Introduction to locally nilpotent derivations

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The aim of this document is to introduce graduate students to the basic theory of locally nilpotent derivations on commutative rings.

Conventions:

- All rings and algebras are tacitly assumed to be commutative and associative and to have an identity element 1.
- "Domain" means integral domain. If A is a ring, A^* is the set of units of A. If A is a domain, $\operatorname{Frac} A$ is the field of fractions of A.
- If A is a ring, $A^{[n]}$ = polynomial ring in n variables over A.
- If $\Bbbk$ is a field, $\Bbbk^{(n)} = \operatorname{Frac}\left(\Bbbk^{[n]}\right)$ = field of fractions of $\Bbbk^{[n]}$.

1. Derivations

1.1. **Definition.** A *derivation* of a ring B is a map $D : B \to B$ satisfying

$$\text{for all } f, g \in B, \qquad D(f+g) = D(f) + D(g) \quad \text{and} \quad D(fg) = D(f)g + fD(g).$$

If $D : B \to B$ is a derivation, we define $\ker D = \{\, x \in B \mid D(x) = 0 \}$.

1.2. **Exercise.** Let B be a ring and $D : B \to B$ a derivation. Verify that $\ker(D)$ is a **subring** of B. Several authors call $\ker(D)$ the *ring of constants* of D, and denote it B^D.

1.3. **Exercise.** Consider a field $\Bbbk$, the polynomial ring $\Bbbk[t] = \Bbbk^{[1]}$, and the derivative $D = \frac{d}{dt} : \Bbbk[t] \to \Bbbk[t]$. Verify that D is a derivation. Show that if $\operatorname{char} \Bbbk = 0$ then $\ker D = \Bbbk$, and that if $\operatorname{char} \Bbbk = p > 0$ then $\ker D = \Bbbk[t^p]$.

The sum of two derivations of a ring B is a derivation of B. If $D : B \to B$ is a derivation and $b \in B$ then the map

$$bD : B \to B, \qquad x \mapsto bD(x)$$

is a derivation of B. It follows that the set

$$\operatorname{Der}(B) = \text{ set of all derivations of } B$$

is a B-module. If $A \subseteq B$ are rings then by an A-*derivation* of B we mean a derivation $D : B \to B$ satisfying $D(A) = \{0\}$. Then the set

$$\operatorname{Der}_A(B) = \text{ set of all } A\text{-derivations of } B$$

is a B-submodule of $\operatorname{Der}(B)$.

1.4. **Example.** Let A be a ring and $B = A[X_1, \dots, X_n] = A^{[n]}$. Then, for each $i = 1, \dots, n$, the partial derivative $\frac{\partial}{\partial X_i}$ is an element of $\operatorname{Der}_A(B)$. Since $\operatorname{Der}_A(B)$ is a B-module, it follows that $\sum_{i=1}^n f_i \frac{\partial}{\partial X_i} \in \operatorname{Der}_A(B)$ for any choice of $f_1, \dots, f_n \in B$.

1.5. **Lemma.** *Let A be a ring and $B = A[X_1, \dots, X_n] = A^{[n]}$. Given any $f_1, \dots, f_n \in B$, there exists a unique $D \in \operatorname{Der}_A(B)$ satisfying $D(X_i) = f_i$ for all $i = 1, \dots, n$.*

Proof. As $\mathrm{Der}_A(B)$ is a B-module, we may define $D \in \mathrm{Der}_A(B)$ by $D = \sum_{i=1}^n f_i \frac{\partial}{\partial X_i}$. Clearly, $D(X_i) = f_i$ for all $i \in \{1, \dots, n\}$. If also $D' \in \mathrm{Der}_A(B)$ satisfies $D'(X_i) = f_i$ for all i, then consider $D_0 = D - D' \in \mathrm{Der}_A(B)$; then $D_0(X_i) = 0$ for all i. Thus $A \cup \{X_1, \dots, X_n\} \subseteq \ker(D_0)$. As $\ker D_0$ is a subring of B, $\ker D_0 = B$; so $D_0 = 0$ and hence $D = D'$. So D is the unique element of $\mathrm{Der}_A(B)$ satisfying $D(X_i) = f_i$ for all $i \in \{1, \dots, n\}$. □

Remark. Lemma 1.5 implies that, when $B = A[X_1, \dots, X_n] = A^{[n]}$, $\mathrm{Der}_A(B)$ is a free B-module with basis $\{\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\}$.

1.6**. Exercise.** Let B be a ring and $D : B \to B$ a derivation. Verify the following claims.

(i) $D(b^n) = nb^{n-1}D(b)$, for all $b \in B$ and $n \in \mathbb{N}$.

(ii) Let $f(T) = \sum_{i=0}^n a_i T^i \in B[T]$ be a polynomial ($a_i \in B$ and T is an indeterminate). If $b \in B$ then $f(b) \in B$, so it makes sense to evaluate D at $f(b)$. Show that

$$D\big(f(b)\big) = f^{(D)}(b) + f'(b)D(b)$$

where we define the polynomial $f^{(D)}(T) \in B[T]$ by $f^{(D)}(T) = \sum_{i=0}^n D(a_i)T^i$, and where $f'(T) \in B[T]$ is the derivative of f, defined by $f'(T) = \sum_{i=1}^n i a_i T^{i-1}$. Note that if all a_i belong to $\ker(D)$ then this formula simplifies to $D\big(f(b)\big) = f'(b)D(b)$.

(iii) More generally, show that if $f \in B[T_1, \dots, T_n]$ and $b_1, \dots, b_n \in B$ then

$$D\big(f(b_1, \dots, b_n)\big) = f^{(D)}(b_1, \dots, b_n) + \textstyle\sum_{i=1}^n f_{T_i}(b_1, \dots, b_n)\, D(b_i),$$

where $f_{T_i} = \frac{\partial f}{\partial T_i} \in B[T_1, \dots, T_n]$.

1.7**. Definition.** Let $A \subseteq B$ be rings. An element $b \in B$ is *algebraic* over A if there exists a nonzero polynomial $f \in A[T] \setminus \{0\}$ such that $f(b) = 0$ (note that f is not required to be monic); if b is not algebraic over A, we say that b is *transcendental* over A.

We say that B is *algebraic* over A if every element of B is algebraic over A. We say that A is *algebraically closed in* B if each element of $B \setminus A$ is transcendental over A.

1.8**. Lemma.** *If B is a domain of characteristic zero and $D \in \mathrm{Der}(B)$ then $\ker D$ is algebraically closed in B.*

Proof. Let $A = \ker D$ and consider $b \in B$ algebraic over A. Let $f \in A[T]$ be a nonzero polynomial of minimal degree such that $f(b) = 0$. Note that $\deg(f) \geq 1$. Then

$$0 = D(f(b)) = f^{(D)}(b) + f'(b)D(b) = f'(b)D(b).$$

We have $f' \neq 0$ (because B is a domain of characteristic zero), so $f'(b) \neq 0$ by minimality of $\deg f$, so $D(b) = 0$. □

1.9**. Exercise.** Let B be a ring of characteristic $n > 0$ and let $0 \neq D \in \mathrm{Der}(B)$. Show that each $x \in B$ satisfies $x^n \in \ker D$. Deduce that $\ker D$ is not algebraically closed in B.

If B is a ring and $D_1, D_2 \in \mathrm{Der}(B)$ then the composition $D_1 \circ D_2 : B \to B$ is a map which preserves addition, but is usually not a derivation. However, a straightforward

verification shows that $D_1 \circ D_2 - D_2 \circ D_1 : B \to B$ is a derivation. One uses the notation $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ for this derivation.
If $D \in \mathrm{Der}(B)$ and $n > 0$, we denote by $D^n : B \to B$ the composition of D with itself n times (for instance $D^3 = D \circ D \circ D$). We also define $D^0 : B \to B$ to be the identity map (even in the case $D = 0$). Note that D^n is usually not a derivation when $n \neq 1$.

