INTRODUCTION TO LOCALLY NILPOTENT DERIVATIONS
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The aim of this document is to introduce students to the basic theory of locally nilpotent derivations on commutative rings.

Conventions:

- All rings and algebras are tacitly assumed to be commutative and associative and to have an identity element 1. For the definitions of ring, subring, homomorphism of rings, and so on, refer to the book [1].
- "Domain" means integral domain. If A is a ring, A^* is the set of units of A. If A is a domain, $\operatorname{Frac} A$ is the field of fractions of A.
- If A is a ring, $A^{[n]}$ = polynomial ring in n variables over A.
- If $\Bbbk$ is a field, $\Bbbk^{(n)} = \operatorname{Frac}\left(\Bbbk^{[n]}\right)$ = field of fractions of $\Bbbk^{[n]}$.

1. DERIVATIONS

1.1. **Definition.** A *derivation* of a ring B is a map $D : B \to B$ satisfying

$$\text{for all } f, g \in B, \qquad D(f + g) = D(f) + D(g) \quad \text{and} \quad D(fg) = D(f)g + fD(g).$$

If $D : B \to B$ is a derivation, we define $\ker D = \{\, x \in B \mid D(x) = 0 \,\}$.

1.2. **Example.** Consider the set of all functions $f : \mathbb{R} \to \mathbb{R}$ which are infinitely differentiable. This set of functions is denoted $C^\infty(\mathbb{R})$. Given $f \in C^\infty(\mathbb{R})$, let f' be the derivative of f, defined in terms of limits as in your analysis courses; then $f' \in C^\infty(\mathbb{R})$. Note that if $f, g \in C^\infty(\mathbb{R})$ then $f + g, fg \in C^\infty(\mathbb{R})$. You can check that $C^\infty(\mathbb{R})$ is a ring, and that the map $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $f \mapsto f'$, is a derivation of that ring.

1.3. **Example.** If B is any ring then the zero map $B \to B$, $x \mapsto 0$, is a derivation (called the zero derivation).

1.4. **Example.** Let A be a ring and $A[t]$ the polynomial ring in one variable t over A. Define a map $D : A[t] \to A[t]$ by declaring that, given $\sum_{i=0}^{n} a_i t^i \in A[t]$ (where $a_i \in A$ for all i),

$$(1) \qquad D\Big(\sum_{i=0}^{n} a_i t^i \Big) = \sum_{i=1}^{n} i a_i t^{i-1}.$$

Verify that the map D is a derivation and show that $A \subseteq \ker D$. We call D "the derivative" and denote it $\frac{d}{dt} : A[t] \to A[t]$.

Remark. Note the differences between 1.2 and 1.4. An element of $C^\infty(\mathbb{R})$ is a function, but an element of $A[t]$ is not a function (a polynomial is a formal sum, not a function). In 1.2, the derivative of f is defined by using limits. In 1.4, we don't use limits to define $\frac{d}{dt}(f)$, we use formula (1). In fact limits do not make sense in the context of 1.4, so we could not use limits even if we wanted to.

1.5. Exercise. Let B be a ring and $D : B \to B$ a derivation. Verify that $\ker(D)$ is a **subring** of B, i.e., prove the following:

- If $x, y \in \ker(D)$ then $x + y \in \ker(D)$ and $xy \in \ker(D)$;
- $1 \in \ker(D)$ and $-1 \in \ker(D)$.

Several authors call $\ker(D)$ the *ring of constants* of D, and denote it B^D.

1.6. Exercise. Show that the only derivation $D : \mathbb{Z} \to \mathbb{Z}$ is the zero derivation. (Hint: the only subring of $\mathbb{Z}$ is $\mathbb{Z}$.)

1.7. Exercise. Let B be a ring and $D : B \to B$ a derivation.

(1) Show that if $f_1, \ldots, f_n \in B$ then $D(f_1 + \cdots + f_n) = D(f_1) + \cdots + D(f_n)$.
(2) Show that if $f, g, h \in B$ then $D(fgh) = ghD(f) + fhD(g) + fgD(h)$. More generally, find a formula for $D(f_1 \cdots f_n)$ where $f_1, \ldots, f_n \in B$ (prove your formula by induction).

1.8. Exercise. Consider the polynomial ring $B = \mathbb{R}[X, Y] = \mathbb{R}^{[2]}$. Suppose that $D : B \to B$ is a derivation satisfying:

$$D(X) = 3, \quad D(Y) = XY \quad \text{and} \quad D(a) = 0 \text{ for all } a \in \mathbb{R}.$$

Let $f = X^2 + Y^2 \in B$. Calculate $D(f)$ and $D\big(D(f)\big)$.

1.9. Exercise. Consider a field $\mathbb{k}$, the polynomial ring $\mathbb{k}[t] = \mathbb{k}^{[1]}$, and the derivative $D = \frac{d}{dt} : \mathbb{k}[t] \to \mathbb{k}[t]$. Show that if $\operatorname{char} \mathbb{k} = 0$ then $\ker D = \mathbb{k}$, and that if $\operatorname{char} \mathbb{k} = p > 0$ then $\ker D = \mathbb{k}[t^p]$.

Given a ring B, define $\operatorname{Der}(B) = $ set of all derivations $D : B \to B$. If $D_1, D_2 \in \operatorname{Der}(B)$ then the map

$$D_1 + D_2 : B \to B, \qquad x \mapsto D_1(x) + D_2(x)$$

is a derivation, i.e., $D_1 + D_2 \in \operatorname{Der}(B)$. If $D \in \operatorname{Der}(B)$ and $b \in B$ then the map

$$bD : B \to B, \qquad x \mapsto bD(x)$$

is a derivation, i.e., $bD \in \operatorname{Der}(B)$. It follows that $\operatorname{Der}(B)$ is a B-module. If $A \subseteq B$ are rings then by an A-*derivation* of B we mean a derivation $D : B \to B$ satisfying $D(a) = 0$ for all $a \in A$. Then the set

$$\operatorname{Der}_A(B) = \text{ set of all } A\text{-derivations } D : B \to B$$

is a B-submodule of $\operatorname{Der}(B)$.

1.10. Exercise. Let A be a ring and $B = A[t] = A^{[1]}$.

(a) Show that if $D_1, D_2 \in \operatorname{Der}_A(B)$ satisfy $D_1(t) = D_2(t)$, then $D_1 = D_2$.
(b) Show that exactly one element $D \in \operatorname{Der}_A(B)$ satisfies $D(t) = 1$. What is this D?

We solve part (a) of this exercise, to demonstrate a useful technique.

Suppose that $D_1, D_2 \in \operatorname{Der}_A(B)$ satisfy $D_1(t) = D_2(t)$. Since $\operatorname{Der}_A(B)$ is a B-module, we have $D_1 - D_2 \in \operatorname{Der}_A(B)$, so $\ker(D_1 - D_2)$ is a subring of B. Since $D_1 - D_2$ is an A-derivation, $A \subseteq \ker(D_1 - D_2)$; since $D_1(t) = D_2(t)$, $t \in \ker(D_1 - D_2)$. The conditions

$$A \subseteq \ker(D_1 - D_2), \quad t \in \ker(D_1 - D_2) \quad \text{and} \quad \ker(D_1 - D_2) \text{ is a subring of } B$$

imply that $A[t] \subseteq \ker(D_1 - D_2)$, so $D_1 - D_2 = 0$, as desired. $\qquad\square$

1.11. Example. Let A be a ring and $B = A[X_1, \ldots, X_n] = A^{[n]}$. Let $j \in \{1, \ldots, n\}$. Define the map $\frac{\partial}{\partial X_j} : B \to B$ by declaring that, given $\sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n} \in B$ (where $a_{i_1, \ldots, i_n} \in A$ for all $i_1, \ldots, i_n$),

$$\frac{\partial}{\partial X_j}\Big(\sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n}\Big) = \sum_{i_1, \ldots, i_n} i_j\, a_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_{j-1}^{i_{j-1}} X_j^{i_j - 1} X_{j+1}^{i_{j+1}} \cdots X_n^{i_n}.$$

You can check that $\frac{\partial}{\partial X_j}$ is an element of $\operatorname{Der}_A(B)$; we call it the partial derivative with respect to X_j. Note that

$$\frac{\partial}{\partial X_j}(X_i) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Since $\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}$ are elements of the B-module $\operatorname{Der}_A(B)$, it follows that $\sum_{i=1}^n f_i \frac{\partial}{\partial X_i} \in \operatorname{Der}_A(B)$ for any choice of $f_1, \ldots, f_n \in B$.

1.12. Lemma. *Let A be a ring and $B = A[X_1, \ldots, X_n] = A^{[n]}$.*

    (1) *Given any $f_1, \ldots, f_n \in B$, there exists a unique $D \in \operatorname{Der}_A(B)$ satisfying $D(X_i) = f_i$ for all $i = 1, \ldots, n$.*

    (2) $\operatorname{Der}_A(B)$ *is a free B-module with basis $\{\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\}$.*

Proof. As $\operatorname{Der}_A(B)$ is a B-module, we may define $D \in \operatorname{Der}_A(B)$ by $D = \sum_{i=1}^n f_i \frac{\partial}{\partial X_i}$. Clearly, $D(X_i) = f_i$ for all $i \in \{1, \ldots, n\}$. If also $D' \in \operatorname{Der}_A(B)$ satisfies $D'(X_i) = f_i$ for all i, then consider $D_0 = D - D' \in \operatorname{Der}_A(B)$; then $D_0(X_i) = 0$ for all i. Thus $A \cup \{X_1, \ldots, X_n\} \subseteq \ker(D_0)$. As $\ker D_0$ is a subring of B, $\ker D_0 = B$; so $D_0 = 0$ and hence $D = D'$. So D is the unique element of $\operatorname{Der}_A(B)$ satisfying $D(X_i) = f_i$ for all $i \in \{1, \ldots, n\}$. This proves (a), and (b) is left to the reader. $\qquad\square$

Remark. Given rings $A \subseteq B$, the B-module $\operatorname{Der}_A(B)$ is not necessarily free. Lemma 1.12 says that it is free when B is a polynomial ring over A.

