraically closed.*

Proof. Assume that $\mathbf{k}$ is algebraically closed and let $\big(A_i\big)_{i \in \mathbb{N}}$ be an infinite descending chain in $\mathcal{A}$ such that $\bigcap_{i=0}^{\infty} A_i \neq \mathbf{k}$. By Prop. 3.20, one of the following holds:

(i) $\big(A_i\big)_{i \in \mathbb{N}}$ is of type ($\boldsymbol{\mathcal{V}}$);

(ii) $\big(A_i\big)_{i \in \mathbb{N}}$ is not of bounded type.

It follows that $\big(A_i\big)_{i \in \mathbb{N}}$ is simple, by L. 3.17 in case (i) and by Prop. 3.23 in case (ii). □

By Lemmas 3.13 and 3.24, the proof of Theorem 3.3 is complete.

4. Infinite chains of inclusions: the general case

Let $\mathbf{k}$ be a field. We consider infinite sequences of rings $\big(A_i\big)_{i\in\mathbb{N}}$ where $A_i = \mathbf{k}^{[2]}$ for all i and $A_0 \supset A_1 \supset A_2 \supset \cdots$ are strict inclusions. We do not assume that the rings have the same field of fractions. If $\bigcap_{i=0}^{\infty} A_i \neq \mathbf{k}$, what can be said? If we are willing to assume that $\operatorname{char}\mathbf{k} = 0$, we obtain a complete answer in Thm 4.15. In arbitrary characteristic, we obtain partial answers in Prop. 4.5 and Prop. 4.14.

Lemma 4.1 is probably well known, but in lack of a reference we provide a proof.

4.1. **Lemma.** *Let $R = \bigcap_{i=0}^{\infty} A_i$, where $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ is a sequence of UFDs satisfying $A_i^* = A_{i+1}^*$ for all i. Then $R^* = A_0^*$, R is a UFD and the following holds:*

$$\text{for each prime element } p \text{ of } R,\ \ p \text{ is a prime element of } A_i \text{ for } i \gg 0. \tag{20}$$

Proof. It is clear that $R \cap A_0^* = R^*$. Define a map $\ell_i : A_i \setminus \{0\} \to \mathbb{N}$ as follows: given $b \in A_i \setminus \{0\}$, consider a factorization $b = \prod_{j=1}^{n} p_j$ where $p_1, \dots, p_n$ are prime elements of A_i; then set $\ell_i(b) = n$ (set $\ell_i(b) = 0$ if $b \in A_i^*$). Now consider an irreducible element $p \in R$. It is clear that $\ell_0(p) \geq \ell_1(p) \geq \ell_2(p) \geq \dots$, so there exist M and n such that $\ell_i(p) = n$ for all $i \geq M$. Consider the prime factorization of p in A_M, $p = q_1 \cdots q_n$. Let $i \geq M$ and consider the prime factorization of p in A_i, $p = q_{i1} \cdots q_{in}$. If some q_{ij} is not prime in A_M then $\ell_i(p) < \ell_M(p)$, which is not the case; so $q_{i1}, \dots, q_{in}$ are prime in A_M and consequently (after permuting $q_{i1}, \dots, q_{in}$ if necessary) we have $q_j = \lambda_j q_{ij}$ ($\lambda_j \in A_M^*$) for all $j = 1, \dots, n$. Since $A_M^* = A_i^*$, we get $q_1, \dots, q_n \in A_i$, and this holds for all $i \geq M$ (so for all $i \in \mathbb{N}$). Thus $q_1, \dots, q_n \in R$, and since p is irreducible in R we have $n = 1$. Since $\ell_i(p) = n = 1$ for all $i \geq M$, p is prime in A_i for all $i \geq M$. We claim that p is prime in R. Indeed, consider $u, v \in R$ such that $p \mid uv$ in R. Then $p \mid uv$ in each A_i. Since p is prime in A_i for all $i \geq M$, there exists $w \in \{u, v\}$ for which the condition "$p \mid w$ in A_i" holds for infinitely many i; then $w/p \in A_i$ for infinitely many i, so $w/p \in R$ and $p \mid w$ in R. We have shown:

(21) if p is an irreducible element of R then p is prime in R
and prime in A_i for $i \gg 0$.

Note that (21) implies (20); let us argue that it also implies that R is a UFD. Let $x \in R \setminus (R^* \cup \{0\})$. Consider all possible factorizations of x in $R \setminus (R^* \cup \{0\})$:

$$x = x_1 \cdot \dots \cdot x_s, \quad \text{with } s \geq 1 \text{ and } x_1, \dots, x_s \in R \setminus (R^* \cup \{0\}). \tag{22}$$

Since $R \cap A_0^* = R^*$, we have $x_1, \dots, x_s \in A_0 \setminus (A_0^* \cup \{0\})$ and consequently $s \leq \ell_0(x)$. So we may choose a factorization (22) of x which maximizes s; in this factorization, each x_i is necessarily an irreducible element of R, hence, by (21), a prime element of R. So R is a UFD. $\square$

The following is well known (see for instance [Miy94, L. 1.39]):

4.2. **Lemma.** *Consider rings $k \subseteq R \subseteq A$ where k is a field, A is an affine k-domain and R has transcendence degree 1 over k. Then R is k-affine.*

4.3. **Lemma.** *Suppose that $k \subseteq R \subseteq A$ where k is a field, $A = k^{[n]}$ for some $n > 0$, and R is a normal domain of transcendence degree 1 over k. Then $R = k^{[1]}$.*

Proof. By L. 4.2, R is k-affine and hence a Dedekind domain; then [Zak71] gives the desired conclusion. $\square$

4.4. **Definition.** Let R be a subring of a domain S. We call R *factorially closed in S* if the conditions $u, v \in S \setminus \{0\}$ and $uv \in R$ imply $u, v \in R$.

4.5. **Proposition.** *Let $\mathbf{k}$ be a field. Let $R = \bigcap_{i=0}^{\infty} A_i$, where $A_0 \supset A_1 \supset A_2 \supset \cdots$ is a strictly decreasing sequence of rings such that $A_i = \mathbf{k}^{[2]}$ for all i. Assume that $R \neq \mathbf{k}$. Then $R = \mathbf{k}^{[1]}$ and R is factorially closed in A_i for $i \gg 0$.*

Proof. If $\operatorname{trdeg}_{\mathbf{k}}(R) \neq 1$ then $\operatorname{Frac} A_0$ is a finite extension of $\operatorname{Frac} R$, so there exists $N \geq 0$ such that $\big(\operatorname{Frac} A_i\big)_{i \geq N}$ is a constant sequence; then Theorem 3.3 applied to $\big(A_i\big)_{i \geq N}$ gives $\bigcap_{i=N}^{\infty} A_i = \mathbf{k}^{[1]}$, which contradicts the assumption that $\operatorname{trdeg}_{\mathbf{k}}(R) \neq 1$. This shows that $\operatorname{trdeg}_{\mathbf{k}}(R) = 1$.

Since $\operatorname{trdeg}_{\mathbf{k}}(R) = 1$ and R is a normal domain included in $A_0 = \mathbf{k}^{[2]}$, L. 4.3 implies
$$R = \mathbf{k}^{[1]}.$$
By L. 4.1,

(23) for each prime element p of R, p is a prime element of A_i for $i \gg 0$.

There remains to show that R is factorially closed in A_i for $i \gg 0$. We first prove the following weaker statement:

(24) There exists $N \geq 0$ such that R is algebraically closed in A_i for all $i \geq N$.

To see this, consider the algebraic closure L of $\operatorname{Frac} R$ in $\operatorname{Frac} A_0$ and define $R_i = L \cap A_i$. For each i we have $R_i = \mathbf{k}^{[1]}$ by L. 4.3; as $\bigcap_{i \in \mathbb{N}} R_i = R = \mathbf{k}^{[1]}$, it follows that the sequence $R_0 \supseteq R_1 \supseteq R_2 \supseteq \cdots$ stabilizes and hence that $R = R_i = L \cap A_i$ for $i \gg 0$. So R is algebraically closed in A_i for $i \gg 0$, which proves (24).