1.10**. Exercise.** Prove **Leibnitz Rule**: *If B is a ring, $D \in \mathrm{Der}(B)$, $x, y \in B$ and $n \in \mathbb{N}$,*

$$D^n(xy) = \sum_{i=0}^{n} \binom{n}{i} D^{n-i}(x) D^i(y).$$

1.11**. Definition.** Given a ring B and $D \in \mathrm{Der}(B)$, define the set

$$\mathrm{Nil}(D) = \{ x \in B \mid \exists_{n \in \mathbb{N}}\ D^n(x) = 0 \}.$$

So $\ker(D) \subseteq \mathrm{Nil}(D) \subseteq B$. By exercise 1.12, $\mathrm{Nil}(D)$ is a subring of B.

1.12**. Exercise.** Use Leibnitz Rule to show that the subset $\mathrm{Nil}(D)$ of B is closed under multiplication. Deduce that $\mathrm{Nil}(D)$ is a subring of B.

1.13**. Example.** Let $B = \mathbb{C}[[T]]$ and $D = d/dT : B \to B$. Then $\ker(D) = \mathbb{C}$ and $\mathrm{Nil}(D) = \mathbb{C}[T]$. Note that $\mathrm{Nil}(D)$ is not integrally closed in B: let $b = \sqrt{1+T} \in B$, then $b \notin \mathrm{Nil}(D)$ but $b^2 \in \mathrm{Nil}(D)$.

1.14**. Exercise.** Let A be a ring and $B = A[X_1, \dots, X_n] = A^{[n]}$. Given $f = (f_1, \dots, f_{n-1}) \in B^{n-1}$, define the map $\Delta_f : B \to B$ by $\Delta_f(g) = \det\left(\frac{\partial(f_1,\dots,f_{n-1},g)}{\partial(X_1,\dots,X_n)}\right)$, for each $g \in B$. Check that $\Delta_f \in \mathrm{Der}_A(B)$ and that $A[f_1, \dots, f_{n-1}] \subseteq \ker(\Delta_f)$. One refers to Δ_f as a "jacobian derivation".

1.15**. Exercise.** Let $B = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$, $f = X^2 + Y^3$ and $g = XY + YZ + XZ$. Give an example of a derivation $D \in \mathrm{Der}_{\mathbb{C}}(B)$ satisfying $D(f) = 0 = D(g)$ and $D \neq 0$.
Hint: use jacobian derivations.

2. Definition of locally nilpotent derivations

2.1**. Definition.** Let B be any ring. A derivation $D : B \to B$ is *locally nilpotent* if it satisfies $\mathrm{Nil}(D) = B$, i.e., if $\forall_{b \in B} \exists_{n \in \mathbb{N}}\, D^n(b) = 0$. We write:

$$\text{LND}(B) = \text{ set of locally nilpotent derivations } B \to B.$$

2.2**. Example.** Let A be a ring and $B = A[X_1, \dots, X_n] = A^{[n]}$. Then $\frac{\partial}{\partial X_i} \in \text{LND}(B)$ for each $i = 1, \dots, n$.

2.3**. Definition.** Let A be a ring and $B = A[X_1, \dots, X_n] = A^{[n]}$. A derivation $D : B \to B$ is *triangular* if $D(A) = \{0\}$ and:

$$\forall\, i \quad D(X_i) \in A[X_1, \dots, X_{i-1}] \quad \text{(in particular } D(X_1) \in A).$$

2.4**. Example.** Let $B = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$ and $D = X^2 \frac{\partial}{\partial Y} + (X^5 + Y^3) \frac{\partial}{\partial Z} \in \mathrm{Der}_{\mathbb{C}}(B)$. Then D is triangular.

2.5. **Lemma.** *Let A be a ring and $B = A[X_1, \dots, X_n] = A^{[n]}$. Then every triangular derivation of B is locally nilpotent.*

Proof. If $D : B \to B$ is triangular then $A \subseteq \ker(D) \subseteq \mathrm{Nil}(D)$, so $\mathrm{Nil}(D)$ is an A-subalgebra of B. As an exercise, prove (by induction on i) that

$$A[X_1, \dots, X_i] \subseteq \mathrm{Nil}(D), \quad \text{for all } i = 1, \dots, n.$$

So $\mathrm{Nil}(D) = B$, i.e., D is locally nilpotent. □

2.6. **Example.** Let $B = \mathbb{C}[X, Y] = \mathbb{C}^{[2]}$, $D_1 = Y \frac{\partial}{\partial X}$ and $D_2 = X \frac{\partial}{\partial Y}$.
Then $D_1, D_2 \in \textsc{lnd}(B)$ (because they are triangular). However, $D_1 + D_2 \notin \textsc{lnd}(B)$ (because $(D_1 + D_2)^2(X) = X$), and you can also check that $[D_1, D_2] \notin \textsc{lnd}(B)$.
Also, $\frac{\partial}{\partial X} \in \textsc{lnd}(B)$ but $X \frac{\partial}{\partial X} \notin \textsc{lnd}(B)$.

2.7. We stress that Example 2.6 shows that the subset $\textsc{lnd}(B)$ of $\mathrm{Der}(B)$ is in general not closed under any of the algebraic operations which are defined in $\mathrm{Der}(B)$:

(i) $D_1, D_2 \in \textsc{lnd}(B) \not\Longrightarrow D_1 + D_2 \in \textsc{lnd}(B)$
(ii) $b \in B$, $D \in \textsc{lnd}(B) \not\Longrightarrow bD \in \textsc{lnd}(B)$
(iii) $D_1, D_2 \in \textsc{lnd}(B) \not\Longrightarrow [D_1, D_2] \in \textsc{lnd}(B)$.

In other words, $\textsc{lnd}(B)$ is just a set. It does not have any algebraic structure.

The facts contained in the following exercise will be needed in the next sections.

2.8. **Exercise.** Let B be a ring, $D \in \textsc{lnd}(B)$ and $A = \ker D$.

(1) If $a \in A$ then $(aD)^n = a^n D^n$ holds for all $n \in \mathbb{N}$. (Make sure that you use the assumption $D(a) = 0$ in your proof!)

(2) If $a \in A$ then $aD \in \textsc{lnd}(B)$ (compare with 2.7(ii)).

(3) Observe that $D : B \to B$ is (in particular) a homomorphism of A-modules. If $S \subset A$ is a multiplicatively closed set, consider the homomorphism of $S^{-1}A$-modules $S^{-1}D : S^{-1}B \to S^{-1}B$ defined by $(S^{-1}D)(x/s) = (Dx)/s$ ($x \in B$, $s \in S$). Show that $S^{-1}D$ is an element of $\textsc{lnd}(S^{-1}B)$ and $\ker(S^{-1}D) = S^{-1}A$.

2.9. **Kernels.** Given a ring B, we write:

$$\textsc{klnd}(B) = \{\ker D \mid D \in \textsc{lnd}(B) \text{ and } D \neq 0\}.$$

Then the problem of describing the set $\textsc{lnd}(B)$ breaks into two steps:

(i) Describe the set $\textsc{klnd}(B)$;
(ii) for each $A \in \textsc{klnd}(B)$, describe the set $\{D \in \textsc{lnd}(B) \mid \ker D = A\}$.

It is often the case that problem (i) is significantly harder than problem (ii). For instance, if $B = \Bbbk^{[3]}$ then (i) is currently an open problem and is considered very hard, but (ii) is completely understood. In these notes, a lot of attention is given to kernels.

3. The exponential map associated to a locally nilpotent derivation

3.1. **Exercise.** If B is a $\mathbb{Q}$-algebra then $\operatorname{Der}(B) = \operatorname{Der}_{\mathbb{Q}}(B)$.

3.2. **Definition.** Let B be a $\mathbb{Q}$-algebra. Given $D \in \textsc{lnd}(B)$, define the map

$$\xi_D : B \longrightarrow B[T], \qquad b \longmapsto \sum_{n \in \mathbb{N}} \frac{1}{n!} D^n(b)\, T^n.$$

We call ξ_D the *exponential map associated to* D (not to be confused with the exponential of D, $\exp(D) : B \to B$, to be defined later). Note that the definition of ξ_D requires us to divide by $n!$, which is (of course) why we need to assume that $\mathbb{Q} \subseteq B$.

The following is a fundamental result in the theory of locally nilpotent derivations. It has several deep consequences.