1.13. Exercise. Consider the polynomial ring $B = \mathbb{R}[X, Y, Z] = \mathbb{R}^{[3]}$. By 1.12, there exists a unique $\mathbb{R}$-derivation $D : B \to B$ satisfying:

$$D(X) = Y, \qquad D(Y) = X \quad \text{and} \quad D(Z) = XY.$$

    (a) Find $f, g, h \in B$ such that $D = f \frac{\partial}{\partial X} + g \frac{\partial}{\partial Y} + h \frac{\partial}{\partial Z}$.

    (b) Calculate $D(X^2 + Y^2 + Z^2)$.

1.14. Exercise. Let B be a ring and $D : B \to B$ a derivation. Verify the following claims.

    (i) $D(b^n) = nb^{n-1}D(b)$, for all $b \in B$ and $n \in \mathbb{N}$.

(ii) Let $f(T) = \sum_{i=0}^{n} a_i T^i \in B[T]$ be a polynomial ($a_i \in B$ and T is an indeterminate). If $b \in B$ then $f(b) \in B$, so it makes sense to evaluate D at $f(b)$. Show that

$$D\big(f(b)\big) = f^{(D)}(b) + f'(b)D(b)$$

where we define the polynomial $f^{(D)}(T) \in B[T]$ by $f^{(D)}(T) = \sum_{i=0}^{n} D(a_i)T^i$, and where $f'(T) \in B[T]$ is the derivative of f, defined by $f'(T) = \sum_{i=1}^{n} i a_i T^{i-1}$. Note that if all a_i belong to $\ker(D)$ then this formula simplifies to $D\big(f(b)\big) = f'(b)D(b)$.

(iii) More generally, show that if $f \in B[T_1, \ldots, T_n]$ and $b_1, \ldots, b_n \in B$ then

$$D\big(f(b_1, \ldots, b_n)\big) = f^{(D)}(b_1, \ldots, b_n) + \sum_{i=1}^{n} f_{T_i}(b_1, \ldots, b_n)\, D(b_i),$$

where $f_{T_i} = \frac{\partial f}{\partial T_i} \in B[T_1, \ldots, T_n]$.

1.15. Exercise. Let B be a ring and $D : B \to B$ a derivation. If $S \subseteq B$ is a multiplicative set, we may consider the ring $S^{-1}B$ obtained by localization (refer to chapter 3 of [1]). Define the map $S^{-1}D : S^{-1}B \to S^{-1}B$ by

$$\big(S^{-1}D\big)(b/s) = \frac{sD(b) - bD(s)}{s^2} \quad \text{for all } b \in B \text{ and } s \in S.$$

(a) Check that the map $S^{-1}D$ is well-defined. You have to show that if $b_1, b_2 \in B$ and $s_1, s_2 \in S$ are such that $\frac{b_1}{s_1} = \frac{b_2}{s_2}$ in $S^{-1}B$, then

$$\frac{s_1 D(b_1) - b_1 D(s_1)}{s_1^2} = \frac{s_2 D(b_2) - b_2 D(s_2)}{s_2^2} \quad \text{in } S^{-1}B.$$

(b) Show that $S^{-1}D$ is a derivation, i.e., $S^{-1}D \in \mathrm{Der}\big(S^{-1}B\big)$.

(c) Remark: we have defined a map $\mathrm{Der}(B) \to \mathrm{Der}(S^{-1}B)$, $D \mapsto S^{-1}D$. We say that every derivation of B can be "extended" to a derivation of $S^{-1}B$.

(d) Remark: in the special case where S is included in $\ker(D)$, the above formula simplifies as follows:

$$\big(S^{-1}D\big)(b/s) = \frac{D(b)}{s} \quad \text{for all } b \in B \text{ and } s \in S.$$

(e) Example: Let $B = \mathbb{C}[X, Y] = \mathbb{C}^{[2]}$ and $S = B \setminus \{0\}$. Then $S^{-1}B = \mathbb{C}(X, Y)$ is the field of "rational functions" in two variables over $\mathbb{C}$. Consider the $\mathbb{C}$-derivation $\frac{\partial}{\partial X} : \mathbb{C}[X, Y] \to \mathbb{C}[X, Y]$ and its extension to a derivation of $\mathbb{C}(X, Y)$,

$$S^{-1}\tfrac{\partial}{\partial X} : \mathbb{C}(X, Y) \to \mathbb{C}(X, Y).$$

Let us simplify the notation, and use the same symbol "$\frac{\partial}{\partial X}$" to represent the derivation $S^{-1}\frac{\partial}{\partial X}$:

$$\tfrac{\partial}{\partial X} : \mathbb{C}(X, Y) \to \mathbb{C}(X, Y).$$

Compute $\frac{\partial}{\partial X}\big(\frac{XY}{X^2+Y^2}\big)$.

1.16. Definition. Let $A \subseteq B$ be rings. An element $b \in B$ is *algebraic* over A if there exists a nonzero polynomial $f \in A[T] \setminus \{0\}$ such that $f(b) = 0$ (note that f is not required to be monic); if b is not algebraic over A, we say that b is *transcendental* over A.

We say that B is *algebraic* over A if every element of B is algebraic over A. We say that A is *algebraically closed in* B if each element of $B \setminus A$ is transcendental over A.

1.17. Lemma. *If B is a domain of characteristic zero and $D \in \mathrm{Der}(B)$ then $\ker D$ is algebraically closed in B.*

Proof. Let $A = \ker D$ and consider $b \in B$ algebraic over A. Let $f \in A[T]$ be a nonzero polynomial of minimal degree such that $f(b) = 0$. Note that $\deg(f) \geq 1$. Then

$$0 = D(f(b)) = f^{(D)}(b) + f'(b)D(b) = f'(b)D(b).$$

We have $f' \neq 0$ (because B is a domain of characteristic zero), so $f'(b) \neq 0$ by minimality of $\deg f$, so $D(b) = 0$. $\qquad\square$

1.18. Exercise. Let B be a ring of characteristic $n > 0$ and let $0 \neq D \in \mathrm{Der}(B)$. Show that each $x \in B$ satisfies $x^n \in \ker D$. Deduce that $\ker D$ is not algebraically closed in B.

If B is a ring and $D_1, D_2 \in \mathrm{Der}(B)$ then the composition $D_1 \circ D_2 : B \to B$ is a map which preserves addition, but is usually not a derivation. However, a straightforward verification shows that $D_1 \circ D_2 - D_2 \circ D_1 : B \to B$ is a derivation. One uses the notation $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ for this derivation.

If $D \in \mathrm{Der}(B)$ and $n > 0$, we denote by $D^n : B \to B$ the composition of D with itself n times (for instance $D^3 = D \circ D \circ D$). We also define $D^0 : B \to B$ to be the identity map (even in the case $D = 0$). Note that D^n is usually not a derivation when $n \neq 1$.

1.19. Exercise. Prove **Leibnitz Rule**: *If B is a ring, $D \in \mathrm{Der}(B)$, $x, y \in B$ and $n \in \mathbb{N}$,*

$$D^n(xy) = \sum_{i=0}^{n} \binom{n}{i} D^{n-i}(x) D^i(y).$$

1.20. Definition. Given a ring B and $D \in \mathrm{Der}(B)$, define the set

$$\mathrm{Nil}(D) = \left\{ x \in B \mid \exists_{n \in \mathbb{N}} \, D^n(x) = 0 \right\}.$$

So $\ker(D) \subseteq \mathrm{Nil}(D) \subseteq B$. By exercise 1.21, $\mathrm{Nil}(D)$ is a subring of B.

1.21. Exercise. Use Leibnitz Rule to show that the subset $\mathrm{Nil}(D)$ of B is closed under multiplication. Deduce that $\mathrm{Nil}(D)$ is a subring of B.

1.22. Example. Let $B = \mathbb{C}[[T]]$ and $D = d/dT : B \to B$. Then $\ker(D) = \mathbb{C}$ and $\mathrm{Nil}(D) = \mathbb{C}[T]$. Note that $\mathrm{Nil}(D)$ is not integrally closed in B: let $b = \sqrt{1+T} \in B$, then $b \notin \mathrm{Nil}(D)$ but $b^2 \in \mathrm{Nil}(D)$.

1.23. Exercise. Let A be a ring and $B = A[X_1, \ldots, X_n] = A^{[n]}$. Given $f = (f_1, \ldots, f_{n-1}) \in B^{n-1}$, define the map $\Delta_f : B \to B$ by $\Delta_f(g) = \det\left(\frac{\partial(f_1, \ldots, f_{n-1}, g)}{\partial(X_1, \ldots, X_n)} \right)$, for each $g \in B$. Check that $\Delta_f \in \mathrm{Der}_A(B)$ and that $A[f_1, \ldots, f_{n-1}] \subseteq \ker(\Delta_f)$. One refers to Δ_f as a "jacobian derivation".

1.24. Exercise. Let $B = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$, $f = X^2 + Y^3$ and $g = XY + YZ + XZ$. Give an example of a derivation $D \in \mathrm{Der}_{\mathbb{C}}(B)$ satisfying $D(f) = 0 = D(g)$ and $D \neq 0$.
Hint: use jacobian derivations.