Let us assume that $\mathbf{k}$ is algebraically closed. Write $R = \mathbf{k}[F]$. Consider the N of (24); then $\mathbf{k}[F]$ is algebraically closed in A_N and consequently
$$E = \big\{\, \lambda \in \mathbf{k} \mid F - \lambda \text{ is not irreducible in } A_N \,\big\}$$
is a finite set (see L. 3.22). So, by (23), there exists $M \geq N$ such that

(25) for all $i \geq M$ and all $\lambda \in E$, $F - \lambda$ is irreducible in A_i.

Then

(26) for all $i \geq M$ and all $\lambda \in \mathbf{k}$, $F - \lambda$ is irreducible in A_i.

Indeed, let $i \geq M$ and $\lambda \in \mathbf{k}$; if $\lambda \in E$, then $F - \lambda$ is irreducible in A_i by (25); if $\lambda \notin E$, then $F - \lambda$ is irreducible in A_N by definition of E, and it follows that it is irreducible in A_i because $A_i \subseteq A_N$ and $A_i^* = A_N^*$. So (26) is proved.

Then (26) implies that, for every $i \geq M$ and every prime element p of R, p is prime in A_i. Thus R is factorially closed in A_i for $i \geq M$ (if $\mathbf{k}$ is algebraically closed).

Now drop the assumption that $\mathbf{k}$ is algebraically closed, and let $\bar{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$. Applying $\bar{\mathbf{k}}\otimes_{\mathbf{k}}(\_)$ to $\big(A_i\big)_{i\in\mathbb{N}}$ produces a sequence $\big(\overline{A}_i\big)_{i\in\mathbb{N}}$ of rings $\overline{A}_i = \bar{\mathbf{k}}^{[2]}$ satisfying all hypotheses of the Proposition. Define $\overline{R} = \bigcap_{i=0}^{\infty} \overline{A}_i$. By the preceding paragraph, there exists M' such that $\overline{R}$ is factorially closed in $\overline{A}_i$ for all $i \geq M'$. Let us deduce that R is factorially closed in A_i for all $i \geq \max(N, M')$, where N is as in (24). Let $i \geq \max(N, M')$ and consider the commutative diagram of injective homomorphisms:

$$\begin{tikzcd} A_i \arrow[r] & \overline{A}_i \\ R \arrow[r] \arrow[u] & \overline{R} \arrow[u] \end{tikzcd}$$

Consider $u, v \in A_i \setminus \{0\}$ satisfying $uv \in R$; then $u, v \in \overline{A}_i \setminus \{0\}$ satisfy $uv \in \overline{R}$, so $u, v \in \overline{R}$ since $\overline{R}$ is factorially closed in $\overline{A}_i$ (because $i \geq M'$). Since $R = \mathbf{k}^{[1]}$ and $\overline{R} = \bar{\mathbf{k}}^{[1]}$, $\overline{R}$ is algebraic over R. Thus u, v are elements of A_i which are algebraic over R; since $i \geq N$, we get $u, v \in R$ by (24), showing that R is factorially closed in A_i. □

4.6**.** Let $\mathbf{k}$ be an algebraically closed field and $f : X \to Y$ a dominant morphism of $\mathbf{k}$-varieties of the same dimension. Let $\mathbf{k}(X)$ and $\mathbf{k}(Y)$ denote the function fields of X and Y respectively. Then $\mathbf{k}(X)/\mathbf{k}(Y)$ is a finite extension of fields and we define $\deg f = [\mathbf{k}(X) : \mathbf{k}(Y)]$, $\deg_s f = [\mathbf{k}(X) : \mathbf{k}(Y)]_s$ and $\deg_i f = [\mathbf{k}(X) : \mathbf{k}(Y)]_i$ (that is, consider the unique field L such that $\mathbf{k}(Y) \subseteq L \subseteq \mathbf{k}(X)$, L is separable over $\mathbf{k}(Y)$ and $\mathbf{k}(X)$ is purely inseparable over L; then $\deg_s f = [L : \mathbf{k}(Y)]$ and $\deg_i f = [\mathbf{k}(X) : L]$). We say that f is a *separable* (resp. *purely inseparable*) morphism if $\mathbf{k}(X)/\mathbf{k}(Y)$ is a separable (resp. purely inseparable) extension.

4.7. **Notation.** Let C be a curve over an algebraically closed field $\mathbf{k}$.

(1) Let $g(C)$ denote the geometric genus of a nonsingular projective model of C.
(2) Suppose that C is affine, let $\mathbf{k}[C]$ be the coordinate algebra of C and $\mathbf{k}(C)$ the function field of C (so $\mathbf{k} \subset \mathbf{k}[C] \subset \mathbf{k}(C)$). The *number of places at infinity* of C is, by definition, the number of valuation rings $\mathcal{O}$ of $\mathbf{k}(C)/\mathbf{k}$ satisfying $\mathbf{k}[C] \nsubseteq \mathcal{O}$. We shall write $n_\infty(C)$ for the number of places at infinity of an affine curve C. Note that $n_\infty(C) \in \mathbb{N} \setminus \{0\}$.

4.8. **Lemma.** *Let X, Y be nonsingular projective curves over an algebraically closed field $\mathbf{k}$, and let $f : X \to Y$ be a finite, separable morphism. Then*

$$2g(X) - 2 \geq n(2g(Y) - 2) + \sum_{x\in X} (e_x - 1)$$

where $n = \deg f$ and e_x is the ramification index of f at x.

Proof. We have $2g(X)-2 = n(2g(Y)-2)+\sum_{x\in X} \operatorname{length}(\Omega_{X/Y})_x$ by Hurwitz's Theorem [Har77, Ch. IV, 2.4], and $\operatorname{length}(\Omega_{X/Y})_x \geq e_x - 1$ by [Har77, Ch. IV, 2.2(c)]. □

4.9. **Lemma.** *Let $\varphi : X \to Y$ be a dominant morphism of affine curves over an algebraically closed field $\mathbf{k}$.*

(1) $n_\infty(X) \geq n_\infty(Y)$

(2) *If φ is purely inseparable, then $g(X) = g(Y)$.*
(3) *If $n_\infty(X) = n_\infty(Y)$ and φ is not purely inseparable, then either $g(X) > g(Y)$ or*

$$g(X) = 0 = g(Y) \text{ and } n_\infty(X) = n_\infty(Y) \leq 2.$$

Proof. (1) Let $\tilde{X}$ and $\tilde{Y}$ be nonsingular projective curves such that $\mathbf{k}(\tilde{X}) = \mathbf{k}(X)$ and $\mathbf{k}(\tilde{Y}) = \mathbf{k}(Y)$. Then $\varphi : X \to Y$ induces a morphism $f : \tilde{X} \to \tilde{Y}$. Let $r = n_\infty(Y)$ and consider the points $Q_1, \dots, Q_r \in \tilde{Y}$ corresponding to the r places at infinity of Y. Then $n_\infty(X) \geq |f^{-1}(\{Q_1, \dots, Q_r\})| \geq r$, so $n_\infty(X) \geq n_\infty(Y)$.

(2) If φ is purely inseparable then $g(X) = g(Y)$ follows from [Har77, Ch. IV, 2.5].

(3) From now-on, assume that φ is not purely inseparable. Then f factors as

$$\tilde{X} \xrightarrow{f_i} C \xrightarrow{f_s} \tilde{Y}$$

where C is a nonsingular projective curve, f_i is purely inseparable, f_s is separable and $\deg(f_s) \geq 2$. Let $n = \deg(f_s)$ and, for each $P \in C$, let $e_P \geq 1$ be the ramification index of f_s at P.

Also assume that $n_\infty(X) = n_\infty(Y)$. Then, for each $j = 1, \dots, r$, $f^{-1}(Q_j)$ is one point of $\tilde{X}$; so $f_s^{-1}(Q_j)$ is one point $P_j \in C$. The ramification index e_{P_j} of f_s at P_j satisfies $e_{P_j} = [\mathbf{k}(C) : \mathbf{k}(Y)] = n$ for all $j = 1, \dots, r$, because the valuation ring $\mathcal{O}_{P_j}$ is the unique extension of the valuation ring $\mathcal{O}_{Q_j}$ via the field extension $\mathbf{k}(C)/\mathbf{k}(Y)$.