3.3. **Theorem.** *Let B be a $\mathbb{Q}$-algebra and $D \in \textsc{lnd}(B)$. Then the exponential map $\xi_D : B \to B[T]$ is an injective homomorphism of A-algebras, where $A = \ker(D)$.*

Proof. If $e_0 : B[T] \to B$ is the map $f(T) \mapsto f(0)$, then the composite $B \xrightarrow{\xi_D} B[T] \xrightarrow{e_0} B$ is the identity map, so ξ_D is injective. It is clear that ξ_D preserves addition and restricts to the identity map on A, so it suffices to verify that

$$\left(\sum_{i \in \mathbb{N}} \frac{1}{i!} D^i(x)\, T^i \right) \left(\sum_{j \in \mathbb{N}} \frac{1}{j!} D^j(y)\, T^j \right) = \sum_{n \in \mathbb{N}} \frac{1}{n!} D^n(xy)\, T^n \tag{1}$$

holds for all $x, y \in B$. In the left hand side of (1), the coefficient of T^n is

$$\sum_{i+j=n} \frac{1}{i!\, j!} D^i(x) D^j(y) = \frac{1}{n!} \sum_{i+j=n} \frac{n!}{i!\, j!} D^i(x) D^j(y),$$

which is equal to $\frac{1}{n!} D^n(xy)$ by Leibnitz Rule. □

3.4. **Exercise.** Let B be a $\mathbb{Q}$-algebra and $D \in \textsc{lnd}(B)$. Check that ξ_D is the inclusion map $B \hookrightarrow B[T]$ if and only if $D = 0$.

3.5. **Exercise.** Let $B = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$ and $D = X \frac{\partial}{\partial Y} + Y \frac{\partial}{\partial Z}$. Then $D \in \textsc{lnd}(B)$. Note that $\xi_D : B \to B[T]$ is a homomorphism of $\mathbb{C}$-algebras:

$$\xi_D : \mathbb{C}[X, Y, Z] \to \mathbb{C}[X, Y, Z, T].$$

Describe ξ_D by giving the images of X, Y, Z.

4. Degree functions

4.1. **Definition.** A *degree function* on a ring B is a map $\deg : B \to \mathbb{N} \cup \{-\infty\}$ satisfying:

(1) $\forall x \in B \quad \deg x = -\infty \iff x = 0$

(2) $\forall x, y \in B \quad \deg(xy) = \deg x + \deg y$

(3) $\forall x, y \in B \quad \deg(x + y) \le \max(\deg x, \deg y)$.

4.2**. Exercise.** Use parts (1) and (2) of 4.1 to show that if a ring B admits a degree function $\deg : B \to \mathbb{N} \cup \{-\infty\}$, then B is a domain.

4.3**. Exercise.** Let $A[t]$ be the polynomial ring in one variable over a ring A, and let $\deg_t : A[t] \to \mathbb{N} \cup \{-\infty\}$ be the usual degree of polynomials. Show that deg is a degree function if and only if A is an integral domain.

4.4**. Exercise.** Let $\deg : B \to \mathbb{N} \cup \{-\infty\}$ be a degree function and $x, y \in B$ such that $\deg(x) \neq \deg(y)$. Show that $\deg(x + y) = \max(\deg x, \deg y)$.

4.5**. Exercise.** If $B \xrightarrow{\varphi} B'$ is an injective ring homomorphism and $B' \xrightarrow{d} \mathbb{N} \cup \{-\infty\}$ is a degree function then $B \xrightarrow{d \circ \varphi} \mathbb{N} \cup \{-\infty\}$ is a degree function.

4.6**. Definition.** Let B be a ring. Then each $D \in \text{LND}(B)$ determines a map

$$\deg_D : B \to \mathbb{N} \cup \{-\infty\}$$

defined as follows: $\deg_D(x) = \max\{n \in \mathbb{N} \mid D^n x \neq 0\}$ for $x \in B \setminus \{0\}$, and $\deg_D(0) = -\infty$. Note that $\ker D = \{x \in B \mid \deg_D(x) \leq 0\}$.

4.7**. Example.** Let A be a domain of characteristic zero and $B = A[t] = A^{[1]}$. Let $D = \frac{d}{dt} : B \to B$, then $D \in \text{LND}(B)$, so D determines the map $\deg_D : A[t] \to \mathbb{N} \cup \{-\infty\}$. It is easy to see that $\deg_D$ is the usual degree of polynomials.

Although we defined $\deg_D$ for any ring B, it is useful mostly in the case of integral domains of characteristic zero:

4.8**. Proposition.** *Let B be a domain of characteristic zero and $D \in \text{LND}(B)$. Then the map $\deg_D : B \to \mathbb{N} \cup \{-\infty\}$ is a degree function.*

Proof. We first prove the special case where $\mathbb{Q} \subseteq B$. In this case we may consider the map $\xi_D : B \to B[T]$, $\xi_D(b) = \sum_{i=0}^{\infty} \frac{D^n(b)}{n!} T^n$, which is an injective ring homomorphism by 3.3. As B is a domain, $B[T] \xrightarrow{\deg_T} \mathbb{N} \cup \{-\infty\}$ is a degree function and consequently the composite $B \xrightarrow{\xi_D} B[T] \xrightarrow{\deg_T} \mathbb{N} \cup \{-\infty\}$ is a degree function. As this composite map is equal to $\deg_D$, $\deg_D$ is a degree function.
Now the general case. Since B has characteristic zero and $\ker D$ is a subring of B, we have $\mathbb{Z} \subseteq \ker D$. Let $S = \mathbb{Z} \setminus \{0\}$ and consider $S^{-1}D : S^{-1}B \to S^{-1}B$, which belongs to $\text{LND}(S^{-1}B)$ by Exercise 2.8. As $\mathbb{Q} \subseteq S^{-1}B$, the first part of the proof implies that $\deg_{S^{-1}D} : S^{-1}B \to \mathbb{N} \cup \{-\infty\}$ is a degree function. We have:

$$\begin{array}{ccc} S^{-1}B & \xrightarrow{S^{-1}D} & S^{-1}B \\ \uparrow & & \uparrow \\ B & \xrightarrow{D} & B \end{array} \qquad\qquad \begin{array}{ccc} S^{-1}B & \xrightarrow{\deg_{S^{-1}D}} & \mathbb{N} \cup \{-\infty\} \\ \uparrow & \nearrow_{\deg_D} & \\ B & & \end{array}$$

Note that $B \to S^{-1}B$ is injective, because B is a domain and $0 \notin S$. So D is the restriction of $S^{-1}D$ and consequently $\deg_D$ is the restriction of $\deg_{S^{-1}D}$; it follows that $\deg_D$ is a degree function. $\square$

4.9**. Exercise.** Let B be a domain of characteristic zero and suppose that $D \in \mathrm{Der}(B)$ satisfies $D^n = 0$ for some $n > 0$. Show that $D = 0$. (*Hint.* Note that D is locally nilpotent, so the map $\deg_D$ exists and is a degree function. If $D \neq 0$ then we can choose $x \in B$ such that $\deg_D(x) \geq 1$; what is $\deg_D(x^n)$? can $D^n(x^n)$ be zero?)

5. Factorially closed subrings

5.1**. Definition.** Let $A \subseteq B$ be domains. We say that A is *factorially closed* in B if:

$$\forall\, x, y \in B \setminus \{0\} \qquad xy \in A \implies x, y \in A.$$

For instance, consider the polynomial ring $R[T]$ in one variable over an integral domain R. Then R is factorially closed in $R[T]$. Note that this example is a special case of:

5.2**. Lemma.** *If B is a domain and $\deg : B \to \mathbb{N} \cup \{-\infty\}$ is a degree function then $\{\, x \in B \mid \deg x \leq 0 \}$ is a factorially closed subring of B.*

Proof. Obvious. □

5.3**. Corollary.** *Let B be a domain of characteristic zero and $D \in \textsc{lnd}(B)$. Then $\ker(D)$ is a factorially closed subring of B.*

Proof. $\{\, x \in B \mid \deg_D(x) \leq 0 \}$ is factorially closed in B by 4.8 and 5.2. As $\ker D = \{\, x \in B \mid \deg_D(x) \leq 0 \}$, we are done. □

Recall the following definitions. Let R be an integral domain and let $p \in R$. We say that p is *irreducible* if $p \notin R^* \cup \{0\}$ and if the condition $p = xy$ (where $x, y \in R$) implies that $\{x, y\} \cap R^* \neq \varnothing$. We say that p is *prime* if $p \notin R^* \cup \{0\}$ and if the condition $p \mid xy$ (where $x, y \in R$) implies that p divides one of x, y (i.e., p is prime if and only if the principal ideal pR is a nonzero prime ideal of R). Recall that every prime element is irreducible but that the converse is not necessarily true. However, if R is a UFD then "irreducible" is equivalent to "prime".