2. Definition of locally nilpotent derivations

2.1. Definition. Let B be any ring. A derivation $D : B \to B$ is *locally nilpotent* if it satisfies $\mathrm{Nil}(D) = B$, i.e., if $\forall_{b \in B} \exists_{n \in \mathbb{N}} D^n(b) = 0$. We write:

$$\mathrm{LND}(B) = \text{ set of locally nilpotent derivations } B \to B.$$

2.2. Example. Let A be a ring and $B = A[X_1, \ldots, X_n] = A^{[n]}$. Then $\frac{\partial}{\partial X_i} \in \mathrm{LND}(B)$ for each $i = 1, \ldots, n$.

2.3. Exercise. Let $B = \mathbb{C}[t] = \mathbb{C}^{[1]}$. Each polynomial $P(t) \in \mathbb{C}[t]$ determines a $\mathbb{C}$-derivation $D = P(t)\frac{d}{dt} \in \mathrm{Der}_{\mathbb{C}}(B)$. Find all $P(t) \in \mathbb{C}[t]$ which have the property that the corresponding derivation $D = P(t)\frac{d}{dt}$ is locally nilpotent.

2.4. Definition. Let A be a ring and $B = A[X_1, \ldots, X_n] = A^{[n]}$. A derivation $D : B \to B$ is *triangular* if $D(A) = \{0\}$ and:

$$\forall i \quad D(X_i) \in A[X_1, \ldots, X_{i-1}] \quad \text{(in particular } D(X_1) \in A\text{)}.$$

2.5. Example. Let $B = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$ and $D = X^2 \frac{\partial}{\partial Y} + (X^5 + Y^3)\frac{\partial}{\partial Z} \in \mathrm{Der}_{\mathbb{C}}(B)$. Then D is triangular.

2.6. Lemma. *Let A be a ring and $B = A[X_1, \ldots, X_n] = A^{[n]}$. Then every triangular derivation of B is locally nilpotent.*

Proof. Let $D : B \to B$ be a triangular derivation. Let us prove (by induction on i) that

$$(*) \qquad\qquad A[X_1, \ldots, X_i] \subseteq \mathrm{Nil}(D), \quad \text{for all } i = 1, \ldots, n.$$

Before we prove this, observe that if $f \in B$ satisfies $D(f) \in \mathrm{Nil}(D)$, then $f \in \mathrm{Nil}(D)$.

Since D is an A-derivation (by definition of triangular derivation), we have $A \subseteq \ker D$, so in particular $A \subseteq \mathrm{Nil}(D)$.

Since $D(X_1) \in A \subseteq \mathrm{Nil}(D)$ it follows that $X_1 \in \mathrm{Nil}(D)$. Using that $\mathrm{Nil}(D)$ is a ring, we get $A[X_1] \subseteq \mathrm{Nil}(D)$.

Suppose that $i < n$ is such that $A[X_1, \ldots, X_i] \subseteq \mathrm{Nil}(D)$.

Since $D(X_{i+1}) \in A[X_1, \ldots, X_i] \subseteq \mathrm{Nil}(D)$ it follows that $X_{i+1} \in \mathrm{Nil}(D)$. Using that $\mathrm{Nil}(D)$ is a ring, we get $A[X_1, \ldots, X_i, X_{i+1}] \subseteq \mathrm{Nil}(D)$.

By induction, this proves $(*)$. So $\mathrm{Nil}(D) = B$, i.e., D is locally nilpotent. $\qquad\square$

2.7. Example. Let $B = \mathbb{C}[X, Y] = \mathbb{C}^{[2]}$, $D_1 = Y\frac{\partial}{\partial X}$ and $D_2 = X\frac{\partial}{\partial Y}$.

Since D_2 is triangular, it is locally nilpotent. Since $D_1 : \mathbb{C}[Y, X] \to \mathbb{C}[Y, X]$ is triangular, it is locally nilpotent. So

$$D_1, D_2 \in \mathrm{LND}(B).$$

However, $D_1 + D_2 \notin \mathrm{LND}(B)$ (because $(D_1 + D_2)^2(X) = X$), and you can also check that $[D_1, D_2] \notin \mathrm{LND}(B)$.

Also, $\frac{\partial}{\partial X} \in \mathrm{LND}(B)$ but $X\frac{\partial}{\partial X} \notin \mathrm{LND}(B)$.

2.8. We stress that Example 2.7 shows that the subset $\mathrm{LND}(B)$ of $\mathrm{Der}(B)$ is in general not closed under any of the algebraic operations which are defined in $\mathrm{Der}(B)$:

(i) $D_1, D_2 \in \text{LND}(B) \not\Longrightarrow D_1 + D_2 \in \text{LND}(B)$
(ii) $b \in B, D \in \text{LND}(B) \not\Longrightarrow bD \in \text{LND}(B)$
(iii) $D_1, D_2 \in \text{LND}(B) \not\Longrightarrow [D_1, D_2] \in \text{LND}(B)$.

In other words, $\text{LND}(B)$ is just a set. It does not have any algebraic structure.

The facts contained in the following exercise will be needed in the next sections.

2.9. Exercise. Let B be a ring, $D \in \text{LND}(B)$ and $A = \ker D$.

(1) If $a \in A$ then $(aD)^n = a^n D^n$ holds for all $n \in \mathbb{N}$. (Make sure that you use the assumption $D(a) = 0$ in your proof!)

(2) If $a \in A$ then $aD \in \text{LND}(B)$ (compare with 2.8(ii)).

(3) Observe that $D : B \to B$ is (in particular) a homomorphism of A-modules. If $S \subset A$ is a multiplicatively closed set, consider the homomorphism of $S^{-1}A$-modules $S^{-1}D : S^{-1}B \to S^{-1}B$ defined by $(S^{-1}D)(x/s) = (Dx)/s$ ($x \in B$, $s \in S$). Show that $S^{-1}D$ is an element of $\text{LND}(S^{-1}B)$ and $\ker(S^{-1}D) = S^{-1}A$.

2.10. Definition. Given a ring B, we write:

$$\text{KLND}(B) = \{\ker D \mid D \in \text{LND}(B) \text{ and } D \neq 0\}.$$

So $\text{KLND}(B)$ is a set of subrings of B.

2.11. Example. Let $B = \mathbb{C}[X, Y] = \mathbb{C}^{[2]}$.

Since $\frac{\partial}{\partial X} \in \text{LND}(B)$ and $\frac{\partial}{\partial X} \neq 0$, $\ker(\frac{\partial}{\partial X}) \in \text{KLND}(B)$. Since $\ker(\frac{\partial}{\partial X}) = \mathbb{C}[Y]$, we have $\mathbb{C}[Y] \in \text{KLND}(B)$.

Similarly, $\mathbb{C}[X] \in \text{KLND}(B)$. In fact, $\text{KLND}(B)$ has many more elements than $\mathbb{C}[X]$ and $\mathbb{C}[Y]$; one can show that $\text{KLND}(B)$ is an infinite set.

3. The exponential map associated to a locally nilpotent derivation

3.1. Exercise. If B is a $\mathbb{Q}$-algebra then $\text{Der}(B) = \text{Der}_{\mathbb{Q}}(B)$.

3.2. Definition. Let B be a $\mathbb{Q}$-algebra. Given $D \in \text{LND}(B)$, define the map

$$\xi_D : B \longrightarrow B[T], \qquad b \longmapsto \sum_{n \in \mathbb{N}} \frac{1}{n!} D^n(b) \, T^n.$$

We call ξ_D the *exponential map associated to D* (not to be confused with the exponential of D, $\exp(D) : B \to B$, to be defined later). Note that the definition of ξ_D requires us to divide by $n!$, which is (of course) why we need to assume that $\mathbb{Q} \subseteq B$.

The following is a fundamental result in the theory of locally nilpotent derivations. It has several deep consequences.

3.3. Theorem. *Let B be a $\mathbb{Q}$-algebra and $D \in \text{LND}(B)$. Then the exponential map $\xi_D : B \to B[T]$ is an injective homomorphism of A-algebras, where $A = \ker(D)$.*

Proof. If $e_0 : B[T] \to B$ is the map $f(T) \mapsto f(0)$, then the composite $B \xrightarrow{\xi_D} B[T] \xrightarrow{e_0} B$ is the identity map, so ξ_D is injective. It is clear that ξ_D preserves addition and restricts to the identity map on A, so it suffices to verify that

$$(2) \qquad \left(\sum_{i \in \mathbb{N}} \frac{1}{i!} D^i(x) \, T^i \right) \left(\sum_{j \in \mathbb{N}} \frac{1}{j!} D^j(y) \, T^j \right) = \sum_{n \in \mathbb{N}} \frac{1}{n!} D^n(xy) \, T^n$$

holds for all $x, y \in B$. In the left hand side of (2), the coefficient of T^n is

$$\sum_{i+j=n} \frac{1}{i! \, j!} D^i(x) D^j(y) = \frac{1}{n!} \sum_{i+j=n} \frac{n!}{i! \, j!} D^i(x) D^j(y),$$

which is equal to $\frac{1}{n!} D^n(xy)$ by Leibnitz Rule. $\qquad \square$

3.4. Exercise. Let B be a $\mathbb{Q}$-algebra and $D \in \mathrm{LND}(B)$. Check that ξ_D is the inclusion map $B \hookrightarrow B[T]$ if and only if $D = 0$.

3.5. Exercise. Let $B = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$ and $D = X \frac{\partial}{\partial Y} + Y \frac{\partial}{\partial Z}$. Then $D \in \mathrm{LND}(B)$. Note that $\xi_D : B \to B[T]$ is a homomorphism of $\mathbb{C}$-algebras:

$$\xi_D : \mathbb{C}[X, Y, Z] \to \mathbb{C}[X, Y, Z, T].$$

Describe ξ_D by giving the images of X, Y, Z.