Note that $g(X) = g(C)$, again by [Har77, Ch. IV, 2.5]. Applying L. 4.8 to the separable morphism $f_s : C \to Y$ gives the first inequality in:

$$\begin{aligned} (27)\quad 2g(X) - 2 = 2g(C) - 2 &\geq n(2g(Y) - 2) + \sum_{P \in C} (e_P - 1) \\ &\geq n(2g(Y) - 2) + \sum_{i=1}^{r} (e_{P_i} - 1) = n(2g(Y) - 2) + r(n - 1) \end{aligned}$$

where $n \geq 2$ and $r \geq 1$. In particular $g(X) - 1 > n(g(Y) - 1)$, so $g(X) > 0$ implies $g(X) > g(Y)$. If $g(X) = 0$ then $-2 \geq n(2g(Y) - 2) + r(n - 1)$, so $g(Y) = 0$ and $r \leq 2$, as desired. ☐

4.10. **Definition.** Let $\mathbf{k}$ be an algebraically closed field and $F \in A = \mathbf{k}^{[2]}$. We say that F is a *generally rational polynomial in A* if, for almost all $\lambda \in \mathbf{k}$, $F - \lambda$ is irreducible in A and the zero-set of $F - \lambda$ in $\operatorname{Spec} A$ is a rational curve.

4.11. **Remark.** Let $\mathbf{k}$ be an algebraically closed field and $F \in A = \mathbf{k}^{[2]}$. If $\operatorname{char} \mathbf{k} = 0$, F is a field generator in A if and only if it is a generally rational polynomial in A (see the introduction of [MS80]). This is not true in positive characteristic. It is shown in [Dai15] that, in arbitrary characteristic, F is a field generator in A if and only if it is a generally rational polynomial in A without moving singularities.

4.12. **Lemma.** *Let $\mathbf{k}$ be an algebraically closed field and $F \in A = \mathbf{k}^{[2]}$ a generally rational polynomial in A. Then*

$$\operatorname{dic}(F, A) - 1 = \textstyle\sum_{\lambda \in \mathbf{k}} (N_\lambda - 1),$$

where N_λ is the number of irreducible components of the zero-set of $F - \lambda$ in $\operatorname{Spec} A$.

Proof. See paragraph 2.3 for the definition of $\operatorname{dic}(F, A)$. If $\mathbf{k} = \mathbb{C}$, this result is either one of [Suz74, Th. 2] or [Kal92, Cor. 2]; if $\operatorname{char} \mathbf{k} = 0$, see [MS80, 1.6]; for the general case, see [Dai15, 1.11]. $\square$

4.13. **Definition.** Let $\mathbf{k}$ be a field, and consider an infinite sequence of rings $\big(A_i\big)_{i\in\mathbb{N}}$ where $A_i = \mathbf{k}^{[2]}$ for all i and $A_0 \supset A_1 \supset A_2 \supset \cdots$ are strict inclusions. We say that $\big(A_i\big)_{i\in\mathbb{N}}$ *is of p.i. type* if there exists $N \in \mathbb{N}$ such that, for all $i \geq N$, $\operatorname{Frac} A_i$ is a purely inseparable extension of $\operatorname{Frac} A_{i+1}$. (Note that if K is a field then the trivial field extension K/K is both separable and purely inseparable. Consequently, if the sequence of fields $\big(\operatorname{Frac} A_i\big)_{i\in\mathbb{N}}$ stabilizes then $\big(A_i\big)_{i\in\mathbb{N}}$ is of p.i. type.)

4.14. **Proposition.** *Let $\mathbf{k}$ be an algebraically closed field. Let $R = \bigcap_{i=0}^{\infty} A_i$, where $A_0 \supset A_1 \supset A_2 \supset \cdots$ is a strictly decreasing sequence of rings such that $A_i = \mathbf{k}^{[2]}$ for all i. Assume that $R \neq \mathbf{k}$ and that $\big(A_i\big)_{i\in\mathbb{N}}$ is not of p.i. type. Then $R = \mathbf{k}[F] = \mathbf{k}^{[1]}$ and, for $i \gg 0$, F is a generally rational polynomial in A_i with one dicritical and at most two places at infinity.*

Remark. The last sentence of the above statement gives some properties of F as an element of A_i; in that context, "F has one dicritical" means $\operatorname{dic}(F, A_i) = 1$, and "$F$ has at most two places at infinity" means:

$$\text{for general } \lambda \in \mathbf{k}, \quad n_\infty(C_{i,\lambda}) \leq 2$$

where $C_{i,\lambda} \subset \operatorname{Spec} A_i$ is the zero-set of $F - \lambda$ in $\operatorname{Spec} A_i$.

Proof of Prop. 4.14. By Prop. 4.5, $R = \mathbf{k}^{[1]}$ and R is factorially closed in A_i for $i \gg 0$. So there exists N such that, for all $i \geq N$ and all $\lambda \in \mathbf{k}$, $F - \lambda$ is an irreducible element of A_i. Replacing $\big(A_i\big)_{i\in\mathbb{N}}$ by $\big(A_i\big)_{i\geq N}$ if necessary, we may assume that

(28) for all $i \in \mathbb{N}$ and all $\lambda \in \mathbf{k}$, $F - \lambda$ is an irreducible element of A_i.

For each $i \in \mathbb{N}$, let $S_i = \operatorname{Spec} A_i \cong \mathbb{A}^2$; for each $\lambda \in \mathbf{k}$, let $C_{i,\lambda} \subset S_i$ be the zero-set of $F - \lambda$ (so $C_{i,\lambda}$ is an integral curve). Let $\varphi_i : S_i \to S_{i+1}$ be the dominant morphism determined by the inclusion $A_i \hookleftarrow A_{i+1}$, and let $\varphi_{i,\lambda} : C_{i,\lambda} \to C_{i+1,\lambda}$ be the restriction of φ_i. For each $i \in \mathbb{N}$, there exist $g_i, n_i \in \mathbb{N}$ satisfying

$$\text{for general } \lambda \in \mathbf{k}, \quad g(C_{i,\lambda}) = g_i \text{ and } n_\infty(C_{i,\lambda}) = n_i$$

(see 4.7 for the notation). By Lemma 4.9, $n_i \geq n_{i+1}$ for all i; so there exists N such that $(n_i)_{i\geq N}$ is a constant sequence; replacing $\big(A_i\big)_{i\in\mathbb{N}}$ by $\big(A_i\big)_{i\geq N}$, we may assume that $(n_i)_{i\in\mathbb{N}}$ is constant. Under that assumption, Lemma 4.9 implies that $g_i \geq g_{i+1}$ for all i; so we may assume that $(g_i)_{i\in\mathbb{N}}$ is a constant sequence.

For each $i \in \mathbb{N}$, we also have

$$\begin{aligned} \text{for general } \lambda \in \mathbf{k}, \quad & \varphi_{i,\lambda} : C_{i,\lambda} \to C_{i+1,\lambda} \text{ is a dominant morphism} \\ & \text{and } \deg_s(\varphi_{i,\lambda}) = [\operatorname{Frac} A_i : \operatorname{Frac} A_{i+1}]_s \end{aligned}$$

(this follows from [Dai15, L. 2.9]). The assumption that $\big(A_i\big)_{i\in\mathbb{N}}$ is not of p.i. type implies that $[\operatorname{Frac} A_i : \operatorname{Frac} A_{i+1}]_s > 1$ for infinitely many $i \in \mathbb{N}$; so there are infinitely many $i \in \mathbb{N}$ satisfying:

(29) $$\deg_s(\varphi_{i,\lambda}) > 1 \quad \text{for almost all } \lambda \in \mathbf{k}.$$

By Lemma 4.9, for each i satisfying (29) we have either $g_i > g_{i+1}$ or

$$g_i = 0 = g_{i+1} \text{ and } n_i = n_{i+1} \leq 2;$$

as $(g_i)_{i\in\mathbb{N}}$ and $(n_i)_{i\in\mathbb{N}}$ are constant sequences, it follows that $g_i = 0$ and $n_i \leq 2$ for all $i \in \mathbb{N}$. Thus F is a generally rational polynomial in A_i with at most two places at infinity. Let $i \in \mathbb{N}$; then L. 4.12 gives

$$\operatorname{dic}(F, A_i) - 1 = \textstyle\sum_{\lambda\in\mathbf{k}}(N_\lambda - 1), \tag{30}$$

where N_λ is the number of irreducible components of the zero-set of $F - \lambda$ in $\operatorname{Spec} A_i$. By (28), the right hand side of (30) is 0, so $\operatorname{dic}(F, A_i) = 1$. □

4.15. **Theorem.** *Let $\mathbf{k}$ be a field of characteristic zero. Let $R = \bigcap_{i=0}^{\infty} A_i$, where $A_0 \supset A_1 \supset A_2 \supset \cdots$ is a strictly decreasing sequence of rings such that $A_i = \mathbf{k}^{[2]}$ for all i. Assume that $R \neq \mathbf{k}$. Then $R = \mathbf{k}^{[1]}$ and $A_i = R^{[1]}$ for $i \gg 0$.*

Remark. Since $R = \mathbf{k}^{[1]}$, we have $R = \mathbf{k}[F]$ for some F. The last assertion states that F is a variable of A_i for all $i \gg 0$.