5.4**. Exercise.** Suppose that A is a factorially closed subring of a domain B. Then:

(1) $A^* = B^*$.

(2) An element of A is irreducible in A iff it is irreducible in B.

(3) If B is a UFD then so is A.

From 5.3 and 5.4, we get:

5.5**. Corollary.** *Let B be a domain of characteristic zero, $D \in \textsc{lnd}(B)$ and $A = \ker(D)$.*

(1) $A^* = B^*$
(2) *If $\Bbbk$ is any field included in B, then D is a $\Bbbk$-derivation.*
(3) *If B is a UFD then so is A.*

5.6**. Exercise.** For domains $A \subseteq B$, the following implications are valid:

$$A \text{ is factorially closed in } B \implies A \text{ is algebraically closed in } B$$
$$\implies A \text{ is integrally closed in } B.$$

5.7. **Exercise.** Let $B = \mathbb{C}[X,Y] = \mathbb{C}^{[2]}$ and $f = XY \in B$. Show that if A is a factorially closed subring of B satisfying $f \in A$, then $A = B$. Deduce the following assertions:

(1) The only $D \in \text{LND}(B)$ satisfying $D(f) = 0$ is the zero derivation.
(2) The jacobian derivation $\Delta_f \in \text{Der}_{\mathbb{C}}(B)$ (refer to 1.14) is not locally nilpotent.
(3) The ring $\ker(\Delta_f)$ is algebraically closed in B but not factorially closed.

6. Transcendence degree

6.1. Given a field extension L/K, we will write $\text{trdeg}_K(L)$ for the *transcendence degree* of L over K (read the definition of that concept in some algebra textbook). Note that transcendence degree has the following properties:

(1) Let $K \subseteq L$ be fields. Then $\text{trdeg}_K(L) = 0$ if and only if L is algebraic over K.
(2) Let K be a field, $t_1, \dots, t_n$ indeterminates over K and $L = K(t_1, \dots, t_n) = K^{(n)}$. Then $\text{trdeg}_K(L) = n$.
(3) Let $K \subseteq L \subseteq M$ be fields. Then $\text{trdeg}_K(M) < \infty$ if and only if both $\text{trdeg}_K(L)$ and $\text{trdeg}_L(M)$ are finite. Moreover, if $\text{trdeg}_K(M) < \infty$ then
$$\text{trdeg}_K(M) = \text{trdeg}_K(L) + \text{trdeg}_L(M).$$

6.2. **Notation.** Given domains $A \subseteq B$, we define $\text{trdeg}_A(B)$ to be equal to the transcendence degree of $\text{Frac}\, B$ over $\text{Frac}\, A$.

6.3. **Exercise.** Use 6.1 to prove the following properties of transcendence degree of integral domains.

(1) Let $A \subseteq B$ be domains. Then $\text{trdeg}_A(B) = 0$ if and only if B is algebraic over A.
(2) Let A be a domain and let $B = A^{[n]}$. Then $\text{trdeg}_A(B) = n$.
(3) Let $A \subseteq B \subseteq C$ be domains. Then $\text{trdeg}_C(A) < \infty$ if and only if both $\text{trdeg}_A(B)$ and $\text{trdeg}_B(C)$ are finite. Moreover, if $\text{trdeg}_A(C) < \infty$ then
$$\text{trdeg}_A(C) = \text{trdeg}_A(B) + \text{trdeg}_B(C).$$

6.4. **Exercise.** Let $A \subseteq B$ be domains, where $\text{trdeg}_A(B) < \infty$ and A is algebraically closed in B. Suppose that A' is a ring such that $A \subseteq A' \subseteq B$ and $\text{trdeg}_A(B) = \text{trdeg}_{A'}(B)$. Show that $A = A'$.

7. Slices and preslices

7.1. **Definition.** Let B be a ring and $D \in \text{LND}(B)$. A *slice* of D is an element $s \in B$ satisfying $D(s) = 1$.

7.2. **Examples.** Let $B = \mathbb{C}[X,Y,Z] = \mathbb{C}^{[3]}$.

(1) X is a slice of $\frac{\partial}{\partial X} \in \text{LND}(B)$.
(2) Define $D \in \text{LND}(B)$ by $DZ = Y$, $DY = X$, $DX = 0$. Then given $f \in B$,
$$D(f) = f_X D(X) + f_Y D(Y) + f_Z D(Z) = f_Y X + f_Z Y,$$
thus $D(B) \subseteq (X,Y)B$, so D does not have a slice.

When a slice exists, the situation is very special:

7.3. **Theorem** ([2, Prop. 2.1]). *Let B be a $\mathbb{Q}$-algebra, $D \in \text{LND}(B)$ and $A = \ker(D)$. If $s \in B$ satisfies $Ds = 1$ then $B = A[s] = A^{[1]}$ and $D = \frac{d}{ds} : A[s] \to A[s]$.*

Proof. Consider $f(T) = \sum_{i=0}^{n} a_i T^i \in A[T] \setminus \{0\}$ (where $n \geq 0$, $a_i \in A$ and $a_n \neq 0$). Then $D^j(f(s)) = f^{(j)}(s)$ for all $j \geq 0$, where $f^{(j)}(T) \in A[T]$ denotes the j-th derivative of f; so $D^n(f(s)) = n!\, a_n \neq 0$ and in particular $f(s) \neq 0$. So s is transcendental over A, i.e., $A[s] = A^{[1]}$.

To show that $B = A[s]$, consider the homomorphism of A-algebra $\xi : B \to B$ obtained by composing the homomorphism $\xi_D : B \to B[T]$ of 3.3 with the evaluation map $B[T] \to B$, $f(T) \mapsto f(-s)$. Explicitely, if $x \in B$ then $\xi(x) = \sum_{j=0}^{\infty} \frac{D^j x}{j!}(-s)^j$. For each $x \in B$,

$$D\big(\xi(x)\big) = \sum_{j=0}^{\infty} \frac{D^{j+1}x}{j!}(-s)^j + \sum_{j=0}^{\infty} \frac{D^j x}{j!} j(-s)^{j-1}(-1) = 0,$$

so $\xi(B) \subseteq A$; since ξ is a A-homomorphism, $\xi(B) = A$.

By induction on $\deg_D(x)$, we show that $\forall_{x \in B}\ x \in A[s]$. This is clear if $\deg_D(x) \leq 0$, so assume that $\deg_D(x) \geq 1$. Since $x = \xi(x) + (x - \xi(x))$ where $\xi(x) \in A$ and $x - \xi(x) \in sB$,

$$x = a + x's, \quad \text{for some } a \in A \text{ and } x' \in B. \tag{2}$$

This implies that $Dx = D(x')s + x'$ and it easily follows by induction that

$$\forall_{m \geq 1}\ \ D^m(x) = D^m(x')s + mD^{m-1}(x'). \tag{3}$$

Choose $m \geq 1$ such that $D^{m-1}(x') \neq 0$ and $D^m(x') = 0$ (such an m exists because $\deg_D(x) \geq 1$, so $x \notin A$, so $x' \neq 0$). Then (3) gives $D^m(x) = mD^{m-1}(x') \neq 0$ and $D^{m+1}(x) = 0$, so $\deg_D(x') = \deg_D(x) - 1$. By the inductive hypothesis we have $x' \in A[s]$; then (2) gives $x \in A[s]$. So $B = A[s] = A^{[1]}$. □

