CLASSIFICATION OF WEIGHTED GRAPHS UP TO BLOWING-UP AND BLOWING-DOWN

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Abstract. We classify weighted forests up to the blowing-up and blowing-down operations which are relevant for the study of algebraic surfaces.

The word "graph" in this text means a finite undirected graph such that no edge connects a vertex to itself and at most one edge joins any given pair of vertices. A *weighted graph* is a graph in which each vertex is assigned an integer (called its weight). Two operations are performed on weighted graphs: The blowing-up and its inverse, the blowing-down. Two weighted graphs are said to be equivalent if one can be obtained from the other by means of a finite sequence of blowings-up and blowings-down (see 1.4–1.7).

These weighted graphs and operations are well known to geometers who study algebraic surfaces. Many problems in the geometry of surfaces can be formulated in graph-theoretic terms and solving these sometimes requires elaborate graph-theoretic considerations. This gives rise to a variety of questions about weighted graphs, all in connection with the equivalence relation generated by blowing-up and blowing-down.

The present paper proposes a classification of weighted forests up to equivalence. In particular, Theorem 8.34 defines an invariant $\bar{Q}(\mathcal{G})$ for any pseudo-minimal (3.8) weighted forest $\mathcal{G}$, and asserts that $\bar{Q}(\mathcal{G}) = \bar{Q}(\mathcal{G}')$ if and only if $\mathcal{G}$ is equivalent to $\mathcal{G}'$. Since $\bar{Q}(\mathcal{G})$ can actually be computed, this yields an algorithm for deciding whether two weighted forests are equivalent (see Problem 5, at the end of section 8). Apparently this decision problem was previously open, even in the special case of "linear chains", i.e., weighted graphs of the form:

$$\overset{x_1}{\bullet}\!\!-\!\!-\!\!-\overset{x_2}{\bullet}\!\!-\!\!-\;\cdots\;-\!\!-\overset{x_q}{\bullet} \qquad (x_i \in \mathbb{Z}).$$

Note that, in the case of linear chains, 8.34 simplifies to 5.4.

We also contribute to the problem of listing all minimal elements in a given equivalence class of weighted forests. Section 9 reduces that problem to the case of linear chains. This special case is given a recursive solution in Section 7 and, in some simple cases, an explicit solution. Incidentally, the cases that we are able to describe explicitely are precisely those which arise from the study of algebraic surfaces.

Section 3 is concerned with topological properties of graphs, but topology is never mentioned. For instance, 3.6 has the following consequence:

> *Let $\mathcal{G}, \mathcal{G}'$ be weighted trees which are minimal and equivalent. If $\mathcal{G}$ is not a linear chain then the two trees are homeomorphic.*

Note, however, that if n is a positive integer then the linear chains

$$\overset{n}{\bullet} \qquad \text{and} \qquad \overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underbrace{\overset{-2}{\bullet}\!\!-\!\!\!-\ \cdots\ -\!\!\!-\!\!\overset{-2}{\bullet}}_{n-1 \text{ vertices}}$$

are minimal and equivalent, but not homeomorphic. Because of this irregularity, and for other reasons as well, the notions of skeleton and skeletal map (Section 2) are better suited than topology for our purpose. The "topological" results of section 3 are of fundamental importance for classifying weighted trees and forests (Section 8), but are not needed for the special case of linear chains. Readers only interested in that case may restrict themselves to sections 1, 4, 5 and 7.[1]

Acknowledgements. Papers [4] and [5] classify weighted forests up to an equivalence relation weaker than the one considered here (the relation is generated by blowing-up, blowing-down and other operations which are not allowed here). Result 3.2.1 of [6] classifies linear chains but, again, this is relative to a weak equivalence relation. Paper [7] uses the same equivalence relation as we do, but only classifies a restricted class of weighted trees.

Proposition 3.2 of [6] almost[2] implies the fact (4.23.1) that each linear chain is equivalent to at least one canonical chain. As we realized *a posteriori,* there is even some similarity between the cited result and our method for proving 4.23.1. We also noticed *a posteriori* a certain resemblance between our skeletons equipped with extra structure (2.1, 8.4, 8.10) and the "W-graphs" briefly described in [5].

1. Basic definitions and facts

1.1. If E is a set then $E^* = \cup_{n=0}^{\infty} E^n$ denotes the set of finite sequences in E, including the empty sequence $\varnothing \in E^*$. We write A^- for the *reversal* of $A \in E^*$, i.e., if $A = (a_1, \dots, a_n)$ then $A^- = (a_n, \dots, a_1)$.