Proof of Thm 4.15. Let $\bar{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$. Applying $\bar{\mathbf{k}} \otimes_{\mathbf{k}} (\_)$ to $\big(A_i\big)_{i\in\mathbb{N}}$ produces a sequence $\big(\overline{A}_i\big)_{i\in\mathbb{N}}$ of rings $\overline{A}_i = \bar{\mathbf{k}}^{[2]}$ satisfying all hypotheses of the Theorem. Define $\overline{R} = \bigcap_{i=0}^{\infty} \overline{A}_i$. We know that $R = \mathbf{k}^{[1]}$ and that $\overline{R} = \bar{\mathbf{k}}^{[1]}$; write $R = \mathbf{k}[F]$ and $\overline{R} = \bar{\mathbf{k}}[u]$.

Assume that the theorem is true in the case where $\mathbf{k}$ is algebraically closed; then $\overline{A}_i = \overline{R}^{[1]}$ for all $i \gg 0$. Moreover, R is factorially closed (hence algebraically closed) in A_i for $i \gg 0$. Let i be large enough so that

$$\overline{A}_i = \overline{R}^{[1]} \text{ and } R \text{ is algebraically closed in } A_i.$$

Then u is a variable of $\overline{A}_i$. Since $F \in \bar{\mathbf{k}}[u]$, the statement

> *Let $K \subseteq L$ be fields of characteristic zero, X, Y indeterminates over L and $\varphi \in K[X,Y] \subseteq L[X,Y]$. If $\varphi \in L[u]$ for some variable u of $L[X,Y]$, then $\varphi \in K[u']$ for some variable u' of $K[X,Y]$.*

(whose proof is left to the reader) implies that $F \in \mathbf{k}[u']$ for some variable u' of A_i. Since $F \in \mathbf{k}[u'] \subset A_i$ and $R = \mathbf{k}[F]$ is algebraically closed in A_i, we have $R = \mathbf{k}[u']$ and hence $A_i = R^{[1]}$. This shows that if the theorem is true in the case where $\mathbf{k}$ is algebraically closed, then it is true in general.

From now-on, assume that $\mathbf{k}$ is algebraically closed.

If the sequence of fields $\big(\operatorname{Frac} A_i\big)_{i\in\mathbb{N}}$ stabilizes, then we are done by Theorem 3.3. So we may assume that that sequence does not stabilize, which (together with $\operatorname{char}\mathbf{k} = 0$) implies that $\big(A_i\big)_{i\in\mathbb{N}}$ is not of p.i. type. Then Prop. 4.14 implies that, for $i \gg 0$, F is a generally rational polynomial in A_i with at most one dicritical.

Since $\operatorname{char}\mathbf{k} = 0$, every generally rational polynomial is a field generator (cf. Rem. 4.11); so, for $i \gg 0$, F is a field generator in A_i with only one dicritical. Then F is a variable of A_i (the fact that *a field generator with one dicritical is a variable* is due to Russell, but see for instance [CND15b, 5.2] for an explicit statement and a proof). □

5. Sets of polynomial rings

5.1. **Lemma.** *Let $\mathbf{k}$ be a field, let $A \subseteq B$ be such that $A = \mathbf{k}^{[2]}$ and $B = \mathbf{k}^{[2]}$, and let*

$$\mathcal{R}^*(A,B) = \big\{\, R \mid A \subseteq R \subseteq B,\ R = \mathbf{k}^{[2]},\ \text{and } \operatorname{Frac} R = \operatorname{Frac} A \,\big\}.$$

Then there exists $N \in \mathbb{N}$ such that all chains of strict inclusions

$$R_0 \subset R_1 \subset \cdots \subset R_s$$

of elements of $\mathcal{R}^*(A,B)$ satisfy $s \leq N$.

Proof. Let $\bar{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$, $\bar{A} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} A = \bar{\mathbf{k}}^{[2]}$ and $\bar{B} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} B = \bar{\mathbf{k}}^{[2]}$. Then $R \mapsto \bar{\mathbf{k}} \otimes_{\mathbf{k}} R$ is an order-preserving injective map from $\mathcal{R}^*(A,B)$ to $\mathcal{R}^*(\bar{A},\bar{B})$. So, we may assume that $\mathbf{k}$ is algebraically closed. This allows us to use the results of [CND14] and [Dai91].

Let $B_0 = B \cap \operatorname{Frac} A$; note that B_0 is a normal domain and, by Zariski's result [Zar54], a finitely generated $\mathbf{k}$-algebra. Thus $S_0 = \operatorname{Spec} B_0$ is a normal affine surface, and we may consider a resolution of singularities $\pi : S \to S_0$ of S_0. In particular, S is a nonsingular surface and $\pi : S \to S_0$ is a morphism which is birational and surjective. Let $h : S \to \mathbb{A}^2$ be the composition $S \xrightarrow{\pi} S_0 = \operatorname{Spec} B_0 \to \operatorname{Spec} A = \mathbb{A}^2$; then h is a birational morphism of nonsingular surfaces, so $n(h) \in \mathbb{N}$ is defined (see [CND14, 2.3]). Each $R \in \mathcal{R}^*(A,B)$ determines a pair $S \xrightarrow{f_R} \mathbb{A}^2 \xrightarrow{g_R} \mathbb{A}^2$ of birational morphisms defined by the commutative diagram

$$\begin{tikzcd}
& S \arrow[d, "\pi"'] \arrow[dr, "f_R"] \arrow[drr, bend left, "h"] & & \\
\operatorname{Spec} B \arrow[r] & \operatorname{Spec} B_0 \arrow[r] & \operatorname{Spec} R \arrow[r, "g_R"] & \operatorname{Spec} A \\
B & B_0 \arrow[l, hook'] & R \arrow[l, hook'] & A \arrow[l, hook']
\end{tikzcd}$$

We claim:

(31) *each curve in* $\operatorname{Spec} R$ *meets the image of* $\operatorname{Spec} B \to \operatorname{Spec} R$.

Indeed, let $C \subset \operatorname{Spec} R$ be a curve; as $R = \mathbf{k}^{[2]}$ is a UFD, C is the zero-set of an irreducible element $p \in R$; as $p \notin \mathbf{k}$ and $p \in R \subseteq B = \mathbf{k}^{[2]}$, it follows that p is a non-unit element of B; so there exists a maximal ideal $\mathfrak{m}$ of B such that $p \in \mathfrak{m} \cap R$, which means that some closed point of $\operatorname{Spec} B$ is mapped to a closed point of C. So (31) is true. As the image of $\operatorname{Spec} B \to \operatorname{Spec} R$ is included in that of $\operatorname{Spec} B_0 \to \operatorname{Spec} R$ (because $R \subseteq B_0 \subseteq B$), which is (by surjectivity of π) equal to that of f_R, we conclude that

each curve in $\operatorname{Spec} R$ *meets the image of* f_R.

So $q_0(f_R) = 0$, where q_0 is defined in [CND14, 2.2]. By [Dai91, 1.3], it follows that

$$n(f_R) + n(g_R) = n(g_R \circ f_R).$$

As $g_R \circ f_R = h$,

(32) $n(g_R) \leq n(h)$, *for all* $R \in \mathcal{R}^*(A,B)$.