7.4. **Corollary.** *Let B be a $\mathbb{Q}$-algebra, $D \in \text{LND}(B)$ and $A = \ker(D)$. If $s \in B$ satisfies $Ds \in A^*$ then $B = A[s] = A^{[1]}$.*

Proof. Let $a \in A$ be such that $aD(s) = 1$. Then as is a slice of D, so $B = A[as] = A[s]$, and s is transcendental over A (since as is). □

7.5. **Corollary.** *Let B be a domain of characteristic zero and suppose that $A \in \text{KLND}(B)$. Then $S^{-1}B = (\operatorname{Frac} A)^{[1]}$, where $S = A \setminus \{0\}$. In particular, $\operatorname{trdeg}_A(B) = 1$.*

Proof. Let $A \in \text{KLND}(B)$. Choose $D \in \text{LND}(B)$ such that $\ker D = A$ (then $D \neq 0$). If we write $S = A \setminus \{0\}$ and $K = \operatorname{Frac}(A)$ then exercise 2.8 gives $S^{-1}D \in \text{LND}(S^{-1}B)$ and $\ker(S^{-1}D) = K$. Note that $S^{-1}D$ has a slice (indeed, choose a preslice $s \in B$ of D and let $a = Ds$, then $a \in S$, so $s/a \in S^{-1}B$, and it is clear that $S^{-1}D(s/a) = 1$). So 7.3 implies that $S^{-1}B = K^{[1]}$, which proves the assertion. □

7.6. **Exercise.** Let B be a domain of characteristic zero. Show that if $A, A' \in \text{KLND}(B)$ satisfy $A \subseteq A'$, then $A = A'$.

7.7**. Definition.** Let B be a domain of characteristic zero. We say that B is *rigid* if it satisfies the following equivalent conditions: (i) $\text{LND}(B) = \{0\}$; (ii) $\text{KLND}(B) = \varnothing$.

7.8**. Exercise.** Let B be a domain of characteristic zero which has transcendence degree 1 over some field $\Bbbk_0 \subseteq B$. Show that if B is not rigid then $B = \Bbbk^{[1]}$ for some field $\Bbbk$ such that $\Bbbk_0 \subseteq \Bbbk \subseteq B$. (Hint: let $D \in \text{LND}(B)$, $D \neq 0$, define $\Bbbk = \ker D$ and consider $\Bbbk_0 \subseteq \Bbbk \subseteq B$. Show that $\Bbbk$ is integral over $\Bbbk_0$ and hence must be a field. Show that there exists $b \in B$ such that $D(b) \in \Bbbk^*$ and conclude that $B = \Bbbk^{[1]}$.)

7.9**. Exercise.** Consider the subring $B = \mathbb{C}[T^2, T^3]$ of $\mathbb{C}[T] = \mathbb{C}^{[1]}$. Show that B is rigid.

7.10**. Exercise.** Let $B = \mathbb{Z}[X, Y] = \mathbb{Z}^{[2]}$ and $D = \frac{\partial}{\partial Y} + Y \frac{\partial}{\partial X}$. Since D is triangular, we have $D \in \text{LND}(B)$. Moreover, $DY = 1$. Show that $\ker D = \mathbb{Z}[2X - Y^2]$ and that B is not a polynomial ring over $\ker D$. (So in 7.3 the hypothesis that B is a $\mathbb{Q}$-algebra is not superfluous.)

7.11**. Definition.** Let B be a ring and $D \in \text{LND}(B)$. A *preslice* of D is an element $s \in B$ satisfying $D(s) \neq 0$ and $D^2(s) = 0$ (i.e., $\deg_D(s) = 1$).

Remark. It is clear that if $D \in \text{LND}(B)$ and $D \neq 0$ then D has a preslice.

Preslices are important because they always exist, and because they have the following nice property:

7.12**. Corollary.** *Let B be a $\mathbb{Q}$-algebra, $D \in \text{LND}(B)$ and $A = \ker(D)$. If $s \in B$ satisfies $Ds \neq 0$ and $D^2 s = 0$, then $B_\alpha = A_\alpha[s] = (A_\alpha)^{[1]}$ where $\alpha = Ds \in A \setminus \{0\}$.*

Proof. Let $S = \{1, \alpha, \alpha^2, \dots\}$ and consider $S^{-1}D : S^{-1}B \to S^{-1}B$. By exercise 2.8, $S^{-1}D \in \text{LND}(S^{-1}B)$ and $\ker(S^{-1}D) = S^{-1}A$. As $(S^{-1}D)(s) = \alpha \in (S^{-1}B)^*$, the result follows from 7.4. ☐

Geometric interpretation of 7.12

Given $A \in \text{KLND}(B)$, the inclusion map $A \hookrightarrow B$ is a ring homomorphism and hence determines a morphism of schemes $\pi : \text{Spec}(B) \to \text{Spec}(A)$. It is natural to ask what are the properties of this morphism. Result 7.12 implies that the general fiber of π is an affine line. More precisely:

7.13**. Corollary.** *Let B be a domain containing $\mathbb{Q}$ and let $A \in \text{KLND}(B)$. Consider the map $\pi : \text{Spec}\, B \to \text{Spec}\, A$ determined by $A \hookrightarrow B$.*

Then there exists a dense open set $U \subseteq \text{Spec}\, A$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \cong\ } & U \times \mathbb{A}^1 \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \text{projection}} & \\ & U & \end{array}$$

In particular, for each $P \in U$, the fiber $\pi^{-1}(P)$ is $\mathbb{A}^1_{\kappa(P)}$, where $\kappa(P)$ is the residue field of $\text{Spec}\, A$ *at the point P.*

Proof. Choose $D \in \operatorname{LND}(B)$ such that $\ker D = A$. Since $D \neq 0$, there exists a preslice $s \in B$ of D. Let $\alpha = D(s) \in A \setminus \{0\}$ and define

$$U = \operatorname{Spec} A \setminus V(\alpha).$$

As A is a domain and $\alpha \neq 0$, U is dense in $\operatorname{Spec}(A)$. We have:

$$\begin{array}{ccc} B & \longrightarrow & B_\alpha \\ \uparrow & & \uparrow \\ A & \longrightarrow & A_\alpha \end{array} \quad \Longleftrightarrow \quad \begin{array}{ccccc} \operatorname{Spec} B & \hookleftarrow & \pi^{-1}(U) & \xleftarrow{\cong} & \operatorname{Spec} B_\alpha \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow \\ \operatorname{Spec} A & \hookleftarrow & U & \xleftarrow{\cong} & \operatorname{Spec} A_\alpha \end{array}$$

As $B_\alpha = (A_\alpha)^{[1]}$ by 7.12, we also have

$$\begin{array}{ccc} \operatorname{Spec}(B_\alpha) & \xrightarrow{\cong} & \operatorname{Spec}(A_\alpha) \times \mathbb{A}^1 \\ & \searrow \quad \swarrow{\scriptstyle \text{projection}} & \\ & \operatorname{Spec}(A_\alpha) & \end{array}$$

so we obtain the desired conclusion. □

7.14**. Example.** Let $B = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$ and $D \in \operatorname{Der}_{\mathbb{C}}(B)$ defined by $D(X) = 0$, $D(Y) = X$ and $D(Z) = -2Y$. Then D is triangular, so $D \in \operatorname{LND}(B)$. Let $A = \ker(D)$.