1.2. If $\mathcal{G}$ is a weighted graph, $\operatorname{Vtx}(\mathcal{G})$ is its vertex set. If $v \in \operatorname{Vtx}(\mathcal{G})$ then $w(v, \mathcal{G})$ denotes the weight of v in $\mathcal{G}$; $\deg(v, \mathcal{G})$ denotes the degree of v in $\mathcal{G}$, that is, the number of neibhors of v. The empty graph is denoted $\boldsymbol{\varnothing}$.

1.3. A *linear chain* is a weighted tree in which every vertex has degree at most two. Given $(x_1, \dots, x_q) \in \mathbb{Z}^*$, the linear chain

$$\overset{x_1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{x_2}{\bullet}\!\!-\!\!\!-\ \cdots\ -\!\!\!-\!\!\overset{x_q}{\bullet}$$

is denoted $[x_1, \dots, x_q]$. So we distinguish between the graph $[x_1, \dots, x_q]$ and the sequence $(x_1, \dots, x_q)$ and we note that $[x_1, \dots, x_q] = [x_q, \dots, x_1]$.

An *admissible chain* is a linear chain in which every weight is strictly less than -1. The empty graph $\boldsymbol{\varnothing}$ is an admissible chain.

1.4. Let $\mathcal{G}$ be a weighted graph. We define three types of "blowing-up of $\mathcal{G}$":

[1] In fact one needs 3.3 to prove 4.3, but 3.3 can be proved independently from sections 2 and 3. Some ideas from Section 6 are used in the proofs of Section 7, but not in a very crucial way.

[2] One also needs 4.17 for the proof.

(1) If v is a vertex of $\mathcal{G}$ then the *blowing-up of* $\mathcal{G}$ *at* v is the weighted graph $\mathcal{G}'$ obtained from $\mathcal{G}$ by adding one vertex e of weight -1, adding one edge joining e to v, and decreasing the weight of v by 1. (This process is called a blowing-up "at a vertex".)
(2) If $\varepsilon = \{v_1, v_2\}$ is an edge of $\mathcal{G}$ (so v_1, v_2 are distinct vertices of $\mathcal{G}$), then the *blowing-up of* $\mathcal{G}$ *at* ε is the weighted graph $\mathcal{G}'$ obtained from $\mathcal{G}$ by adding one vertex e of weight -1, deleting the edge $\varepsilon = \{v_1, v_2\}$, adding the two edges $\{v_1, e\}$ and $\{e, v_2\}$, and decreasing the weights of v_1 and v_2 by 1. (This is called a blowing-up "at an edge", or a "subdivisional" blowing-up.)
(3) The *free blowing-up of* $\mathcal{G}$ is the weighted graph $\mathcal{G}'$ obtained by taking the disjoint union of $\mathcal{G}$ and of a vertex e of weight -1.

In each of the above three cases, we call e the vertex *created* by the blowing-up. If $\mathcal{G}'$ is a blowing-up of $\mathcal{G}$ then there is a natural way to identify $\mathrm{Vtx}(\mathcal{G})$ with a subset of $\mathrm{Vtx}(\mathcal{G}')$ (whose complement is $\{e\}$). It is understood that, whenever a blowing-up is performed, such an injective map $\mathrm{Vtx}(\mathcal{G}) \hookrightarrow \mathrm{Vtx}(\mathcal{G}')$ is chosen. We stress that if $\mathcal{G}'$ is a blowing-up of $\mathcal{G}$ and $\mathcal{G}''$ is a weighted graph isomorphic to $\mathcal{G}'$, then $\mathcal{G}''$ is a blowing-up of $\mathcal{G}$.

1.5. A vertex e of a weighted graph $\mathcal{G}'$ is said to be *contractible* if the following three conditions hold: (i) e has weight -1; (ii) e has at most two neighbors; (iii) if v_1 and v_2 are distinct neighbors of e then v_1, v_2 are not neighbors of each other.

If e is a contractible vertex of $\mathcal{G}'$ then $\mathcal{G}'$ is the blowing-up of some weighted graph $\mathcal{G}$ in such a way that e is the vertex created by this process. Up to isomorphism of weighted graphs, $\mathcal{G}$ is uniquely determined by $\mathcal{G}'$ and e. We say that $\mathcal{G}$ is obtained by *blowing-down* $\mathcal{G}'$ at e. The blowing-down is the inverse operation of the blowing-up.