Also note that

(33) *if $R' \subset R''$ is a strict inclusion where $R', R'' \in \mathcal{R}^*(A,B)$, then $n(g_{R'}) < n(g_{R''})$.*

Indeed,

$$\begin{tikzcd} \operatorname{Spec} R'' \arrow[r, "u"'] \arrow[rr, bend left, "g_{R''}"] & \operatorname{Spec} R' \arrow[r, "g_{R'}"'] & \operatorname{Spec} A \end{tikzcd}$$

so $n(g_{R''}) = n(g_{R'} \circ u) = n(g_{R'}) + n(u)$ by [CND14, 2.5], and $n(u) > 0$ by [CND14, 2.6(b)] and $R' \neq R''$. So (33) is true. By (32) and (33), we obtain that any chain of strict inclusions $R_0 \subset R_1 \subset \cdots \subset R_s$ of elements of $\mathcal{R}^*(A,B)$ satisfies $s \le n(h)$. $\square$

5.2. **Proposition.** *Let $\mathbf{k}$ be a field and $A = \mathbf{k}^{[2]}$.*

(a) *Let $\mathcal{R}^-(A) = \big\{\, R \mid \mathbf{k} \subset R \subseteq A$ and $R = \mathbf{k}^{[2]} \,\big\}$. Then the poset $\big(\mathcal{R}^-(A), \subseteq\big)$ satisfies ACC.*

(b) *Let Ω be an algebraic closure of $\operatorname{Frac} A$ and define*

$$\mathcal{R}^+(A) = \big\{\, R \mid A \subseteq R \subseteq \Omega \text{ and } R = \mathbf{k}^{[2]} \,\big\}.$$

Then the poset $\big(\mathcal{R}^+(A), \subseteq\big)$ satisfies DCC.

Proof. If $\mathcal{R}^+(A)$ does not satisfy DCC then there exists an infinite strictly decreasing sequence $R_0 \supset R_1 \supset R_2 \supset \cdots$ of elements of $\mathcal{R}^+(A)$. Then Prop. 4.5 implies that $\operatorname{trdeg}_{\mathbf{k}}(R_*) \le 1$, where we define $R_* = \bigcap_{i\in\mathbb{N}} R_i$. This is absurd, since $A \subseteq R_*$. So $\mathcal{R}^+(A)$ satisfies DCC.

Next, let $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$ be an increasing sequence of elements of $\mathcal{R}^-(A)$, and let us prove that it stabilizes. Clearly, $\big(\operatorname{Frac} R_i\big)_{i\in\mathbb{N}}$ stabilizes, because $\operatorname{Frac} A$ is a finite extension of $\operatorname{Frac} R_0$. So there exists $i_0 \in \mathbb{N}$ such that $\operatorname{Frac} R_i = \operatorname{Frac} R_{i_0}$ for all $i \ge i_0$. Observe that $\big(R_i\big)_{i\ge i_0}$ is a sequence in $\mathcal{R}^*(R_{i_0}, A)$, where

$$\mathcal{R}^*(R_{i_0}, A) = \big\{\, R' \mid R_{i_0} \subseteq R' \subseteq A,\ R' = \mathbf{k}^{[2]} \text{ and } \operatorname{Frac} R' = \operatorname{Frac} R_{i_0} \,\big\}.$$

By L. 5.1, $\mathcal{R}^*(R_{i_0}, A)$ satisfies a condition stronger than ACC; so $\big(R_i\big)_{i\ge i_0}$ stabilizes and so does $\big(R_i\big)_{i\in\mathbb{N}}$. This proves the proposition. $\square$

5.3. **Definition.** Let $(X, \le)$ be a poset. A *saturated chain* in X is a finite sequence $x_0 < x_1 < \cdots < x_n$ of elements of X such that, for each $i \in \{1, \dots, n\}$, no element $s \in X$ satisfies $x_{i-1} < s < x_i$. A *maximal chain* in X is a saturated chain $x_0 < \cdots < x_n$ in X such that no element $s \in X$ satisfies $s < x_0$ or $x_n < s$. The natural number n is called the *length* of the chain $x_0 < \cdots < x_n$.

5.4. **Remark.** Let $(X, \le)$ be a poset satisfying ACC and DCC. Then it is easy to see that, given any $a, b \in X$ be such that $a \le b$, there exists a saturated chain $x_0 < \cdots < x_n$ in X satisfying $x_0 = a$ and $x_n = b$. However, the pair (a, b) being fixed, the set

$$\big\{\, n \in \mathbb{N} \mid \exists \text{ a saturated chain } x_0 < \cdots < x_n \text{ s.t. } x_0 = a \text{ and } x_n = b \,\big\}$$

is not necessarily bounded.

5.5. **Remark.** *Any poset $(X, \leq)$ satisfying ACC and DCC has at least one maximal chain.* Indeed, we may choose $a, b \in X$ such that $a \leq b$, a is a minimal element of X and b is a maximal element of X. By Rem. 5.4, there exists a saturated chain $x_0 < \cdots < x_n$ in X satisfying $x_0 = a$ and $x_n = b$. This is a maximal chain in X.

5.6. **Theorem.** *Let $\mathbf{k}$ be a field, let $A \subseteq B$ be such that $A = \mathbf{k}^{[2]}$ and $B = \mathbf{k}^{[2]}$, and let*

$$\mathcal{R}(A,B) = \big\{\, R \mid A \subseteq R \subseteq B \text{ and } R = \mathbf{k}^{[2]} \,\big\}.$$

(a) *The poset $(\mathcal{R}(A,B), \subseteq)$ satisfies ACC and DCC.*
(b) *There exists a maximal chain in $\mathcal{R}(A,B)$.*

Remark. A maximal chain in $\mathcal{R}(A,B)$ is the same thing as a saturated chain $R_0 \subset R_1 \subset \cdots \subset R_n$ in $\mathcal{R}(A,B)$ satisfying $R_0 = A$ and $R_n = B$.

Proof of Thm 5.6. Since $\mathcal{R}(A,B) \subseteq \mathcal{R}^-(B)$ and $\mathcal{R}(A,B) \subseteq \mathcal{R}^+(A)$, assertion (a) follows from Prop. 5.2. Assertion (b) follows from (a) and Rem. 5.5. □

5.7. **Remark.** The setup being as in Thm 5.6, we note:

(1) By Ex. 5.9, $\mathcal{R}(A,B)$ is not necessarily a finite set.
(2) By Thm 5.6, $\mathcal{R}(A,B)$ has at least one maximal chain.
(3) By Ex. 6.3, $\mathcal{R}(A,B)$ may have maximal chains of different lengths.
(4) We don't know if the lengths of maximal chains in $\mathcal{R}(A,B)$ are bounded.

5.8. **Remark.** Let $\mathbf{k}$ be a field, let $A \subseteq B$ be such that $A = \mathbf{k}^{[2]}$ and $B = \mathbf{k}^{[2]}$, and let L be a field such that $\operatorname{Frac} A \subseteq L \subseteq \operatorname{Frac} B$. Define

$$\mathcal{R}^L(A,B) = \big\{\, R \mid A \subseteq R \subseteq B,\ R = \mathbf{k}^{[2]}, \text{ and } \operatorname{Frac} R = L \,\big\}.$$

Then consider the following question:

> *Given A, B and L as above, does there exist $N \in \mathbb{N}$ such that all chains of strict inclusions $R_0 \subset R_1 \subset \cdots \subset R_s$ of elements of $\mathcal{R}^L(A,B)$ satisfy $s \leq N$?*

By L. 5.1, the answer is affirmative in the special case $L = \operatorname{Frac} A$. However, we don't know the answer in the general case. Let us observe that if the answer is always affirmative then we can settle part (4) of Rem. 5.7. Indeed, we claim:

> *Let $\mathbf{k}$ be a field of characteristic zero and let $A \subseteq B$ be such that $A = \mathbf{k}^{[2]}$ and $B = \mathbf{k}^{[2]}$. Suppose that for each field L such that $\operatorname{Frac} A \subseteq L \subseteq \operatorname{Frac} B$, there exists $N_L \in \mathbb{N}$ such that all chains of strict inclusions $R_0 \subset R_1 \subset \cdots \subset R_s$ of elements of $\mathcal{R}^L(A,B)$ satisfy $s \leq N_L$. Then the lengths of maximal chains in $\mathcal{R}(A,B)$ are bounded.*

To see this, consider the set $\mathcal{F}$ of fields L such that $\operatorname{Frac} A \subseteq L \subseteq \operatorname{Frac} B$. Since $\operatorname{char} \mathbf{k} = 0$, $\operatorname{Frac} B$ is a finite separable extension of $\operatorname{Frac} A$, so $\mathcal{F}$ is a finite set. Then it is not hard to see that every chain $R_0 \subset R_1 \subset \cdots \subset R_s$ in $\mathcal{R}(A,B)$ satisfies $s < \sum_{L \in \mathcal{F}} (N_L + 1)$.