Observe that $D(B)$ is included in the ideal (X, Y) of B, so in particular D does not have a slice. However Y is a preslice of D, since $D(Y) = X \neq 0$ and $D^2(Y) = 0$. Then, according to 7.12, we have $B_X = (A_X)[Y] = (A_X)^{[1]}$. We would like to study the morphism $\pi : \operatorname{Spec} B \to \operatorname{Spec} A$ determined by $A \hookrightarrow B$, but for that we need to know exactly what A is. We claim:

$$A = \mathbb{C}[X, XZ + Y^2]$$

but we omit the proof ("$\supseteq$" is easy, "$\subseteq$" is more difficult). Observe that $A \cong \mathbb{C}^{[2]}$, since it is a $\mathbb{C}$-algebra generated by two elements, and these two generators are algebraically independent over $\mathbb{C}$. So we have $\operatorname{Spec}(B) = \mathbb{A}^3$, $\operatorname{Spec}(A) = \mathbb{A}^2$ and

$$\pi : \mathbb{A}^3 \to \mathbb{A}^2, \qquad \pi(x, y, z) = (x, xz + y^2).$$

Now it is easy to calculate $\pi^{-1}(a, b)$ for any $(a, b) \in \mathbb{C}^2$, and we find:

$$\pi^{-1}(a, b) = \begin{cases} \text{an affine line,} & \text{if } a \neq 0 \\ \text{a union of two affine lines,} & \text{if } a = 0 \text{ and } b \neq 0 \\ \text{a nonreduced scheme,} & \text{if } (a, b) = (0, 0). \end{cases}$$

Also observe that the open subset $U \subseteq \operatorname{Spec} A$ of 7.13 is the set $\{X \neq 0\}$ in $\mathbb{A}^2$ (see how U is obtained in the proof of 7.13).

As a final remark, note that $B \neq A^{[1]}$. Indeed, if B were a polynomial ring in one variable over A then *every* fiber of π would be an affine line, which is not the case. (In particular, note the following obvious but important remark: $A \subset B$, $A = \Bbbk^{[2]}$, $B = \Bbbk^{[3]} \not\Longrightarrow B = A^{[1]}$.)

8. Locally nilpotent derivations and automorphisms

If B is not an integral domain, it may happen that a nonzero polynomial $f(T) \in B[T]$ have infinitely many roots in B. However note the following fact, which is needed in the proof of 8.3, below:

8.1**. Lemma.** *Let B be a ring and $f(T) \in B[T]$, where T is an indeterminate. If there exists a field $K \subseteq B$ which contains infinitely many roots of $f(T)$, then $f(T) = 0$.*

Proof. By induction on $\deg_T(f)$. The result is trivial if $\deg_T(f) \le 0$, so assume that $\deg_T(f) > 0$. Pick $a \in K$ such that $f(a) = 0$; since $T - a \in B[T]$ is a monic polynomial, $f(T) = (T - a)g(T)$ for some $g(T) \in B[T]$ such that $\deg_T(g) < \deg_T(f)$. If $b \in K \setminus \{a\}$ is such that $f(b) = 0$, then $(b - a)g(b) = 0$ and $b - a \in B^*$, so $g(b) = 0$. So $g(b) = 0$ holds for infinitely many $b \in K$ and, by the inductive hypothesis, $g(T) = 0$. It follows that $f(T) = 0$. □

We have seen that the subset $\text{LND}(B)$ of $\text{Der}(B)$ is usually not closed under addition. However:

8.2**. Lemma.** *Let B be a ring. If $D_1, D_2 \in \text{LND}(B)$ satisfy $D_2 \circ D_1 = D_1 \circ D_2$, then $D_1 + D_2 \in \text{LND}(B)$.*

Proof. Let $D_1, D_2 \in \text{LND}(B)$ such that $D_2 \circ D_1 = D_1 \circ D_2$ and let $b \in B$. Choose $m, n \in \mathbb{N}$ such that $D_1^m(b) = 0 = D_2^n(b)$. The hypothesis $D_2 \circ D_1 = D_1 \circ D_2$ has the following three consequences:

$$\forall_{i \in \mathbb{N}}\ \forall_{j \ge n}\quad (D_1^i \circ D_2^j)(b) = D_1^i(0) = 0,$$

$$\forall_{i \ge m}\ \forall_{j \in \mathbb{N}}\quad (D_1^i \circ D_2^j)(b) = (D_2^j \circ D_1^i)(b) = D_2^j(0) = 0,$$

$$(D_1 + D_2)^{m+n-1} = \textstyle\sum_{i+j=m+n-1} \binom{m+n-1}{i} D_1^i \circ D_2^j,$$

so $(D_1 + D_2)^{m+n-1}(b) = 0$. Hence, $D_1 + D_2 \in \text{LND}(B)$. □

If $\theta : B \to B$ is an automorphism of a ring B, then the set $B^\theta = \{ b \in B \mid \theta(b) = b \}$ is a subring of B called the *fixed ring* of θ. The following is another consequence of 3.3.

8.3**. Proposition.** *Let B be a $\mathbb{Q}$-algebra. Given $D \in \text{LND}(B)$, define the map*

$$\exp(D) : B \to B, \qquad b \longmapsto \textstyle\sum_{n \in \mathbb{N}} \frac{D^n(b)}{n!}.$$

(a) $\exp(D)$ *is an automorphism of the $\mathbb{Q}$-algebra B*

(b) *the fixed ring $B^{\exp(D)} = \{ b \in B \mid \exp(D)(b) = b \}$ is equal to* $\ker(D)$

(c) *if $D_1, D_2 \in \text{LND}(B)$ are such that $D_2 \circ D_1 = D_1 \circ D_2$, then $D_1 + D_2 \in \text{LND}(B)$ and*

$$\exp(D_1 + D_2) = \exp(D_1) \circ \exp(D_2) = \exp(D_2) \circ \exp(D_1)$$

Proof. If $D \in \text{LND}(B)$ then $\exp(D)$ is equal to the composite map $B \xrightarrow{\xi_D} B[T] \xrightarrow{e_1} B$, where ξ_D is defined in 3.2 and where e_1 is the evaluation homomorphism at $T = 1$, i.e., $e_1(f) = f(1)$. Since ξ_D is a ring homomorphism by 3.3, $\exp(D)$ is a ring homomorphism.

As any ring homomorphism $B \to B$ is in fact a $\mathbb{Q}$-homomorphism, it follows that $\exp(D)$ is a homomorphism of $\mathbb{Q}$-algebras. Before proving that $\exp(D)$ is bijective, we prove assertion (c).

Consider $D_1, D_2 \in \text{LND}(B)$ such that $D_2 \circ D_1 = D_1 \circ D_2$. By 8.2, $D_1 + D_2 \in \text{LND}(B)$ so it makes sense to consider the ring homomorphism $\exp(D_1 + D_2) : B \to B$. As an abbreviation, we write $\epsilon_i = \exp(D_i)$ for $i = 1, 2$. If $b \in B$,

$$(\epsilon_1 \circ \epsilon_2)(b) = \epsilon_1 \left(\sum_{j \in \mathbb{N}} \frac{D_2^j(b)}{j!} \right) = \sum_{j \in \mathbb{N}} \frac{1}{j!} \epsilon_1\big(D_2^j(b)\big) = \sum_{j \in \mathbb{N}} \frac{1}{j!} \left(\sum_{i \in \mathbb{N}} \frac{D_1^i\big(D_2^j(b)\big)}{i!} \right)$$
$$= \sum_{i,j \in \mathbb{N}} \frac{(D_1^i \circ D_2^j)(b)}{i!j!} = \sum_{n \in \mathbb{N}} \frac{1}{n!} \sum_{i+j=n} \binom{n}{i} (D_1^i \circ D_2^j)(b).$$

Since $D_2 \circ D_1 = D_1 \circ D_2$, we have $(D_1 + D_2)^n = \sum_{i+j=n} \binom{n}{i} D_1^i \circ D_2^j$ for each $n \in \mathbb{N}$ and consequently

$$(\epsilon_1 \circ \epsilon_2)(b) = \sum_{n \in \mathbb{N}} \frac{1}{n!} (D_1 + D_2)^n(b) = \exp(D_1 + D_2)(b).$$

So $\exp(D_1) \circ \exp(D_2) = \exp(D_1 + D_2)$, and since $D_1 + D_2 = D_2 + D_1$ it follows that $\exp(D_1) \circ \exp(D_2) = \exp(D_2) \circ \exp(D_1)$, so assertion (c) is proved.

Consider $D \in \text{LND}(B)$. Since $(-D) \circ D = D \circ (-D)$, part (c) implies $\exp(D) \circ \exp(-D) = \exp(-D) \circ \exp(D) = \exp(0) = \text{id}_B$, so $\exp(D)$ is bijective and the proof of (a) is complete.