4. Degree functions

4.1. Definition. A *degree function* on a ring B is a map $\deg : B \to \mathbb{N} \cup \{-\infty\}$ satisfying:

(1) $\forall x \in B \quad \deg x = -\infty \iff x = 0$

(2) $\forall x, y \in B \quad \deg(xy) = \deg x + \deg y$

(3) $\forall x, y \in B \quad \deg(x + y) \leq \max(\deg x, \deg y)$.

4.2. Exercise. Use parts (1) and (2) of 4.1 to show that if a ring B admits a degree function $\deg : B \to \mathbb{N} \cup \{-\infty\}$, then either $B = 0$ or B is a domain.

4.3. Exercise. Let $A[t]$ be the polynomial ring in one variable over a ring A, and let $\deg_t : A[t] \to \mathbb{N} \cup \{-\infty\}$ be the usual degree of polynomials. Show that $\deg$ is a degree function if and only if A is an integral domain or the zero ring.

4.4. Exercise. Let $\deg : B \to \mathbb{N} \cup \{-\infty\}$ be a degree function and $x, y \in B$ such that $\deg(x) \neq \deg(y)$. Show that $\deg(x + y) = \max(\deg x, \deg y)$.

4.5. Exercise. If $B \xrightarrow{\varphi} B'$ is an injective ring homomorphism and $B' \xrightarrow{d} \mathbb{N} \cup \{-\infty\}$ is a degree function then $B \xrightarrow{d \circ \varphi} \mathbb{N} \cup \{-\infty\}$ is a degree function.

4.6. Definition. Let B be a ring. Then each $D \in \mathrm{LND}(B)$ determines a map

$$\deg_D : B \to \mathbb{N} \cup \{-\infty\}$$

defined as follows: $\deg_D(x) = \max \{ n \in \mathbb{N} \mid D^n x \neq 0 \}$ for $x \in B \setminus \{0\}$, and $\deg_D(0) = -\infty$. Note that $\ker D = \{ x \in B \mid \deg_D(x) \leq 0 \}$.
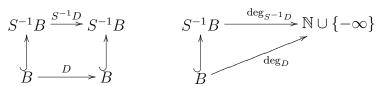
4.7. Example. Let A be a domain of characteristic zero and $B = A[t] = A^{[1]}$. Let $D = \frac{d}{dt} : B \to B$, then $D \in \text{LND}(B)$, so D determines the map $\deg_D : A[t] \to \mathbb{N} \cup \{-\infty\}$. It is easy to see that $\deg_D$ is the usual degree of polynomials.

Although we defined $\deg_D$ for any ring B, it is useful mostly in the case of integral domains of characteristic zero:

4.8. Proposition. *Let B be a domain of characteristic zero and $D \in \text{LND}(B)$. Then the map $\deg_D : B \to \mathbb{N} \cup \{-\infty\}$ is a degree function.*

Proof. We first prove the special case where $\mathbb{Q} \subseteq B$. In this case we may consider the map $\xi_D : B \to B[T]$, $\xi_D(b) = \sum_{i=0}^{\infty} \frac{D^n(b)}{n!} T^n$, which is an injective ring homomorphism by 3.3. As B is a domain, $B[T] \xrightarrow{\deg_T} \mathbb{N} \cup \{-\infty\}$ is a degree function and consequently the composite $B \xrightarrow{\xi_D} B[T] \xrightarrow{\deg_T} \mathbb{N} \cup \{-\infty\}$ is a degree function. As this composite map is equal to $\deg_D$, $\deg_D$ is a degree function.

Now the general case. Since B has characteristic zero and $\ker D$ is a subring of B, we have $\mathbb{Z} \subseteq \ker D$. Let $S = \mathbb{Z} \setminus \{0\}$ and consider $S^{-1}D : S^{-1}B \to S^{-1}B$, which belongs to $\text{LND}(S^{-1}B)$ by Exercise 2.9. As $\mathbb{Q} \subseteq S^{-1}B$, the first part of the proof implies that $\deg_{S^{-1}D} : S^{-1}B \to \mathbb{N} \cup \{-\infty\}$ is a degree function. We have:

$$
\begin{array}{ccc}
S^{-1}B & \xrightarrow{S^{-1}D} & S^{-1}B \\
\uparrow & & \uparrow \\
B & \xrightarrow{\ D\ } & B
\end{array}
\qquad\qquad
\begin{array}{c}
S^{-1}B \xrightarrow{\ \deg_{S^{-1}D}\ } \mathbb{N} \cup \{-\infty\} \\
\uparrow \qquad \nearrow {\scriptstyle \deg_D} \\
B
\end{array}
$$

Note that $B \to S^{-1}B$ is injective, because B is a domain and $0 \notin S$. So D is the restriction of $S^{-1}D$ and consequently $\deg_D$ is the restriction of $\deg_{S^{-1}D}$; it follows that $\deg_D$ is a degree function. $\qquad\square$

4.9. Exercise. Let B be a domain of characteristic zero and suppose that $D \in \text{Der}(B)$ satisfies $D^n = 0$ for some $n > 0$. Show that $D = 0$. (*Hint.* Note that D is locally nilpotent, so the map $\deg_D$ exists and is a degree function. If $D \neq 0$ then we can choose $x \in B$ such that $\deg_D(x) \geq 1$; what is $\deg_D(x^n)$? can $D^n(x^n)$ be zero?)

5. FACTORIALLY CLOSED SUBRINGS

5.1. Definition. Let $A \subseteq B$ be domains. We say that A is *factorially closed* in B if:
$$\forall\, x, y \in B \setminus \{0\} \quad xy \in A \implies x, y \in A.$$

For instance, consider the polynomial ring $R[T]$ in one variable over an integral domain R. Then R is factorially closed in $R[T]$. Note that this example is a special case of:

5.2. Lemma. *If B is a domain and $\deg : B \to \mathbb{N} \cup \{-\infty\}$ is a degree function then $\{x \in B \mid \deg x \leq 0\}$ is a factorially closed subring of B.*

Proof. Obvious. $\qquad\square$

5.3. Corollary. *Let B be a domain of characteristic zero and $D \in \mathrm{LND}(B)$. Then $\ker(D)$ is a factorially closed subring of B.*

Proof. $\{x \in B \mid \deg_D(x) \leq 0\}$ is factorially closed in B by 4.8 and 5.2. As $\ker D = \{x \in B \mid \deg_D(x) \leq 0\}$, we are done. $\qquad\square$

Recall the following definitions. Let R be an integral domain and let $p \in R$. We say that p is *irreducible* if $p \notin R^* \cup \{0\}$ and if the condition $p = xy$ (where $x, y \in R$) implies that $\{x, y\} \cap R^* \neq \varnothing$. We say that p is *prime* if $p \notin R^* \cup \{0\}$ and if the condition $p \mid xy$ (where $x, y \in R$) implies that p divides one of x, y (i.e., p is prime if and only if the principal ideal pR is a nonzero prime ideal of R). Recall that every prime element is irreducible but that the converse is not necessarily true.

A *unique factorization domain* (UFD) is a domain R such that each element of $R \setminus (R^* \cup \{0\})$ is a finite product of prime elements.

If R is a UFD then an element of R is irreducible if and only if it is prime.

5.4. Exercise. Suppose that A is a factorially closed subring of a domain B. Then:

    (1) $A^* = B^*$.

    (2) An element of A is irreducible in A iff it is irreducible in B.

    (3) If B is a UFD then so is A.

From 5.3 and 5.4, we get:

5.5. Corollary. *Let B be a domain of characteristic zero, $D \in \mathrm{LND}(B)$ and $A = \ker(D)$.*

    (1) *$A^* = B^*$*

    (2) *If $\Bbbk$ is any field included in B, then D is a $\Bbbk$-derivation.*

    (3) *If B is a UFD then so is A.*

5.6. Exercise. Suppose that $\mathbb{C} \subseteq B$ where B is a domain. Show that each element A of $\mathrm{KLND}(B)$ satisfies $\mathbb{C} \subseteq A$.

5.7. Exercise. For domains $A \subseteq B$, the following implications are valid:

A is factorially closed in $B \implies A$ is algebraically closed in B

$$\implies A \text{ is integrally closed in } B.$$

5.8. Exercise. Let $B = \mathbb{C}[X, Y] = \mathbb{C}^{[2]}$ and $f = XY \in B$. Show that if A is a factorially closed subring of B satisfying $f \in A$, then $A = B$. Deduce the following assertions:

    (1) The only $D \in \mathrm{LND}(B)$ satisfying $D(f) = 0$ is the zero derivation.

    (2) The jacobian derivation $\Delta_f \in \mathrm{Der}_{\mathbb{C}}(B)$ (refer to 1.23) is not locally nilpotent.

    (3) The ring $\ker(\Delta_f)$ is algebraically closed in B but not factorially closed.

6. TRANSCENDENCE DEGREE

6.1. Given a field extension L/K, we will write $\mathrm{trdeg}_K(L)$ for the *transcendence degree* of L over K (read the definition of that concept in some algebra textbook). Note that transcendence degree has the following properties:

(1) Let $K \subseteq L$ be fields. Then $\operatorname{trdeg}_K(L) = 0$ if and only if L is algebraic over K.

(2) Let K be a field, $t_1, \ldots, t_n$ indeterminates over K and $L = K(t_1, \ldots, t_n) = K^{(n)}$. Then $\operatorname{trdeg}_K(L) = n$.