5.9. **Example.** Assume that $\operatorname{char} \mathbf{k} = p > 0$ and consider $B = \mathbf{k}[x,y] = \mathbf{k}^{[2]}$ and $A = \mathbf{k}[x^p, y^p] = \mathbf{k}^{[2]}$. Then $\mathcal{R}(A,B)$ is an infinite set. Indeed, $n \mapsto \mathbf{k}[x^p, y + x^{p^n+1}]$ is an injective map from $\mathbb{N}$ to $\mathcal{R}(A,B)$, and $\lambda \mapsto \mathbf{k}[x^p, y + \lambda x]$ is an injective map from $\mathbf{k}$ to $\mathcal{R}(A,B)$. Furthermore, the main result of [Dai93] (or of [Gan82] if $\mathbf{k}$ is algebraically closed) implies that $\mathcal{R}(A,B) = \{A,B\} \cup \big\{\, \mathbf{k}[u^p,v] \mid (u,v) \in B^2 \text{ is s.t. } B = \mathbf{k}[u,v] \,\big\}$. This description of $\mathcal{R}(A,B)$ implies that every maximal chain $R_0 \subset R_1 \subset \cdots \subset R_n$ in $\mathcal{R}(A,B)$ satisfies $n=2$.

5.10. **Definition.** Let $\mathbf{k}$ be a field, $R = \mathbf{k}^{[2]}$ and $F \in R$. We say that F is *univariate in R* if there exist X, Y such that $R = \mathbf{k}[X,Y]$ and $F \in \mathbf{k}[X]$.

5.11. **Theorem.** *Let $\mathbf{k}$ be a field, $A = \mathbf{k}^{[2]}$, $F \in A \setminus \mathbf{k}$ and*

$$\mathcal{U}(A,F) = \big\{\, R \mid R = \mathbf{k}^{[2]},\ F \in R \subseteq A \text{ and } F \text{ is not univariate in } R \,\big\}.$$

Then $\big(\mathcal{U}(A,F), \subseteq\big)$ satisfies ACC. If $\operatorname{char}\mathbf{k} = 0$, then it also satisfies DCC.

Proof. ACC follows from Prop. 5.2 and $\mathcal{U}(A,F) \subseteq \mathcal{R}^-(A)$; DCC (if $\operatorname{char}\mathbf{k} = 0$) follows from Thm 4.15. □

It is interesting to note the following variant of Thm 5.11, valid over an arbitrary field $\mathbf{k}$:

5.12. **Theorem.** *Let $\mathbf{k}$ be a field, $A = \mathbf{k}^{[2]}$, $F \in A \setminus \mathbf{k}$ and*

$$\mathcal{U}^*(A,F) = \big\{\, R \in \mathcal{U}(A,F) \mid \operatorname{Frac} R = \operatorname{Frac} A \,\big\}.$$

Then $\big(\mathcal{U}^*(A,F), \subseteq\big)$ satisfies ACC and DCC.

Proof. Since $\mathcal{U}^*(A,F) \subseteq \mathcal{U}(A,F)$ and $\mathcal{U}(A,F)$ satisfies ACC by Thm 5.11, $\mathcal{U}^*(A,F)$ satisfies ACC. By Thm 1.1(i), $\mathcal{U}^*(A,F)$ satisfies DCC. □

5.13**. Some examples.** We conclude this section with some examples of minimal elements of $\mathcal{U}(A,F)$. In L. 5.13.1 and Ex. 5.13.2, $\mathbf{k}$ is an algebraically closed field of characteristic zero.

5.13.1. **Lemma.** *Let $A = \mathbf{k}[x,y] = \mathbf{k}^{[2]}$. Then $\mathcal{U}(A,F) = \{A\}$ in each of the following cases:*

(a) *$F = xy$;*
(b) *$F = x^2 + y^3$.*

Proof. In both cases (a) and (b), it is clear that $A \in \mathcal{U}(A,F)$. Consider any $R = \mathbf{k}^{[2]}$ such that $F \in R \subset A$ (where "$\subset$" is strict); to prove the claim, we have to show that F is a variable of R.

Let $h : \operatorname{Spec} A \to \operatorname{Spec} R$ be the morphism determined by the inclusion $R \hookrightarrow A$.

For each $\lambda \in \mathbf{k}$ such that $F - \lambda$ is irreducible in A (and hence in R), let $C_\lambda^A \subset \operatorname{Spec} A$ be the curve "$F = \lambda$" in $\operatorname{Spec} A$, and let $C_\lambda^R \subset \operatorname{Spec} R$ be the curve "$F = \lambda$" in $\operatorname{Spec} R$ (both are irreducible curves). Note that these curves are defined for all $\lambda \in \mathbf{k}^*$ in case (a), and for all $\lambda \in \mathbf{k}$ in case (b).

Consider case (a). Since h is a dominant morphism, there are at most finitely many $\lambda \in \mathbf{k}^*$ such that $h(C_\lambda^A)$ is a point. So, for general $\lambda \in \mathbf{k}^*$, we have a dominant morphism $C_\lambda^A \to C_\lambda^R$; since C_λ^A is a rational curve, it follows that C_λ^R is rational (for general λ). Thus F is a generally rational polynomial in R (and hence a field generator in R, cf. Rem. 4.11). We noted that $F - \lambda$ is irreducible in R for each $\lambda \in \mathbf{k}^*$, but in fact it is irreducible in R also when $\lambda = 0$ (otherwise $x, y \in R$ contradicts $R \neq A$). It follows from L. 4.12 that $\operatorname{dic}(F, R) = 1$, so F is a field generator of R with only one dicritical; as explained in the last sentence of the proof of Thm 4.15, it follows that F is a variable of R.

Case (b). Suppose that h is birational. Since h is not an isomorphism (because $R \neq A$), [CND14, L. 2.6(b)] implies that there exists a curve $C \subset \operatorname{Spec} A$ such that $h(C)$ is a point. By statement (4) in paragraph 2.3 of [CND14], the curve C is nonsingular and rational. On the other hand, C must be included in (and hence equal to) C^A_λ for some $\lambda \in \mathbf{k}$; this is impossible because there are no values of λ such that C^A_λ is both nonsingular and rational. This shows that h is not birational.

It follows that, for general $\lambda \in \mathbf{k}$, h restricts to a morphism $h_\lambda : C^A_\lambda \to C^R_\lambda$ that is dominant but not birational (hence not purely inseparable); since $g(C^A_\lambda) = 1$ and $n_\infty(C^A_\lambda) = 1$, L. 4.9 implies that $n_\infty(C^R_\lambda) = 1$ and that either $g(C^A_\lambda) > g(C^R_\lambda)$ (in which case C^R_λ is rational) or $g(C^A_\lambda) = 0 = g(C^R_\lambda)$ (which cannot happen); so C^R_λ is rational and has one place at infinity (for general λ). Thus F is a generally rational polynomial (hence a field generator) in R, with one place at infinity; so F is a variable of R. $\square$

In the following example $\mathcal{U}(A, F)$ has several minimal elements (R and R') and F has a completely different "personality" depending on whether it is viewed as an element of R or of R'.

5.13.2. **Example.** *Let $A = \mathbf{k}[x, y] = \mathbf{k}^{[2]}$ and $F = x^2 + x^3y^3 \in A$. Then $R = \mathbf{k}[x, xy]$ and $R' = \mathbf{k}[x^2, xy^3]$ are distinct minimal elements of $\mathcal{U}(A, F)$. Moreover, F is a field generator of R' but not of R.*

Indeed, let $w = xy$; then $R = \mathbf{k}[x, w]$ and $F = x^2 + w^3$, so L. 5.13.1 implies that $\mathcal{U}(R, F) = \{R\}$. With $u = x^2$ and $v = 1 + xy^3$ we have $R' = \mathbf{k}[u, v]$ and $F = uv$, so L. 5.13.1 implies that $\mathcal{U}(R', F) = \{R'\}$. So R, R' are minimal elements of $\mathcal{U}(A, F)$, and it is clear that F is a field generator of R' but not of R.