It is clear that $\ker(D) \subseteq B^{\exp(D)}$. To prove the reverse inclusion, consider $b \in B$ such that $\exp(D)(b) = b$. Then for every integer $n > 0$ we have

$$b = (\exp D)^n(b) = \exp(nD)(b) = \sum_{j=0}^{\infty} \frac{1}{j!} (nD)^j(b) = \sum_{j=0}^{\infty} \frac{1}{j!} D^j(b) n^j = b + f(n),$$

where we define $f(T) \in B[T]$ by $f(T) = \sum_{j=1}^{\infty} \frac{1}{j!} D^j(b) T^j$. As $\mathbb{Q} \subseteq B$ and $\mathbb{Q}$ contains infinitely many roots of $f(T)$, we have $f(T) = 0$ by 8.1, so in particular $D(b) = 0$, and we have shown that $B^{\exp(D)} \subseteq \ker(D)$. So (b) is proved. □

8.4. **Exercise.** Let $B = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$ and $D \in \text{Der}_{\mathbb{C}}(B)$ defined by $D(X) = 0$, $D(Y) = X$ and $D(Z) = -2Y$. Then D is triangular, so $D \in \text{LND}(B)$ and we may consider the $\mathbb{Q}$-automorphism $\exp(D) : B \to B$. Note that $\exp(D)$ is actually a $\mathbb{C}$-automorphism, because $\mathbb{C} \subset \ker(D) = B^{\exp(D)}$. Compute the images of X, Y and Z by $\exp(D)$.

8.5. Let B be a domain containing a field $\mathbb{k}$ of characteristic zero. Note that if $D \in \text{LND}(B)$ then $\exp(D) : B \to B$ is a $\mathbb{k}$-automorphism of B (because $\mathbb{k} \subseteq \ker(D) = B^{\exp(D)}$). So we have a well-defined set map,

$$\text{LND}(B) \longrightarrow \text{Aut}_{\mathbb{k}}(B), \qquad D \longmapsto \exp(D).$$

Of course this map is not a homomorphism, since $\text{LND}(B)$ is only a set. Consider the subgroup $\langle E \rangle$ of $\text{Aut}_{\mathbb{k}}(B)$ generated by the set $E = \{\exp(D) \mid D \in \text{LND}(B)\}$.

8.6. **Lemma.** *Let B be a domain containing a field $\mathbb{k}$ of characteristic zero and consider the subgroup $\langle E \rangle$ of $\text{Aut}_{\mathbb{k}}(B)$ generated by the set $E = \{\exp(D) \mid D \in \text{LND}(B)\}$. Then $\langle E \rangle$ is a normal subgroup of $\text{Aut}_{\mathbb{k}}(B)$.*

Proof. If $\theta \in \mathrm{Aut}_{\Bbbk}(B)$ and $D \in \mathrm{LND}(B)$, then $\theta^{-1} \circ D \circ \theta \in \mathrm{Der}(B)$ and $(\theta^{-1} \circ D \circ \theta)^n = \theta^{-1} \circ D^n \circ \theta$, so $\theta^{-1} \circ D \circ \theta \in \mathrm{LND}(B)$. It is easily verified that $\theta^{-1} \circ \exp(D) \circ \theta = \exp(\theta^{-1} \circ D \circ \theta)$, so $\theta^{-1} E \theta \subseteq E$ holds for all $\theta \in \mathrm{Aut}_{\Bbbk}(B)$. It follows that $\langle E \rangle \lhd \mathrm{Aut}_{\Bbbk}(B)$. $\square$

It is interesting to ask how big $\langle E \rangle$ is, or how small $\mathrm{Aut}_{\Bbbk}(B)/\langle E \rangle$ is. This of course depends on B. For instance, if $B = \Bbbk^{[n]}$ and $n > 2$ then it is an open problem to determine the structure of the group $\mathrm{Aut}_{\Bbbk}(B)$, and it is believed that $\langle E \rangle$ is almost all of $\mathrm{Aut}_{\Bbbk}(B)$ (it is conjectured that $\mathrm{Aut}_{\Bbbk}(B)$ is generated by its subgroups $\langle E \rangle$ and $GL_n(\Bbbk)$). On the other hand if B is a rigid ring then $\langle E \rangle = \{1\}$.

8.7. **Exercise.** Let B be a domain containing a field $\Bbbk$ of characteristic zero. Fix a derivation $D \in \mathrm{LND}(B)$ and consider the map

$$\Bbbk \longrightarrow \mathrm{Aut}_{\Bbbk}(B), \qquad \lambda \longmapsto \exp(\lambda D).$$

Show that this is a group homomorphism $(\Bbbk, +) \to \mathrm{Aut}_{\Bbbk}(B)$. Show that this homomorphism is injective whenever $D \neq 0$.

8.8. **Exercise.** Let B be a domain containing a field $\Bbbk$ of characteristic zero. Fix a derivation $D \in \mathrm{LND}(B)$ and consider the map

$$\Bbbk \times B \longrightarrow B, \qquad (\lambda, b) \longmapsto \lambda \oplus b,$$

where we define $\lambda \oplus b = \exp(\lambda D)(b)$ (so the operation $\oplus$ depends on the choice of D). Show that this is an action of the group $(\Bbbk, +)$ on the $\Bbbk$-algebra B, i.e., verify the following conditions:

- $0 \oplus b = b$ for all $b \in B$
- $(\lambda_1 + \lambda_2) \oplus b = \lambda_1 \oplus (\lambda_2 \oplus b)$ for all $\lambda_1, \lambda_2 \in \Bbbk$ and all $b \in B$
- for each $\lambda \in \Bbbk$, the map $B \to B$, $b \mapsto \lambda \oplus b$, is an automorphism of B as a $\Bbbk$-algebra.

9. Locally nilpotent derivations and $\mathbb{G}_a$-actions

Throughout this section, $\Bbbk$ is an algebraically closed field of characteristic zero and $\mathbb{G}_a(\Bbbk)$, or simply $\mathbb{G}_a$, denotes the group $(\Bbbk, +)$ viewed as an algebraic group. Let X be an affine algebraic variety over $\Bbbk$ and let B be the coordinate algebra of X (or if you prefer, let B be an integral domain which is finitely generated as a $\Bbbk$-algebra, and let $X = \mathrm{Spec}(B)$).[1]

9.1. **Definition.** An *action* of $\mathbb{G}_a(\Bbbk)$ on X (also called a $\mathbb{G}_a$-action on X) is a morphism $\alpha : \Bbbk \times X \to X$ of $\Bbbk$-varieties satisfying:

(1) $\alpha(0, x) = x$ for all $x \in X$

(2) $\alpha(a + b, x) = \alpha(a, \alpha(b, x))$ for all $a, b \in \Bbbk$ and $x \in X$.

In other words, a $\mathbb{G}_a$-action on X is a morphism $\alpha : \Bbbk \times X \to X$ satisfying:

(1+2) The map $a \mapsto \alpha(a, \_\_)$ is a group homomorphism $(\Bbbk, +) \to \mathrm{Aut}_{\Bbbk}(X)$.

[1] The theory could be developed in the more general setting where $\Bbbk$ is any $\mathbb{Q}$-algebra and B is any $\Bbbk$-algebra (see for instance [1]).

See 8.3 for the definition of $\exp(D) : B \to B$, where $D \in \text{LND}(B)$. In the following, this $\Bbbk$-automorphism is conveniently denoted $e^D : B \to B$.