(3) Let $K \subseteq L \subseteq M$ be fields. Then $\operatorname{trdeg}_K(M) < \infty$ if and only if both $\operatorname{trdeg}_K(L)$ and $\operatorname{trdeg}_L(M)$ are finite. Moreover, if $\operatorname{trdeg}_K(M) < \infty$ then

$$\operatorname{trdeg}_K(M) = \operatorname{trdeg}_K(L) + \operatorname{trdeg}_L(M).$$

6.2. **Notation.** Given domains $A \subseteq B$, we define $\operatorname{trdeg}_A(B)$ to be equal to the transcendence degree of $\operatorname{Frac} B$ over $\operatorname{Frac} A$.

6.3. **Exercise.** Use 6.1 to prove the following properties of transcendence degree of integral domains.

(1) Let $A \subseteq B$ be domains. Then $\operatorname{trdeg}_A(B) = 0$ if and only if B is algebraic over A.

(2) Let A be a domain and let $B = A^{[n]}$. Then $\operatorname{trdeg}_A(B) = n$.

(3) Let $A \subseteq B \subseteq C$ be domains. Then $\operatorname{trdeg}_A(C) < \infty$ if and only if both $\operatorname{trdeg}_A(B)$ and $\operatorname{trdeg}_B(C)$ are finite. Moreover, if $\operatorname{trdeg}_A(C) < \infty$ then

$$\operatorname{trdeg}_A(C) = \operatorname{trdeg}_A(B) + \operatorname{trdeg}_B(C).$$

6.4. **Exercise.** Let $A \subseteq B$ be domains, where $\operatorname{trdeg}_A(B) < \infty$ and A is algebraically closed in B. Suppose that A' is a ring such that $A \subseteq A' \subseteq B$ and $\operatorname{trdeg}_A(B) = \operatorname{trdeg}_{A'}(B)$. Show that $A = A'$.

7. Slices and preslices

7.1. **Definition.** Let B be a ring and $D \in \operatorname{LND}(B)$. A *slice* of D is an element $s \in B$ satisfying $D(s) = 1$.

7.2. **Examples.** Let $B = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$.

(1) X is a slice of $\frac{\partial}{\partial X} \in \operatorname{LND}(B)$.

(2) Define $D \in \operatorname{LND}(B)$ by $DZ = Y$, $DY = X$, $DX = 0$. Then given $f \in B$,

$$D(f) = f_X D(X) + f_Y D(Y) + f_Z D(Z) = f_Y X + f_Z Y,$$

thus $D(B) \subseteq (X, Y)B$, so D does not have a slice.

When a slice exists, the situation is very special:

7.3. **Theorem** ([3, Prop. 2.1]). *Let B be a $\mathbb{Q}$-algebra, $D \in \operatorname{LND}(B)$ and $A = \ker(D)$. If $s \in B$ satisfies $Ds = 1$ then $B = A[s] = A^{[1]}$ and $D = \frac{d}{ds} : A[s] \to A[s]$.*

Proof. Consider $f(T) = \sum_{i=0}^n a_i T^i \in A[T] \setminus \{0\}$ (where $n \geq 0$, $a_i \in A$ and $a_n \neq 0$). Then $D^j(f(s)) = f^{(j)}(s)$ for all $j \geq 0$, where $f^{(j)}(T) \in A[T]$ denotes the j-th derivative of f; so $D^n(f(s)) = n!\, a_n \neq 0$ and in particular $f(s) \neq 0$. So s is transcendental over A, i.e., $A[s] = A^{[1]}$.

To show that $B = A[s]$, consider the homomorphism of A-algebra $\xi : B \to B$ obtained by composing the homomorphism $\xi_D : B \to B[T]$ of 3.3 with the evaluation map $B[T] \to B$, $f(T) \mapsto f(-s)$. Explicitely, if $x \in B$ then $\xi(x) = \sum_{j=0}^{\infty} \frac{D^j x}{j!} (-s)^j$. For each $x \in B$,

$$D\big(\xi(x)\big) = \sum_{j=0}^{\infty} \frac{D^{j+1}x}{j!} (-s)^j + \sum_{j=0}^{\infty} \frac{D^j x}{j!} j(-s)^{j-1}(-1) = 0,$$

so $\xi(B) \subseteq A$; since ξ is a A-homomorphism, $\xi(B) = A$.

By induction on $\deg_D(x)$, we show that $\forall_{x \in B} \; x \in A[s]$. This is clear if $\deg_D(x) \leq 0$, so assume that $\deg_D(x) \geq 1$. Since $x = \xi(x) + (x - \xi(x))$ where $\xi(x) \in A$ and $x - \xi(x) \in sB$,

$$(3) \qquad\qquad x = a + x's, \quad \text{for some } a \in A \text{ and } x' \in B.$$

This implies that $Dx = D(x')s + x'$ and it easily follows by induction that

$$(4) \qquad\qquad \forall_{m \geq 1} \; D^m(x) = D^m(x')s + mD^{m-1}(x').$$

Choose $m \geq 1$ such that $D^{m-1}(x') \neq 0$ and $D^m(x') = 0$ (such an m exists because $\deg_D(x) \geq 1$, so $x \notin A$, so $x' \neq 0$). Then (4) gives $D^m(x) = mD^{m-1}(x') \neq 0$ and $D^{m+1}(x) = 0$, so $\deg_D(x') = \deg_D(x) - 1$. By the inductive hypothesis we have $x' \in A[s]$; then (3) gives $x \in A[s]$. So $B = A[s] = A^{[1]}$. $\qquad\square$

7.4. Corollary. *Let B be a $\mathbb{Q}$-algebra, $D \in \mathrm{LND}(B)$ and $A = \ker(D)$. If $s \in B$ satisfies $Ds \in A^*$ then $B = A[s] = A^{[1]}$.*

Proof. Let $a \in A$ be such that $aD(s) = 1$. Then as is a slice of D, so $B = A[as] = A[s]$, and s is transcendental over A (since as is). $\qquad\square$

7.5. Corollary. *Let B be a domain of characteristic zero and suppose that $A \in \mathrm{KLND}(B)$. Then $S^{-1}B = (\mathrm{Frac}\,A)^{[1]}$, where $S = A \setminus \{0\}$. In particular, $\mathrm{trdeg}_A(B) = 1$.*

Proof. Let $A \in \mathrm{KLND}(B)$. Choose $D \in \mathrm{LND}(B)$ such that $\ker D = A$ (then $D \neq 0$). If we write $S = A \setminus \{0\}$ and $K = \mathrm{Frac}(A)$ then exercise 2.9 gives $S^{-1}D \in \mathrm{LND}(S^{-1}B)$ and $\ker(S^{-1}D) = K$. Note that $S^{-1}D$ has a slice (indeed, choose a preslice $s \in B$ of D and let $a = Ds$, then $a \in S$, so $s/a \in S^{-1}B$, and it is clear that $S^{-1}D(s/a) = 1$). So 7.3 implies that $S^{-1}B = K^{[1]}$, which proves the assertion. $\qquad\square$

7.6. Exercise. Let B be a domain of characteristic zero. Show that if $A, A' \in \mathrm{KLND}(B)$ satisfy $A \subseteq A'$, then $A = A'$.

7.7. Definition. Let B be a domain of characteristic zero. We say that B is *rigid* if it satisfies the following equivalent conditions: (i) $\mathrm{LND}(B) = \{0\}$; (ii) $\mathrm{KLND}(B) = \varnothing$.

7.8. Exercise. Let B be a domain of characteristic zero which has transcendence degree 1 over some field $\mathbb{k}_0 \subseteq B$. Show that if B is not rigid then $B = \mathbb{k}^{[1]}$ for some field $\mathbb{k}$ such that $\mathbb{k}_0 \subseteq \mathbb{k} \subseteq B$. (Hint: let $D \in \mathrm{LND}(B)$, $D \neq 0$, define $\mathbb{k} = \ker D$ and consider $\mathbb{k}_0 \subseteq \mathbb{k} \subseteq B$. Show that $\mathbb{k}$ is integral over $\mathbb{k}_0$ and hence must be a field. Show that there exists $b \in B$ such that $D(b) \in \mathbb{k}^*$ and conclude that $B = \mathbb{k}^{[1]}$.)

7.9. Exercise. Consider the subring $B = \mathbb{C}[T^2, T^3]$ of $\mathbb{C}[T] = \mathbb{C}^{[1]}$. Show that B is rigid.

7.10. Exercise. Let $B = \mathbb{Z}[X, Y] = \mathbb{Z}^{[2]}$ and $D = \frac{\partial}{\partial Y} + Y\frac{\partial}{\partial X}$. Since D is triangular, we have $D \in \text{LND}(B)$. Moreover, $DY = 1$. Show that $\ker D = \mathbb{Z}[2X - Y^2]$ and that B is not a polynomial ring over $\ker D$. (So in 7.3 the hypothesis that B is a $\mathbb{Q}$-algebra is not superfluous.)

7.11. Definition. Let B be a ring and $D \in \text{LND}(B)$. A *preslice* of D is an element $s \in B$ satisfying $D(s) \neq 0$ and $D^2(s) = 0$ (i.e., $\deg_D(s) = 1$).

Remark. It is clear that if $D \in \text{LND}(B)$ and $D \neq 0$ then D has a preslice.