6. Two applications

We fix an arbitrary field $\mathbf{k}$ throughout this section. We write $\mathbb{A}^n$ for the affine n-space over $\mathbf{k}$, i.e., $\mathbb{A}^n = \operatorname{Spec} A$ where $A = \mathbf{k}^{[n]}$. See the introduction for the notations $\operatorname{Mor}(X, Y)$, $\operatorname{Dom}(X)$, $\operatorname{Bir}(X)$, and $\operatorname{Aut}(X)$.

This section is in two parts. The first one, very short, contains a few remarks about the monoid $\operatorname{Dom}(\mathbb{A}^2)$ of dominant endomorphisms of $\mathbb{A}^2$. The second part is concerned with lean factorizations of dominant morphisms $\mathbb{A}^2 \to \mathbb{A}^n$, $n \in \{1, 2\}$.

The monoid $\operatorname{Dom}(\mathbb{A}^2)$.

6.1. **Definition.** Let M be a multiplicative monoid with identity element denoted 1. An element u of M is *invertible* if there exists $v \in M$ satisfying $uv = 1 = vu$. An element p of M is *irreducible* if it is not invertible and for each factorization $p = xy$ with $x, y \in M$, at least one of x, y is invertible. The monoid M is *atomic* if each non invertible element of M is a finite product of irreducible elements.

We have $\operatorname{Aut}(\mathbb{A}^2) \subseteq \operatorname{Bir}(\mathbb{A}^2) \subseteq \operatorname{Dom}(\mathbb{A}^2)$. It is known that the monoid $\operatorname{Bir}(\mathbb{A}^2)$ is cancellative and atomic (cancellation is trivial, atomicity is implicit in [Dai91], explicit in [CND14], [CND15a]); more results about the structure of $\mathrm{Bir}(\mathbb{A}^2)$ can be found in [CND14], [CND15a].

The results of the present paper allow us to say something about $\mathrm{Dom}(\mathbb{A}^2)$:

6.2. **Corollary.** *The monoid* $\mathrm{Dom}(\mathbb{A}^2)$ *is atomic.*

Proof. Trivial consequence of Thm 5.6(b). □

Remark. The monoid $\mathrm{Mor}(\mathbb{A}^2, \mathbb{A}^2)$ is not atomic, and $\mathrm{Dom}(\mathbb{A}^2)$ is a submonoid of it.

As noted in the introduction, it is known that $\mathrm{Dom}(\mathbb{A}^1)$ is atomic and that factorizations into irreducibles (in $\mathrm{Dom}(\mathbb{A}^1)$) have certain uniqueness properties. One of these properties is the fact that any two irreducible factorizations of a given noninvertible element have the same number of factors (see [Rit22]). This does not hold for $\mathrm{Dom}(\mathbb{A}^2)$:

6.3. **Example.** Define $f, g \in \mathrm{Dom}(\mathbb{A}^2)$ by

$$\begin{array}{ccc} \mathbb{A}^2 & \xrightarrow{f} & \mathbb{A}^2 \\ (x,y) & \mapsto & (x^2,y) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{A}^2 & \xrightarrow{g} & \mathbb{A}^2 \\ (x,y) & \mapsto & (x,xy). \end{array}$$

Then it is easy to see that f and g are irreducible elements of $\mathrm{Dom}(\mathbb{A}^2)$, and that $g \circ f = f \circ g \circ g$.

Since we mentioned that $\mathrm{Bir}(\mathbb{A}^2)$ is cancellative, it is perhaps not superfluous to point out that $\mathrm{Dom}(\mathbb{A}^2)$ is not. The reader may verify the following claim:

For each $n \geq 1$*, the monoid* $\mathrm{Dom}(\mathbb{A}^n)$ *is right-cancellative but not left-cancellative.*

Lean factorizations. See the introduction for the definitions of *lean morphism* and *lean factorization*, and for an explanation of why these notions are relevant. Our aim, here, is to study lean factorizations of dominant morphisms $\mathbb{A}^2_{\mathbf{k}} \to \mathbb{A}^n_{\mathbf{k}}$ ($n = 1, 2$). We give two results, Thms 6.5 and 6.7.

6.4. **Remarks.** Assume that $\mathbf{k}$ is algebraically closed.

(1) If a morphism $f : \mathbb{A}^2 \to \mathbb{A}^1$ is not lean, then there exists a nonsingular rational curve $C \subset \mathbb{A}^2$ such that $f(C)$ is a point.

(2) If a dominant morphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ is not lean, then there exists a nonsingular rational curve $C \subset \mathbb{A}^2$ such that $f(C)$ is a point, or a rational curve $C \subset \mathbb{A}^2$ with one place at infinity such that $C \cap f(\mathbb{A}^2)$ is a finite set.

Indeed, consider (2) for instance. Since f is not lean there exists a factorization $\mathbb{A}^2 \xrightarrow{\alpha} \mathbb{A}^2 \to \mathbb{A}^2 \xrightarrow{\beta} \mathbb{A}^2$ of f with $\alpha, \beta \in \mathrm{Bir}(\mathbb{A}^2)$ and $\{\alpha, \beta\} \nsubseteq \mathrm{Aut}(\mathbb{A}^2)$. If $\alpha \notin \mathrm{Aut}(\mathbb{A}^2)$ then $\mathrm{Cont}(\alpha) \neq \varnothing$, and any $C \in \mathrm{Cont}(\alpha)$ is a nonsingular rational curve $C \subset \mathbb{A}^2$ such that $f(C)$ is a point; if $\beta \notin \mathrm{Aut}(\mathbb{A}^2)$ then $\mathrm{Miss}(\beta) \neq \varnothing$, and any $C \in \mathrm{Miss}(\beta)$ is a rational curve $C \subset \mathbb{A}^2$ with one place at infinity such that $C \cap f(\mathbb{A}^2)$ is a finite set.

6.5. **Theorem.** *Every dominant morphism* $\mathbb{A}^2 \to \mathbb{A}^2$ *has a lean factorization.*

Proof. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a dominant morphism. Let us view f as the morphism $f : \operatorname{Spec} A \to \operatorname{Spec} R$ determined by an inclusion of rings $A \hookleftarrow R$, where $A = \mathbf{k}^{[2]}$ and $R = \mathbf{k}^{[2]}$. The pair $R \subseteq A$ determines the set of rings

$$\mathcal{A} = \big\{\, A' \mid A \supseteq A' \supseteq R,\ A' = \mathbf{k}^{[2]},\ \text{and}\ \operatorname{Frac} A' = \operatorname{Frac} A \,\big\}.$$

With notation as in Prop. 5.2, we have $\mathcal{A} \subseteq \mathcal{R}^+(R)$, so (by that result) $\mathcal{A}$ satisfies DCC. Choose a minimal element A' of $\mathcal{A}$ and consider the set of rings

$$\mathcal{R} = \big\{\, R' \mid A' \supseteq R' \supseteq R,\ R' = \mathbf{k}^{[2]},\ \text{and}\ \operatorname{Frac} R' = \operatorname{Frac} R \,\big\}.$$

Then $\mathcal{R} \subseteq \mathcal{R}^-(A')$, so $\mathcal{R}$ satisfies ACC by Prop. 5.2. Choose a maximal element R' of $\mathcal{R}$. Then we have inclusions $A \hookleftarrow A' \hookleftarrow R' \hookleftarrow R$ and the corresponding morphisms

$$\operatorname{Spec} A \xrightarrow{\alpha} \operatorname{Spec} A' \xrightarrow{f'} \operatorname{Spec} R' \xrightarrow{\beta} \operatorname{Spec} R$$

constitute a lean factorization of f. $\square$

We shall now discuss lean factorizations of morphisms $\mathbb{A}^2 \to \mathbb{A}^1$. See Def. 2.4.2 for the notion of very good field generator.