1.6. A weighted graph is *minimal* if it does not have a contractible vertex.

1.7. Two weighted graphs $\mathcal{G}$ and $\mathcal{H}$ are *equivalent* (notation: $\mathcal{G} \sim \mathcal{H}$) if one can be obtained from the other by a finite sequence of blowings-up and blowings-down.

1.8. Given a weighted graph $\mathcal{G}$, consider the real vector space V with basis $\mathrm{Vtx}(\mathcal{G})$ and define a symmetric bilinear form $B_{\mathcal{G}} : V \times V \to \mathbb{R}$ by:

$$B_{\mathcal{G}}(u,v) = \begin{cases} w(u,\mathcal{G}), & \text{if } u = v \in \mathrm{Vtx}(\mathcal{G}), \\ 1, & \text{if } u, v \in \mathrm{Vtx}(\mathcal{G}) \text{ are distinct and joined by an edge,} \\ 0, & \text{if } u, v \in \mathrm{Vtx}(\mathcal{G}) \text{ are distinct and not joined by an edge.} \end{cases}$$

One calls $B_{\mathcal{G}}$ the *intersection form* of $\mathcal{G}$. Then define the natural number $\|\mathcal{G}\| = \max_W \dim W$, where W runs in the set of subspaces of V satisfying

$$\forall_{x \in W} \quad B_{\mathcal{G}}(x,x) \geq 0.$$

Note that $\|\mathcal{G}\| = 0$ iff $B_{\mathcal{G}}$ is negative definite, in which case we say that $\mathcal{G}$ is negative definite.

1.9. *For weighted graphs* $\mathcal{G}$ *and* $\mathcal{G}'$, $\mathcal{G} \sim \mathcal{G}' \implies \|\mathcal{G}\| = \|\mathcal{G}'\|$.

Proof. See for instance 1.14 of [6]. □

1.10. Consider a weighted graph $\mathcal{G}$ and its intersection form $B_{\mathcal{G}} : V \times V \to \mathbb{R}$ (see 1.8). Let $v_1, \dots, v_n$ be the distinct vertices of $\mathcal{G}$ (enumerated in any order) and let M be the $n \times n$ matrix representing $B_{\mathcal{G}}$ with respect to the basis $(v_1, \dots, v_n)$ of V. That is, $M_{ii} = w(v_i, \mathcal{G})$ and, if $i \neq j$, $M_{ij} = 1$ (resp. 0) if v_i, v_j are neighbors (resp. are not neighbors) in $\mathcal{G}$. Note that $\det(-M)$ is independent of the choice of an ordering for $\mathrm{Vtx}(\mathcal{G})$. One defines the *determinant* of the weighted graph $\mathcal{G}$ by:

$$\det(\mathcal{G}) = \det(-M).$$

Note that $\det(\mathcal{G}) \in \mathbb{Z}$. By convention, $\det(\boldsymbol{\varnothing}) = 1$.

The following is well-known, and easily verified:

1.11. *For weighted graphs $\mathcal{G}$ and $\mathcal{G}'$, $\mathcal{G} \sim \mathcal{G}' \implies \det(\mathcal{G}) = \det(\mathcal{G}')$.*

Remark. Without the minus sign in $\det(-M)$, 1.11 would only be true up to sign.

2. Skeletons and skeletal maps

In this section, all graphs are forests and (except in 2.8) no graph is weighted.

2.1. **Definition.** A *skeleton* is a forest which contains no vertex of degree zero or two.

2.2. **Definition.** Given a forest G, let $P(G)$ be the set of nonempty finite sequences $\gamma = (v_0, \dots, v_n)$ of vertices of G satisfying either

(1) $n = 0$ and $\deg(v_0, G) = 0$; or
(2) $n > 0$ and the following hold:
   (a) $v_0, \dots, v_n$ are distinct
   (b) $\{v_0, v_1\}, \dots, \{v_{n-1}, v_n\}$ are edges in G
   (c) $\{\, i \mid \deg(v_i, G) \neq 2 \,\} = \{0, n\}$.

Note that $P(G)$ is the empty set if and only if $G = \boldsymbol{\varnothing}$. If $\gamma = (v_0, \dots, v_n) \in P(G)$, then $\gamma^- = (v_n, \dots, v_0)$ also belongs to $P(G)$.