Preslices are important because they always exist, and because they have the following nice property:

7.12. Corollary. *Let B be a $\mathbb{Q}$-algebra, $D \in \text{LND}(B)$ and $A = \ker(D)$. If $s \in B$ satisfies $Ds \neq 0$ and $D^2 s = 0$, then $B_\alpha = A_\alpha[s] = (A_\alpha)^{[1]}$ where $\alpha = Ds \in A \setminus \{0\}$.*
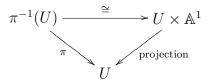
Proof. Let $S = \{1, \alpha, \alpha^2, \dots\}$ and consider $S^{-1}D : S^{-1}B \to S^{-1}B$. By exercise 2.9, $S^{-1}D \in \text{LND}(S^{-1}B)$ and $\ker(S^{-1}D) = S^{-1}A$. As $(S^{-1}D)(s) = \alpha \in (S^{-1}B)^*$, the result follows from 7.4. $\qquad\square$

<div align="center">GEOMETRIC INTERPRETATION OF 7.12</div>

Given $A \in \text{KLND}(B)$, the inclusion map $A \hookrightarrow B$ is a ring homomorphism and hence determines a morphism of schemes $\pi : \text{Spec}(B) \to \text{Spec}(A)$. It is natural to ask what are the properties of this morphism. Result 7.12 implies that the general fiber of π is an affine line. More precisely:

7.13. Corollary. *Let B be a domain containing $\mathbb{Q}$ and let $A \in \text{KLND}(B)$. Consider the map $\pi : \text{Spec}\, B \to \text{Spec}\, A$ determined by $A \hookrightarrow B$.*

Then there exists a dense open set $U \subseteq \text{Spec}\, A$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\ \cong\ } & U \times \mathbb{A}^1 \\
& \searrow{\scriptstyle \pi} \quad \swarrow{\scriptstyle \text{projection}} & \\
& U &
\end{array}
$$

In particular, for each $P \in U$, the fiber $\pi^{-1}(P)$ is $\mathbb{A}^1_{\kappa(P)}$, where $\kappa(P)$ is the residue field of $\text{Spec}\, A$ at the point P.

Proof. Choose $D \in \text{LND}(B)$ such that $\ker D = A$. Since $D \neq 0$, there exists a preslice $s \in B$ of D. Let $\alpha = D(s) \in A \setminus \{0\}$ and define
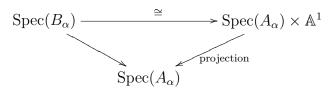
$$U = \text{Spec}\, A \setminus V(\alpha).$$

As A is a domain and $\alpha \neq 0$, U is dense in $\text{Spec}(A)$. We have:

$$
\begin{array}{ccc}
B & \longrightarrow & B_\alpha \\
\uparrow & & \uparrow \\
A & \longrightarrow & A_\alpha
\end{array}
\qquad \Longleftrightarrow \qquad
\begin{array}{ccccc}
\text{Spec}\, B & \hookleftarrow & \pi^{-1}(U) & \xleftarrow{\cong} & \text{Spec}\, B_\alpha \\
\pi \downarrow & & \pi \downarrow & & \downarrow \\
\text{Spec}\, A & \hookleftarrow & U & \xleftarrow{\cong} & \text{Spec}\, A_\alpha
\end{array}
$$

As $B_\alpha = (A_\alpha)^{[1]}$ by 7.12, we also have

$$\text{Spec}(B_\alpha) \xrightarrow{\cong} \text{Spec}(A_\alpha) \times \mathbb{A}^1$$

$$\searrow \qquad \swarrow \text{ projection}$$

$$\text{Spec}(A_\alpha)$$

so we obtain the desired conclusion. $\hfill\square$

7.14. Example. Let $B = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$ and $D \in \text{Der}_\mathbb{C}(B)$ defined by $D(X) = 0$, $D(Y) = X$ and $D(Z) = -2Y$. Then D is triangular, so $D \in \text{LND}(B)$. Let $A = \ker(D)$.

Observe that $D(B)$ is included in the ideal (X, Y) of B, so in particular D does not have a slice. However Y is a preslice of D, since $D(Y) = X \neq 0$ and $D^2(Y) = 0$. Then, according to 7.12, we have $B_X = (A_X)[Y] = (A_X)^{[1]}$. We would like to study the morphism $\pi : \text{Spec}\, B \to \text{Spec}\, A$ determined by $A \hookrightarrow B$, but for that we need to know exactly what A is. We claim:
$$A = \mathbb{C}[X, XZ + Y^2]$$
but we omit the proof ("$\supseteq$" is easy, "$\subseteq$" is more difficult). Observe that $A \cong \mathbb{C}^{[2]}$, since it is a $\mathbb{C}$-algebra generated by two elements, and these two generators are algebraically independent over $\mathbb{C}$. So we have $\text{Spec}(B) = \mathbb{A}^3$, $\text{Spec}(A) = \mathbb{A}^2$ and
$$\pi : \mathbb{A}^3 \to \mathbb{A}^2, \qquad \pi(x, y, z) = (x, xz + y^2).$$
Now it is easy to calculate $\pi^{-1}(a, b)$ for any $(a, b) \in \mathbb{C}^2$, and we find:

$$\pi^{-1}(a, b) = \begin{cases} \text{an affine line,} & \text{if } a \neq 0 \\ \text{a union of two affine lines,} & \text{if } a = 0 \text{ and } b \neq 0 \\ \text{a nonreduced scheme,} & \text{if } (a, b) = (0, 0). \end{cases}$$

Also observe that the open subset $U \subseteq \text{Spec}\, A$ of 7.13 is the set $\{X \neq 0\}$ in $\mathbb{A}^2$ (see how U is obtained in the proof of 7.13).

As a final remark, note that $B \neq A^{[1]}$. Indeed, if B were a polynomial ring in one variable over A then *every* fiber of π would be an affine line, which is not the case. (In particular, note the following obvious but important remark: $A \subset B$, $A = \Bbbk^{[2]}$, $B = \Bbbk^{[3]} \not\Longrightarrow B = A^{[1]}$.)

8. Locally nilpotent derivations and automorphisms

If B is not an integral domain, it may happen that a nonzero polynomial $f(T) \in B[T]$ have infinitely many roots in B. However note the following fact, which is needed in the proof of 8.3, below:

8.1. Lemma. *Let B be a ring and $f(T) \in B[T]$, where T is an indeterminate. If there exists a field $K \subseteq B$ which contains infinitely many roots of $f(T)$, then $f(T) = 0$.*

Proof. By induction on $\deg_T(f)$. The result is trivial if $\deg_T(f) \leq 0$, so assume that $\deg_T(f) > 0$. Pick $a \in K$ such that $f(a) = 0$; since $T - a \in B[T]$ is a monic polynomial, $f(T) = (T - a)g(T)$ for some $g(T) \in B[T]$ such that $\deg_T(g) < \deg_T(f)$. If $b \in K \setminus \{a\}$

is such that $f(b) = 0$, then $(b - a)g(b) = 0$ and $b - a \in B^*$, so $g(b) = 0$. So $g(b) = 0$ holds for infinitely many $b \in K$ and, by the inductive hypothesis, $g(T) = 0$. It follows that $f(T) = 0$. $\qquad\square$

We have seen that the subset $\mathrm{LND}(B)$ of $\mathrm{Der}(B)$ is usually not closed under addition. However:

8.2. Lemma. *Let B be a ring. If $D_1, D_2 \in \mathrm{LND}(B)$ satisfy $D_2 \circ D_1 = D_1 \circ D_2$, then $D_1 + D_2 \in \mathrm{LND}(B)$.*

Proof. Let $D_1, D_2 \in \mathrm{LND}(B)$ such that $D_2 \circ D_1 = D_1 \circ D_2$ and let $b \in B$. Choose $m, n \in \mathbb{N}$ such that $D_1^m(b) = 0 = D_2^n(b)$. The hypothesis $D_2 \circ D_1 = D_1 \circ D_2$ has the following three consequences:

$$\forall_{i \in \mathbb{N}} \ \forall_{j \geq n} \ (D_1^i \circ D_2^j)(b) = D_1^i(0) = 0,$$

$$\forall_{i \geq m} \ \forall_{j \in \mathbb{N}} \ (D_1^i \circ D_2^j)(b) = (D_2^j \circ D_1^i)(b) = D_2^j(0) = 0,$$

$$(D_1 + D_2)^{m+n-1} = \sum_{i+j=m+n-1} \binom{m+n-1}{i} D_1^i \circ D_2^j,$$

so $(D_1 + D_2)^{m+n-1}(b) = 0$. Hence, $D_1 + D_2 \in \mathrm{LND}(B)$. $\qquad\square$

If $\theta : B \to B$ is an automorphism of a ring B, then the set $B^\theta = \{b \in B \mid \theta(b) = b\}$ is a subring of B called the *fixed ring* of θ. The following is another consequence of 3.3.

8.3. Proposition. *Let B be a $\mathbb{Q}$-algebra. Given $D \in \mathrm{LND}(B)$, define the map*

$$\exp(D) : B \to B, \qquad b \longmapsto \sum_{n \in \mathbb{N}} \frac{D^n(b)}{n!}.$$

   (a) *$\exp(D)$ is an automorphism of the $\mathbb{Q}$-algebra B*
   (b) *the fixed ring $B^{\exp(D)} = \{b \in B \mid \exp(D)(b) = b\}$ is equal to $\ker(D)$*
   (c) *if $D_1, D_2 \in \mathrm{LND}(B)$ are such that $D_2 \circ D_1 = D_1 \circ D_2$, then $D_1 + D_2 \in \mathrm{LND}(B)$ and*

$$\exp(D_1 + D_2) = \exp(D_1) \circ \exp(D_2) = \exp(D_2) \circ \exp(D_1)$$

Proof. If $D \in \mathrm{LND}(B)$ then $\exp(D)$ is equal to the composite map $B \xrightarrow{\xi_D} B[T] \xrightarrow{e_1} B$, where ξ_D is defined in 3.2 and where e_1 is the evaluation homomorphism at $T = 1$, i.e., $e_1(f) = f(1)$. Since ξ_D is a ring homomorphism by 3.3, $\exp(D)$ is a ring homomorphism. As any ring homomorphism $B \to B$ is in fact a $\mathbb{Q}$-homomorphism, it follows that $\exp(D)$ is a homomorphism of $\mathbb{Q}$-algebras. Before proving that $\exp(D)$ is bijective, we prove assertion (c).