9.2**.** We proceed to define a set map

$$\begin{array}{rcl} \text{LND}(B) & \longrightarrow & \text{set of actions of } \mathbb{G}_a(\Bbbk) \text{ on } \operatorname{Spec}(B). \\ D & \longmapsto & \alpha_D \end{array} \tag{4}$$

Let $D \in \text{LND}(B)$. By Exercise 8.7, we have the group homomorphism

$$(\Bbbk, +) \longrightarrow \operatorname{Aut}_{\Bbbk}(B), \qquad \lambda \longmapsto e^{\lambda D};$$

applying the functor Spec, we obtain the group homomorphism

$$(\Bbbk, +) \longrightarrow \operatorname{Aut}_{\Bbbk}(\operatorname{Spec} B), \qquad \lambda \longmapsto \operatorname{Spec}\big(e^{\lambda D}\big).$$

Then define the following map:

$$\alpha_D \;:\; \Bbbk \times \operatorname{Spec} B \longrightarrow \operatorname{Spec} B, \qquad (\lambda, x) \longmapsto \big(\operatorname{Spec} e^{\lambda D}\big)(x).$$

To conclude that α_D is an action of $\mathbb{G}_a$ on $\operatorname{Spec}(B)$, there only remains to verify that it is a morphism in the sense of algebraic geometry. Note that we may identify $\Bbbk \times \operatorname{Spec} B$ with $\operatorname{Spec}\big(\Bbbk[T] \otimes_{\Bbbk} B\big) = \operatorname{Spec}(B[T])$ where T is an indeterminate. By 3.3, D determines the homomorphism of $\Bbbk$-algebras $\xi_D : B \to B[T]$, $\xi_D(b) = \sum_{j \in \mathbb{N}} \frac{D^j b}{j!} T^j$, and one can verify that $\operatorname{Spec}(\xi_D) = \alpha_D$; so α_D is a morphism and hence an action.
This shows that (4) is a well-defined map.

9.3**. Theorem.** *The set map*

$$\begin{array}{rcl} \text{LND}(B) & \longrightarrow & \textit{set of actions of } \mathbb{G}_a(\Bbbk) \textit{ on } \operatorname{Spec}(B), \\ D & \longmapsto & \alpha_D \end{array}$$

defined in 9.2*, is bijective.*

We refer to [1] for the proof of 9.3. Note that, under this bijection, the zero derivation corresponds to the trivial action. So B is rigid (see 7.7) if and only if the only $\mathbb{G}_a$-action on $X = \operatorname{Spec}(B)$ is the trivial one.

9.4**. Example.** Let $B = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$ and $D = X\frac{\partial}{\partial Y} + (Y^2 + XY)\frac{\partial}{\partial Z} \in \text{LND}(B)$. Then D determines an action $\alpha_D : \mathbb{C} \times \mathbb{C}^3 \to \mathbb{C}^3$ which we now compute (note that we are identifying $\operatorname{Spec}(B)$ with $\mathbb{C}^3$). We have

$$\begin{aligned} e^{\lambda D}(Z) = \textstyle\sum_{n=0}^{\infty} \frac{(\lambda D)^n(Z)}{n!} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} D^n(Z) \\ = Z + \lambda(Y^2 + XY) + \frac{\lambda^2}{2}(2XY + X^2) + \frac{\lambda^3}{6}(2X^2), \end{aligned}$$

and similarly $e^{\lambda D}(X) = X$ and $e^{\lambda D}(Y) = Y + (\lambda D)(Y) = Y + \lambda X$. So, given $\lambda \in \mathbb{C}$ and $(x, y, z) \in \mathbb{C}^3$,

$$\alpha_D \;:\; \big(\lambda, (x, y, z)\big) \;\longmapsto\; \Big(x,\; y + \lambda x,\; z + \lambda(y^2 + xy) + \frac{\lambda^2}{2}(2xy + x^2) + \frac{\lambda^3}{3}x^2\Big).$$

Our next objective is to describe the fixed points and the ring of invariants of a $\mathbb{G}_a$-action. Let us first define these notions.

9.5. **Definition.** Suppose that $\alpha : G \times X \to X$, $(g, x) \mapsto gx$, is an action of an algebraic group G on $X = \mathrm{Spec}(B)$.

(1) A *fixed point* of α is a closed point $x \in X$ satisfying $\forall_{g\in G}\ gx = x$.
(2) The action α determines an action
$$G \times B \longrightarrow B, \qquad (g, b) \longmapsto gb,$$
of G on B, which, in turn, determines the subring $B^G = \{ b \in B \mid \forall_{g\in G}\ gb = b \}$ of B. We call B^G the *ring of invariants* of the G-action α on X.

9.6. **Definition.** Given $D \in \mathrm{LND}(B)$, define
$$\mathrm{Fix}(D) = \{ \mathfrak{p} \in \mathrm{Spec}(B) \mid \mathfrak{p} \supseteq D(B) \} .$$
Note that $\mathrm{Fix}(D)$ is a closed subset of $\mathrm{Spec}(B)$.

9.7. **Proposition.** *Let $D \in \mathrm{LND}(B)$ and consider the $\mathbb{G}_a$-action α_D on $X = \mathrm{Spec}(B)$ defined in 9.2 and 9.3.*

(1) *The ring of invariants of α_D is the subring $\ker(D)$ of B.*
(2) *The fixed points of α_D are precisely the closed points which belong to $\mathrm{Fix}(D)$.*

Proof. The action α_D of $\mathbb{G}_a$ on X determines the following action of $\mathbb{G}_a$ on B:
$$\Bbbk \times B \longrightarrow B, \qquad (\lambda, b) \longmapsto e^{\lambda D}(b).$$
For any $b \in B$ we have
$$b \in B^{\mathbb{G}_a} \iff \forall_{\lambda\in\Bbbk}\ e^{\lambda D}(b) = b \overset{8.3}{\iff} \forall_{\lambda\in\Bbbk}\ b \in \ker(\lambda D) \iff b \in \ker(D),$$
which proves assertion (1). Assertion (2) is a corrolary of 9.8, below. $\square$

9.8. **Proposition.** *Let $\Bbbk$ and B be rings such that $\mathbb{Q} \subseteq \Bbbk \subseteq B$, and let $D \in \mathrm{LND}_{\Bbbk}(B)$. Then, for a maximal ideal $\mathfrak{m}$ of B, the following are equivalent:*

(1) *For all $\lambda \in \Bbbk$, $e^{\lambda D}(\mathfrak{m}) = \mathfrak{m}$*
(2) *$\mathfrak{m} \supseteq D(B)$.*

Proof. Suppose that (2) holds. Given $\lambda \in \Bbbk$ and $b \in \mathfrak{m}$, we have $D^j(b) \in \mathfrak{m}$ for all $j \in \mathbb{N}$, so $e^{\lambda D}(b) = \sum_{j=0}^{\infty} \frac{D^j(b)}{j!}\lambda^j \in \mathfrak{m}$; this shows that $e^{\lambda D}(\mathfrak{m}) \subseteq \mathfrak{m}$, and since $e^{\lambda D}$ is an automorphism we must have $e^{\lambda D}(\mathfrak{m}) = \mathfrak{m}$. So (2) implies (1).

Conversely, suppose that (1) holds. The first step is to prove that
$$D(\mathfrak{m}) \subseteq \mathfrak{m}. \tag{5}$$
Let $b \in \mathfrak{m}$. Define $f(T) = \sum_{j=0}^{\infty} \frac{D^j(b)}{j!} T^j \in B[T]$ and note that $f(\lambda) = e^{\lambda D}(b)$ for all $\lambda \in \Bbbk$. Since (1) holds, we have $f(\lambda) \in \mathfrak{m}$ for all $\lambda \in \Bbbk$, so in particular this holds for all $\lambda \in \mathbb{Q}$. Consider the field $\kappa = B/\mathfrak{m}$, the canonical epimorphism $\pi : B \to \kappa$ and the polynomial $f^{(\pi)} \in \kappa[T]$. Then $\mathbb{Q} \subseteq \kappa$ and $f^{(\pi)}(\lambda) = 0$ for all $\lambda \in \mathbb{Q}$; so $f^{(\pi)} = 0$, i.e., all coefficients of $f(T)$ belong to $\mathfrak{m}$. In particular $D(b) \in \mathfrak{m}$, which proves (5).

By (5), $\delta(b + \mathfrak{m}) = D(b) + \mathfrak{m}$ is a well-defined locally nilpotent derivation $\delta : \kappa \to \kappa$. By 5.5, $\delta = 0$; this means that $D(B) \subseteq \mathfrak{m}$, i.e., (2) holds. $\square$

9.9. **Exercise.** Let $B = \mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$ and $D = X\frac{\partial}{\partial Y} + (Y^2 + XY)\frac{\partial}{\partial Z} \in \text{LND}(B)$. Then D determines an action α_D on $\mathbb{C}^3$, see 9.4. Find the set of fixed points of this action.

References

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