Each element $\gamma = (v_0, \dots, v_n)$ of $P(G)$ is of one of four types, defined as follows:

γ is of type $(-,-)$ if $\deg(v_0, G) < 2$ and $\deg(v_n, G) < 2$;
γ is of type $(+,-)$ if $\deg(v_0, G) > 2$ and $\deg(v_n, G) < 2$;
γ is of type $(-,+)$ if $\deg(v_0, G) < 2$ and $\deg(v_n, G) > 2$;
γ is of type $(+,+)$ if $\deg(v_0, G) > 2$ and $\deg(v_n, G) > 2$.

Observe that if $n = 0$ then γ is of type $(-,-)$.

2.3. **Notation.** Let G be a graph and let $\mathcal{P}$ be a condition on the degree of a vertex; then we define:

$$\mathrm{Vtx}_{\mathcal{P}}(G) = \big\{\, x \in \mathrm{Vtx}(G) \mid \deg(x, G) \text{ satisfies } \mathcal{P} \,\big\}.$$

For instance, $\mathrm{Vtx}_{\neq 2}(G) = \big\{\, x \in \mathrm{Vtx}(G) \mid \deg(x, G) \neq 2 \,\big\}$.

2.4. **Definition.** Let G and G' be forests.

(1) A *pre-skeletal map* from G to G' is a set map
$$f : \mathrm{Vtx}_{\neq 2}(G) \to \mathrm{Vtx}_{\neq 2}(G')$$
such that, given any $\gamma = (v_0, \dots, v_m) \in P(G)$, there exists an element $\gamma' = (v'_0, \dots, v'_n)$ of $P(G')$ satisfying $f(v_0) = v'_0$ and $f(v_m) = v'_n$. Then $\gamma \mapsto \gamma'$ is a well-defined map which we denote $\vec{f} : P(G) \to P(G')$. This map satisfies $\vec{f}(\gamma^-) = \vec{f}(\gamma)^-$ for all $\gamma \in P(G)$.

(2) A *skeletal map* from G to G' is a pre-skeletal map f from G to G' which satisfies the following additional conditions:
   (a) $\vec{f} : P(G) \to P(G')$ is surjective
   (b) if u, v are distinct elements of the domain of f satisfying $f(u) = f(v)$, then there exists $(v_0, \dots, v_m) \in P(G)$ such that $v_0 = u$ and $v_m = v$.

We write $f : G \dashrightarrow G'$ to indicate that f is a skeletal map from G to G'. The composition of two skeletal maps is a skeletal map. If G is a forest then the identity map on the set $\mathrm{Vtx}_{\neq 2}(G)$ is a skeletal map from G to itself.

We leave it to the reader to verify:

2.5. *If $f : G \dashrightarrow G'$ (where G and G' are forests) then $\vec{f} : P(G) \to P(G')$ preserves type. That is, if $\gamma \in P(G)$ then γ and $\vec{f}(\gamma)$ are of the same type (see* 2.2*).*

2.6. **Lemma.** (1) *Suppose that $G \overset{f}{\dashrightarrow} G' \overset{\sigma}{\dashleftarrow} S$ are skeletal maps, where G, G' are forests and S is a skeleton. Then σ factors through f, i.e., there exists $\theta : S \dashrightarrow G$ such that $f \circ \theta = \sigma$.*

(2) *If S and S' are skeletons and $f : S \dashrightarrow S'$ is a skeletal map then f is actually an isomorphism of graphs.*

Proof. Let G, G', S, f and σ be as in statement (1). The definition of skeletal map implies that $f : \mathrm{Vtx}_{\neq 2}(G) \to \mathrm{Vtx}_{\neq 2}(G')$ is surjective and is almost injective:

(i) If $z \in \mathrm{Vtx}_{\neq 2}(G')$ is such that $|f^{-1}(z)| > 1$, then $\deg(z, G') = 0$ and $|f^{-1}(z)| = 2$.
(ii) If $z \in \mathrm{Vtx}_{=0}(G')$ is such that $f^{-1}(z) = \{v\}$, then $\deg(v, G) = 0$.