6.6. **Remark.** Because $\operatorname{Bir}(\mathbb{A}^1) = \operatorname{Aut}(\mathbb{A}^1)$, the definitions of lean morphism and lean factorization (for dominant morphisms $\mathbb{A}^2 \to \mathbb{A}^1$) simplify as follows. Let $f : \mathbb{A}^2 \to \mathbb{A}^1$ be a dominant morphism. We say that f is *lean* if for every diagram $\mathbb{A}^2 \xrightarrow{\alpha} \mathbb{A}^2 \xrightarrow{f'} \mathbb{A}^1$ satisfying $f = f' \circ \alpha$, $\alpha \in \operatorname{Bir}(\mathbb{A}^2)$ and $f' \in \operatorname{Mor}(\mathbb{A}^2, \mathbb{A}^1)$, one has $\alpha \in \operatorname{Aut}(\mathbb{A}^2)$. By a *lean factorization* of f, we mean a diagram $\mathbb{A}^2 \xrightarrow{\alpha} \mathbb{A}^2 \xrightarrow{f'} \mathbb{A}^1$ such that $f = f' \circ \alpha$, with $\alpha \in \operatorname{Bir}(\mathbb{A}^2)$ and f' a lean morphism.

6.7. **Theorem.** *Let $A = \mathbf{k}^{[2]}$, let $F \in A \setminus \mathbf{k}$, and consider the morphism $f : \mathbb{A}^2 \to \mathbb{A}^1$ determined by the inclusion homomorphism $\mathbf{k}[F] \hookrightarrow A$. The following are equivalent:*

(a) *f does not have a lean factorization;*
(b) *there exists a very good field generator G of A such that $F \in \mathbf{k}[G]$.*

Proof. For any $H \in K = \operatorname{Frac} A$, define $\mathcal{A}(A, H) = \big\{\, R \mid H \in R \preceq A \,\big\}$ (see 2.4.1 for the notation $\preceq$). Observe that f has a lean factorization if and only if $\mathcal{A}(A, F)$ has a minimal element. Also,

$$\text{(34)} \qquad \textit{for every } H \in K \textit{ satisfying } F \in \mathbf{k}[H], \quad \mathcal{A}(A, H) = \mathcal{A}(A, F).$$

Indeed, consider R satisfying $R \preceq A$. If $F \in R$ then H is integral over the subring $\mathbf{k}[F]$ of R, so $H \in R$ (because $H \in \operatorname{Frac} R$ and $R = \mathbf{k}^{[2]}$ is normal). Conversely, if $H \in R$ then $F \in \mathbf{k}[H] \subseteq R$ implies $F \in R$. So (34) is true.

Suppose that f does not have a lean factorization. It is shown in the proof of L. 3.21 that the set $\big\{\, \mathbf{k}[H] \mid H \in K \,\big\}$ satisfies ACC. So we may consider a maximal element $\mathbf{k}[G]$ of the set $\Sigma_F = \big\{\, \mathbf{k}[H] \mid H \in K \text{ and } F \in \mathbf{k}[H] \,\big\}$. Then $\mathbf{k}[F] \subseteq \mathbf{k}[G]$ and, by (34), $\mathcal{A}(A, G) = \mathcal{A}(A, F)$. Let us show that G is a very good field generator of A. Consider an A' satisfying $G \in A' \preceq A$. Since $A' \in \mathcal{A}(A, G) = \mathcal{A}(A, F)$ and $\mathcal{A}(A, F)$ does not have a minimal element (because f does not have a lean factorization), there exists an infinite strictly descending sequence $A' = A_0 \supset A_1 \supset A_2 \supset \cdots$ with $A_i \in \mathcal{A}(A, F)$ for all i. By Thm 3.3, there exists a good field generator H of $A_0 = A'$ such that

$\bigcap_{i=0}^{\infty} A_i = \mathbf{k}[H]$. Since $\mathcal{A}(A,F) = \mathcal{A}(A,G)$, we have $\mathbf{k}[G] \subseteq \bigcap_{i=0}^{\infty} A_i = \mathbf{k}[H]$, so $\mathbf{k}[G] = \mathbf{k}[H]$ by maximality of $\mathbf{k}[G] \in \Sigma_F$, so G is a good field generator of $A_0 = A'$. This shows that G is a very good field generator of A, so (a) implies (b).

Conversely, suppose that there exists a very good field generator G of A such that $F \in \mathbf{k}[G]$. Consider $R \in \mathcal{A}(A,F)$. We have $\mathcal{A}(A,G) = \mathcal{A}(A,F)$ by (34), so $G \in R \preceq A$ and hence G is a good field generator of R. So there exists H such that $\mathbf{k}[G,H] \subseteq R$ and $\mathbf{k}(G,H) = K$. Then $\mathbf{k}[G,GH]$ belongs to $\mathcal{A}(A,F)$ and is strictly included in R, showing that R is not a minimal element of $\mathcal{A}(A,F)$. So $\mathcal{A}(A,F)$ does not have a minimal element, which implies that f does not have a lean factorization. Hence, (b) implies (a). $\square$

References

[AEH72] S.S. Abhyankar, P. Eakin and W. Heinzer. On the uniqueness of the coefficient ring in a polynomial ring. *J. Algebra*, 23:310–342, 1972.

[AM75] S. Abhyankar and T.T. Moh. Embeddings of the line in the plane. *J. reine angew. Math.*, 276:148–166, 1975.

[CN05] P. Cassou-Noguès. Bad field generators. In *Affine algebraic geometry*, volume 369 of *Contemp. Math.*, pages 77–83. Amer. Math. Soc., Providence, RI, 2005.

[CND14] P. Cassou-Noguès and D. Daigle. Compositions of birational endomorphisms of the affine plane. *Pacific J. Math.*, 272:353–394, 2014.

[CND15a] P. Cassou-Noguès and D. Daigle. Field generators in two variables and birational endomorphisms of $\mathbb{A}^2$. *J. Algebra Appl.*, 14:1540005, 35, 2015.

[CND15b] P. Cassou-Noguès and D. Daigle. Very good and very bad field generators. *Kyoto J. Math.*, 55:187–218, 2015.

[Dai91] D. Daigle. Birational endomorphisms of the affine plane. *J. Math. Kyoto Univ.*, 31(2):329–358, 1991.

[Dai93] D. Daigle. Plane Frobenius sandwiches of degree p. *Proc. Amer. Math. Soc.*, 117:885–889, 1993.

[Dai15] D. Daigle. Generally rational polynomials in two variables. *Osaka J. Math.*, 52:139–159, 2015.

[Gan82] R. Ganong. Plane Frobenius sandwiches. *Proc. Amer. Math. Soc.*, 84:474–478, 1982.

[Gan11] R. Ganong. The pencil of translates of a line in the plane. In *Affine algebraic geometry*, volume 54 of *CRM Proc. Lecture Notes*, pages 57–71. Amer. Math. Soc., Providence, RI, 2011.

[Har77] R. Hartshorne. *Algebraic Geometry*, volume 52 of *GTM*. Springer-Verlag, 1977.

[Kal92] S. Kaliman. Two remarks on polynomials in two variables. *Pacific J. Math.*, 154:285–295, 1992.

[Mat50] Teruhisa Matsusaka. The theorem of Bertini on linear systems in modular fields. *Mem. Coll. Sci. Univ. Kyoto Ser. A. Math.*, 26:51–62, 1950.

[Miy94] Masayoshi Miyanishi. *Algebraic geometry*, volume 136 of *Translations of Mathematical Monographs*. American Mathematical Society, 1994. Translated from the 1990 Japanese original by the author.

[MS80] M. Miyanishi and T. Sugie. Generically rational polynomials. *Osaka J. Math.*, 17:339–362, 1980.

[Rit22] J. F. Ritt. Prime and composite polynomials. *Trans. Amer. Math. Soc.*, 23:51–66, 1922.

[Rus75] K.P. Russell. Field generators in two variables. *J. Math. Kyoto Univ.*, 15:555–571, 1975.

[Rus76] K.P. Russell. Simple birational extensions of two dimensional affine rational domains. *Compositio Math.*, 33:197–208, 1976.

[Rus77] K.P. Russell. Good and bad field generators. *J. Math. Kyoto Univ.*, 17:319–331, 1977.

[Sch00] A. Schinzel. *Polynomials with special regard to reducibility*, volume 77 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2000.

[Suz74] M. Suzuki. Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace C^2. *J. Math. Soc. Japan*, 26:241–257, 1974.

[Zak71] A. Zaks. Dedekind subrings of $k[x_1, \ldots, x_n]$ are rings of polynomials. *Israel J. of Mathematics*, 9:285–289, 1971.

[Zar54] O. Zariski. Interprétations algébro-géométriques du quatorzième problème de Hilbert. *Bull. Sci. Math.*, 78:155–168, 1954.

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