Consider $D_1, D_2 \in \mathrm{LND}(B)$ such that $D_2 \circ D_1 = D_1 \circ D_2$. By 8.2, $D_1 + D_2 \in \mathrm{LND}(B)$ so it makes sense to consider the ring homomorphism $\exp(D_1 + D_2) : B \to B$. As an abbreviation, we write $\epsilon_i = \exp(D_i)$ for $i = 1, 2$. If $b \in B$,

$$(\epsilon_1 \circ \epsilon_2)(b) = \epsilon_1 \left(\sum_{j \in \mathbb{N}} \frac{D_2^j(b)}{j!} \right) = \sum_{j \in \mathbb{N}} \frac{1}{j!} \epsilon_1 \left(D_2^j(b) \right) = \sum_{j \in \mathbb{N}} \frac{1}{j!} \left(\sum_{i \in \mathbb{N}} \frac{D_1^i \left(D_2^j(b) \right)}{i!} \right)$$

$$= \sum_{i,j \in \mathbb{N}} \frac{(D_1^i \circ D_2^j)(b)}{i! j!} = \sum_{n \in \mathbb{N}} \frac{1}{n!} \sum_{i+j=n} \binom{n}{i} (D_1^i \circ D_2^j)(b).$$

Since $D_2 \circ D_1 = D_1 \circ D_2$, we have $(D_1 + D_2)^n = \sum_{i+j=n} \binom{n}{i} D_1^i \circ D_2^j$ for each $n \in \mathbb{N}$ and consequently

$$(\epsilon_1 \circ \epsilon_2)(b) = \sum_{n \in \mathbb{N}} \frac{1}{n!}(D_1 + D_2)^n(b) = \exp(D_1 + D_2)(b).$$

So $\exp(D_1) \circ \exp(D_2) = \exp(D_1 + D_2)$, and since $D_1 + D_2 = D_2 + D_1$ it follows that $\exp(D_1) \circ \exp(D_2) = \exp(D_2) \circ \exp(D_1)$, so assertion (c) is proved.

Consider $D \in \mathrm{LND}(B)$. Since $(-D) \circ D = D \circ (-D)$, part (c) implies $\exp(D) \circ \exp(-D) = \exp(-D) \circ \exp(D) = \exp(0) = \mathrm{id}_B$, so $\exp(D)$ is bijective and the proof of (a) is complete. It is clear that $\ker(D) \subseteq B^{\exp(D)}$. To prove the reverse inclusion, consider $b \in B$ such that $\exp(D)(b) = b$. Then for every integer $n > 0$ we have

$$b = (\exp D)^n(b) = \exp(nD)(b) = \sum_{j=0}^{\infty} \frac{1}{j!}(nD)^j(b) = \sum_{j=0}^{\infty} \frac{1}{j!}D^j(b)n^j = b + f(n),$$

where we define $f(T) \in B[T]$ by $f(T) = \sum_{j=1}^{\infty} \frac{1}{j!}D^j(b)T^j$. As $\mathbb{Q} \subseteq B$ and $\mathbb{Q}$ contains infinitely many roots of $f(T)$, we have $f(T) = 0$ by 8.1, so in particular $D(b) = 0$, and we have shown that $B^{\exp(D)} \subseteq \ker(D)$. So (b) is proved. $\square$

8.4. Exercise. Let $B = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$ and $D \in \mathrm{Der}_{\mathbb{C}}(B)$ defined by $D(X) = 0$, $D(Y) = X$ and $D(Z) = -2Y$. Then D is triangular, so $D \in \mathrm{LND}(B)$ and we may consider the $\mathbb{Q}$-automorphism $\exp(D) : B \to B$. Note that $\exp(D)$ is actually a $\mathbb{C}$-automorphism, because $\mathbb{C} \subset \ker(D) = B^{\exp(D)}$. Compute the images of X, Y and Z by $\exp(D)$.

8.5. Let B be a domain containing a field $\Bbbk$ of characteristic zero. Note that if $D \in \mathrm{LND}(B)$ then $\exp(D) : B \to B$ is a $\Bbbk$-automorphism of B (because $\Bbbk \subseteq \ker(D) = B^{\exp(D)}$). So we have a well-defined set map,

$$\mathrm{LND}(B) \longrightarrow \mathrm{Aut}_{\Bbbk}(B), \qquad D \longmapsto \exp(D).$$

Of course this map is not a homomorphism, since $\mathrm{LND}(B)$ is only a set. Consider the subgroup $\langle E \rangle$ of $\mathrm{Aut}_{\Bbbk}(B)$ generated by the set $E = \{\exp(D) \mid D \in \mathrm{LND}(B)\}$.

8.6. Lemma. *Let B be a domain containing a field $\Bbbk$ of characteristic zero and consider the subgroup $\langle E \rangle$ of $\mathrm{Aut}_{\Bbbk}(B)$ generated by the set $E = \{\exp(D) \mid D \in \mathrm{LND}(B)\}$. Then $\langle E \rangle$ is a normal subgroup of $\mathrm{Aut}_{\Bbbk}(B)$.*

Proof. If $\theta \in \mathrm{Aut}_{\Bbbk}(B)$ and $D \in \mathrm{LND}(B)$, then $\theta^{-1} \circ D \circ \theta \in \mathrm{Der}(B)$ and $(\theta^{-1} \circ D \circ \theta)^n = \theta^{-1} \circ D^n \circ \theta$, so $\theta^{-1} \circ D \circ \theta \in \mathrm{LND}(B)$. It is easily verified that $\theta^{-1} \circ \exp(D) \circ \theta = \exp(\theta^{-1} \circ D \circ \theta)$, so $\theta^{-1}E\theta \subseteq E$ holds for all $\theta \in \mathrm{Aut}_{\Bbbk}(B)$. It follows that $\langle E \rangle \lhd \mathrm{Aut}_{\Bbbk}(B)$. $\square$

It is interesting to ask how big $\langle E \rangle$ is, or how small $\mathrm{Aut}_{\Bbbk}(B)/\langle E \rangle$ is. This of course depends on B. For instance, if $B = \Bbbk^{[n]}$ and $n > 2$ then it is an open problem to determine the structure of the group $\mathrm{Aut}_{\Bbbk}(B)$, and it is believed that $\langle E \rangle$ is almost all of $\mathrm{Aut}_{\Bbbk}(B)$ (it is conjectured that $\mathrm{Aut}_{\Bbbk}(B)$ is generated by its subgroups $\langle E \rangle$ and $GL_n(\Bbbk)$). On the other hand if B is a rigid ring then $\langle E \rangle = \{1\}$.

8.7. Exercise. Let B be a domain containing a field $\Bbbk$ of characteristic zero. Fix a derivation $D \in \mathrm{LND}(B)$ and consider the map

$$\Bbbk \longrightarrow \mathrm{Aut}_{\Bbbk}(B), \qquad \lambda \longmapsto \exp(\lambda D).$$

Show that this is a group homomorphism $(\Bbbk, +) \to \mathrm{Aut}_{\Bbbk}(B)$. Show that this homomorphism is injective whenever $D \neq 0$.

8.8. Exercise. Let B be a domain containing a field $\Bbbk$ of characteristic zero. Fix a derivation $D \in \mathrm{LND}(B)$ and consider the map

$$\Bbbk \times B \longrightarrow B, \qquad (\lambda, b) \longmapsto \lambda \oplus b,$$

where we define $\lambda \oplus b = \exp(\lambda D)(b)$ (so the operation $\oplus$ depends on the choice of D). Show that this is an action of the group $(\Bbbk, +)$ on the $\Bbbk$-algebra B, i.e., verify the following conditions:

- $0 \oplus b = b$ for all $b \in B$
- $(\lambda_1 + \lambda_2) \oplus b = \lambda_1 \oplus (\lambda_2 \oplus b)$ for all $\lambda_1, \lambda_2 \in \Bbbk$ and all $b \in B$
- for each $\lambda \in \Bbbk$, the map $B \to B$, $b \mapsto \lambda \oplus b$, is an automorphism of B as a $\Bbbk$-algebra.

9. Locally nilpotent derivations and $\mathbb{G}_a$-actions

Throughout this section, $\Bbbk$ is an algebraically closed field of characteristic zero and $\mathbb{G}_a(\Bbbk)$, or simply $\mathbb{G}_a$, denotes the group $(\Bbbk, +)$ viewed as an algebraic group. Let X be an affine algebraic variety over $\Bbbk$ and let B be the coordinate algebra of X (or if you prefer, let B be an integral domain which is finitely generated as a $\Bbbk$-algebra, and let $X = \mathrm{Spec}(B)$).[1]

9.1. Definition. An *action* of $\mathbb{G}_a(\Bbbk)$ on X (also called a $\mathbb{G}_a$-action on X) is a morphism $\alpha : \Bbbk \times X \to X$ of $\Bbbk$-varieties satisfying:

(1) $\alpha(0, x) = x$ for all $x \in X$

(2) $\alpha(a + b, x) = \alpha(a, \alpha(b, x))$ for all $a, b \in \Bbbk$ and $x \in X$.