Of course, σ has similar properties. So, if we define $Z = \mathrm{Vtx}_{=0}(G')$, f and σ restrict to bijections:
$$f^{-1}\big(\mathrm{Vtx}_{\neq 2}(G') \setminus Z\big) \xrightarrow{f_1} \mathrm{Vtx}_{\neq 2}(G') \setminus Z \xleftarrow{\sigma_1} \sigma^{-1}\big(\mathrm{Vtx}_{\neq 2}(G') \setminus Z\big),$$
so we may define the bijection $\theta_1 = f_1^{-1} \circ \sigma_1 : \sigma^{-1}\big(\mathrm{Vtx}_{\neq 2}(G') \setminus Z\big) \to f^{-1}\big(\mathrm{Vtx}_{\neq 2}(G') \setminus Z\big)$. Moreover, for each $z \in Z$ we have $|\sigma^{-1}(z)| = 2$ and $|f^{-1}(z)| \in \{1, 2\}$, so we may define a surjection $\theta_z : \sigma^{-1}(z) \to f^{-1}(z)$. Gluing θ_1 with the various θ_z gives a surjection θ from $\mathrm{Vtx}_{\neq 2}(S) = \mathrm{Vtx}(S)$ to $\mathrm{Vtx}_{\neq 2}(G)$ satisfying $f \circ \theta = \sigma$; it is easily verified that θ is a skeletal map, $\theta : S \dashrightarrow G$, so assertion (1) is true.

Let $f : S \dashrightarrow S'$ be as in assertion (2). By the above properties (i) and (ii), it follows that f is a bijection $\mathrm{Vtx}(S) \to \mathrm{Vtx}(S')$; consequently, $\vec{f} : P(S) \to P(S')$ is bijective. Since S is a skeleton, $P(S)$ is exactly the set of ordered pairs (u, v) such that $\{u, v\}$

is an edge of S (and similarly for $P(S')$). So the bijectivity of $\vec{f}$ implies that f is an isomorphism of graphs, which proves (2). $\square$

2.7. **Lemma.** *Given a forest G, there exist a skeleton S and a skeletal map $\sigma : S \dashrightarrow G$. Moreover, the pair (S, σ) is unique up to isomorphism of graphs, i.e., if S, S' are skeletons and $S \overset{\sigma}{\dashrightarrow} G \overset{\sigma'}{\dashleftarrow} S'$ are skeletal maps then there exists an isomorphism of graphs $\theta : S' \to S$ such that $\sigma' = \sigma \circ \theta$.*

Proof. To prove uniqueness, consider skeletal maps $S \overset{\sigma}{\dashrightarrow} G \overset{\sigma'}{\dashleftarrow} S'$ where S and S' are skeletons. Part (1) of 2.6 gives $\sigma' = \sigma \circ \theta$ for some $\theta : S' \dashrightarrow S$; then θ is an isomorphism of graphs, by assertion (2) of 2.6. We prove the existence of (S, σ) by induction on $|\operatorname{Vtx}_{\in\{0,2\}}(G)|$; it suffices to show that if $|\operatorname{Vtx}_{\in\{0,2\}}(G)| > 0$ then there exists a pair (G', σ) where G' is a forest satisfying $|\operatorname{Vtx}_{\in\{0,2\}}(G')| < |\operatorname{Vtx}_{\in\{0,2\}}(G)|$ and $\sigma : G' \dashrightarrow G$ is a skeletal map.

If v is a vertex of degree two in G with neighbors v_1 and v_2, then remove v and the edges $\{v_1, v\}$ and $\{v, v_2\}$ and add the edge $\{v_1, v_2\}$; let G' be the resulting graph and note that $\operatorname{Vtx}_{\neq 2}(G) = \operatorname{Vtx}_{\neq 2}(G')$ and that the identity map of $\operatorname{Vtx}_{\neq 2}(G)$ is a skeletal map $G' \dashrightarrow G$. Clearly, $|\operatorname{Vtx}_{\in\{0,2\}}(G')| < |\operatorname{Vtx}_{\in\{0,2\}}(G)|$.

If v is a vertex of degree zero in G, then add a new vertex v^* and an edge $\{v, v^*\}$; let G' be the resulting graph and note that $\operatorname{Vtx}_{\neq 2}(G') = \operatorname{Vtx}_{\neq 2}(G) \cup \{v^*\}$. Define a map $\sigma : \operatorname{Vtx}_{\neq 2}(G') \to \operatorname{Vtx}_{\neq 2}(G)$ by $\sigma(v^*) = v$ and, for all $u \in \operatorname{Vtx}_{\neq 2}(G)$, $\sigma(u) = u$. Then $\sigma : G' \dashrightarrow G$ is a skeletal map and, again, $|\operatorname{Vtx}_{\in\{0,2\}}(G')| < |\operatorname{Vtx}_{\in\{0,2\}}(G)|$. $\square$