In other words, a $\mathbb{G}_a$-action on X is a morphism $\alpha : \Bbbk \times X \to X$ satisfying:

(1+2) The map $a \mapsto \alpha(a, \_\_)$ is a group homomorphism $(\Bbbk, +) \to \mathrm{Aut}_{\Bbbk}(X)$.

See 8.3 for the definition of $\exp(D) : B \to B$, where $D \in \mathrm{LND}(B)$. In the following, this $\Bbbk$-automorphism is conveniently denoted $e^D : B \to B$.

9.2. We proceed to define a set map

(5)
$$\begin{array}{rcl} \mathrm{LND}(B) & \longrightarrow & \text{set of actions of } \mathbb{G}_a(\Bbbk) \text{ on } \mathrm{Spec}(B). \\ D & \longmapsto & \alpha_D \end{array}$$

Let $D \in \mathrm{LND}(B)$. By Exercise 8.7, we have the group homomorphism

$$(\Bbbk, +) \longrightarrow \mathrm{Aut}_{\Bbbk}(B), \qquad \lambda \longmapsto e^{\lambda D};$$

applying the functor Spec, we obtain the group homomorphism

$$(\Bbbk, +) \longrightarrow \mathrm{Aut}_{\Bbbk}(\mathrm{Spec}\, B), \qquad \lambda \longmapsto \mathrm{Spec}\left(e^{\lambda D}\right).$$

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[1]The theory could be developed in the more general setting where $\Bbbk$ is any $\mathbb{Q}$-algebra and B is any $\Bbbk$-algebra (see for instance [2]).

Then define the following map:

$$\alpha_D \; : \; \Bbbk \times \operatorname{Spec} B \longrightarrow \operatorname{Spec} B, \qquad (\lambda, x) \longmapsto \big(\operatorname{Spec} e^{\lambda D}\big)(x).$$

To conclude that α_D is an action of $\mathbb{G}_a$ on $\operatorname{Spec}(B)$, there only remains to verify that it is a morphism in the sense of algebraic geometry. Note that we may identify $\Bbbk \times \operatorname{Spec} B$ with $\operatorname{Spec}\big(\Bbbk[T] \otimes_\Bbbk B\big) = \operatorname{Spec}(B[T])$ where T is an indeterminate. By 3.3, D determines the homomorphism of $\Bbbk$-algebras $\xi_D : B \to B[T]$, $\xi_D(b) = \sum_{j \in \mathbb{N}} \frac{D^j b}{j!} T^j$, and one can verify that $\operatorname{Spec}(\xi_D) = \alpha_D$; so α_D is a morphism and hence an action.

This shows that (5) is a well-defined map.

9.3. Theorem. *The set map*

$$\begin{aligned} \operatorname{LND}(B) &\longrightarrow & \text{set of actions of } \mathbb{G}_a(\Bbbk) \text{ on } \operatorname{Spec}(B), \\ D &\longmapsto & \alpha_D \end{aligned}$$

defined in 9.2, *is bijective.*

We refer to [2] for the proof of 9.3. Note that, under this bijection, the zero derivation corresponds to the trivial action. So B is rigid (see 7.7) if and only if the only $\mathbb{G}_a$-action on $X = \operatorname{Spec}(B)$ is the trivial one.

9.4. Example. Let $B = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$ and $D = X \frac{\partial}{\partial Y} + (Y^2 + XY) \frac{\partial}{\partial Z} \in \operatorname{LND}(B)$. Then D determines an action $\alpha_D : \mathbb{C} \times \mathbb{C}^3 \to \mathbb{C}^3$ which we now compute (note that we are identifying $\operatorname{Spec}(B)$ with $\mathbb{C}^3$). We have

$$e^{\lambda D}(Z) = \sum_{n=0}^\infty \frac{(\lambda D)^n (Z)}{n!} = \sum_{n=0}^\infty \frac{\lambda^n}{n!} D^n(Z)$$

$$= Z + \lambda(Y^2 + XY) + \frac{\lambda^2}{2}(2XY + X^2) + \frac{\lambda^3}{6}(2X^2),$$

and similarly $e^{\lambda D}(X) = X$ and $e^{\lambda D}(Y) = Y + (\lambda D)(Y) = Y + \lambda X$. So, given $\lambda \in \mathbb{C}$ and $(x, y, z) \in \mathbb{C}^3$,

$$\alpha_D \; : \; \big(\lambda, (x, y, z)\big) \; \longmapsto \; \Big(x, \; y + \lambda x, \; z + \lambda(y^2 + xy) + \frac{\lambda^2}{2}(2xy + x^2) + \frac{\lambda^3}{3} x^2\Big).$$

Our next objective is to describe the fixed points and the ring of invariants of a $\mathbb{G}_a$-action. Let us first define these notions.

9.5. Definition. Suppose that $\alpha : G \times X \to X$, $(g, x) \mapsto gx$, is an action of an algebraic group G on $X = \operatorname{Spec}(B)$.

(1) A *fixed point* of α is a closed point $x \in X$ satisfying $\forall_{g \in G} \; gx = x$.
(2) The action α determines an action

$$G \times B \longrightarrow B, \qquad (g, b) \longmapsto gb,$$

of G on B, which, in turn, determines the subring $B^G = \{b \in B \mid \forall_{g \in G} \; gb = b\}$ of B. We call B^G the *ring of invariants* of the G-action α on X.

9.6. Definition. Given $D \in \mathrm{LND}(B)$, define

$$\mathrm{Fix}(D) = \{\, \mathfrak{p} \in \mathrm{Spec}(B) \mid \mathfrak{p} \supseteq D(B)\,\}.$$

Note that $\mathrm{Fix}(D)$ is a closed subset of $\mathrm{Spec}(B)$.

9.7. Proposition. *Let $D \in \mathrm{LND}(B)$ and consider the $\mathbb{G}_a$-action α_D on $X = \mathrm{Spec}(B)$ defined in 9.2 and 9.3.*

(1) *The ring of invariants of α_D is the subring $\ker(D)$ of B.*
(2) *The fixed points of α_D are precisely the closed points which belong to $\mathrm{Fix}(D)$.*

Proof. The action α_D of $\mathbb{G}_a$ on X determines the following action of $\mathbb{G}_a$ on B:

$$\Bbbk \times B \longrightarrow B, \qquad (\lambda, b) \longmapsto e^{\lambda D}(b).$$

For any $b \in B$ we have

$$b \in B^{\mathbb{G}_a} \iff \forall_{\lambda \in \Bbbk}\; e^{\lambda D}(b) = b \overset{8.3}{\iff} \forall_{\lambda \in \Bbbk}\; b \in \ker(\lambda D) \iff b \in \ker(D),$$

which proves assertion (1). Assertion (2) is a corollary of 9.8, below. $\qquad\square$

9.8. Proposition. *Let $\Bbbk$ and B be rings such that $\mathbb{Q} \subseteq \Bbbk \subseteq B$, and let $D \in \mathrm{LND}_{\Bbbk}(B)$. Then, for a maximal ideal $\mathfrak{m}$ of B, the following are equivalent:*

(1) *For all $\lambda \in \Bbbk$, $e^{\lambda D}(\mathfrak{m}) = \mathfrak{m}$*
(2) $\mathfrak{m} \supseteq D(B)$.

Proof. Suppose that (2) holds. Given $\lambda \in \Bbbk$ and $b \in \mathfrak{m}$, we have $D^j(b) \in \mathfrak{m}$ for all $j \in \mathbb{N}$, so $e^{\lambda D}(b) = \sum_{j=0}^{\infty} \frac{D^j(b)}{j!}\lambda^j \in \mathfrak{m}$; this shows that $e^{\lambda D}(\mathfrak{m}) \subseteq \mathfrak{m}$, and since $e^{\lambda D}$ is an automorphism we must have $e^{\lambda D}(\mathfrak{m}) = \mathfrak{m}$. So (2) implies (1).

Conversely, suppose that (1) holds. The first step is to prove that

$$(6) \qquad\qquad\qquad\qquad D(\mathfrak{m}) \subseteq \mathfrak{m}.$$

Let $b \in \mathfrak{m}$. Define $f(T) = \sum_{j=0}^{\infty} \frac{D^j(b)}{j!}T^j \in B[T]$ and note that $f(\lambda) = e^{\lambda D}(b)$ for all $\lambda \in \Bbbk$. Since (1) holds, we have $f(\lambda) \in \mathfrak{m}$ for all $\lambda \in \Bbbk$, so in particular this holds for all $\lambda \in \mathbb{Q}$. Consider the field $\kappa = B/\mathfrak{m}$, the canonical epimorphism $\pi : B \to \kappa$ and the polynomial $f^{(\pi)} \in \kappa[T]$. Then $\mathbb{Q} \subseteq \kappa$ and $f^{(\pi)}(\lambda) = 0$ for all $\lambda \in \mathbb{Q}$; so $f^{(\pi)} = 0$, i.e., all coefficients of $f(T)$ belong to $\mathfrak{m}$. In particular $D(b) \in \mathfrak{m}$, which proves (6).

By (6), $\delta(b + \mathfrak{m}) = D(b) + \mathfrak{m}$ is a well-defined locally nilpotent derivation $\delta : \kappa \to \kappa$. By 5.5, $\delta = 0$; this means that $D(B) \subseteq \mathfrak{m}$, i.e., (2) holds. $\qquad\square$

9.9. Exercise. Let $B = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$ and $D = X\frac{\partial}{\partial Y} + (Y^2 + XY)\frac{\partial}{\partial Z} \in \mathrm{LND}(B)$. Then D determines an action α_D on $\mathbb{C}^3$, see 9.4. Find the set of fixed points of this action.

References

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