So far, we defined skeletal maps for forests which are not weighted. We need that notion in the weighted case as well:

2.8. **Definition.** By a *skeletal map from $\mathcal{G}$ to $\mathcal{G}'$*, where $\mathcal{G}$ and $\mathcal{G}'$ are forests which may or may not be weighted, we mean a skeletal map from the underlying graph of $\mathcal{G}$ to the underlying graph of $\mathcal{G}'$. The symbol $f : \mathcal{G} \dashrightarrow \mathcal{G}'$ means that f is a skeletal map from $\mathcal{G}$ to $\mathcal{G}'$. If $\mathcal{G}$ is a weighted forest then, by 2.7, there exists a pair (S, σ) where S is a skeleton (so S is not weighted) and $\sigma : S \dashrightarrow \mathcal{G}$ is a skeletal map, and moreover (S, σ) is unique up to isomorphism of graphs. By *the skeleton of a weighted forest* $\mathcal{G}$, we mean a skeleton S such that there exists a skeletal map $S \dashrightarrow \mathcal{G}$. So the skeleton of $\mathcal{G}$ is not weighted.

3. Strict equivalence of weighted graphs

3.1. **Definition.** Let $\mathcal{G}$ and $\mathcal{G}'$ be weighted graphs, where $\mathcal{G}'$ is a blowing-up of $\mathcal{G}$. If $\mathcal{G}'$ is either a subdivisional blowing-up of $\mathcal{G}$, or a blowing-up of $\mathcal{G}$ at a vertex v such that $\deg(v, \mathcal{G}) \leq 1$, we say that $\mathcal{G}'$ is a *strict blowing-up* of $\mathcal{G}$ and that $\mathcal{G}$ is a *strict blowing-down* of $\mathcal{G}'$. Two weighted graphs are *strictly equivalent* if one can be transformed into the other by a sequence of strict blowings-up and strict blowings-down.

3.2. **Theorem.** *For minimal weighted graphs, equivalence implies strict equivalence.*

Although this result is stated and proved for general weighted graphs, we will only use it on forests. The result is of fundamental importance for the remainder of this paper because it allows us to restrict our attention to strict equivalence, which has the effect (as we will see) of fixing the skeleton. See also 3.9, 3.10. For the proof of 3.2 we need:

3.2.1. **Lemma.** *Consider a sequence* $(\mathcal{G}_0, \dots, \mathcal{G}_n)$ *of weighted graphs satisfying:*

(1) *There exists an integer* m *satisfying* $0 < m < n$ *and such that* $\mathcal{G}_i$ *is a blowing-up (resp. blowing-down) of* $\mathcal{G}_{i-1}$ *for all* i *such that* $0 < i \le m$ *(resp.* $m < i \le n$*);*
(2) $\mathcal{G}_1$ *is a blowing-up of* $\mathcal{G}_0$ *at a vertex* v*;*
(3) $v \in \mathrm{Vtx}(\mathcal{G}_n)$ *and* $\deg(v, \mathcal{G}_{n-1}) > \deg(v, \mathcal{G}_n)$.

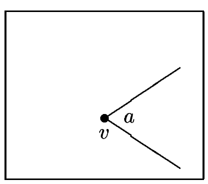
Then there exists a sequence of blowings-up and blowings-down which transforms $\mathcal{G}_0$ *into* $\mathcal{G}_n$ *in fewer than* n *operations.*

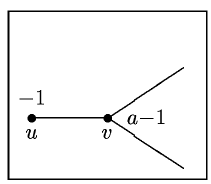
Regarding the statement of 3.2.1, two remarks are in order. First, $(\mathcal{G}_0, \dots, \mathcal{G}_n)$ is a sequence of blowings-up and blowings-down which transforms $\mathcal{G}_0$ into $\mathcal{G}_n$ in exactly n operations; the lemma claims that there is a shorter sequence achieving the same thing, but not necessarely satisfying (1–3). Secondly, the blowings-up come with injective maps

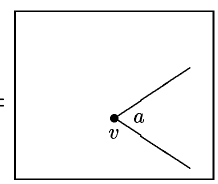
$$\mathrm{Vtx}(\mathcal{G}_{i-1}) \hookrightarrow \mathrm{Vtx}(\mathcal{G}_i) \quad (0 < i \le m) \qquad \text{and} \qquad \mathrm{Vtx}(\mathcal{G}_{i-1}) \hookleftarrow \mathrm{Vtx}(\mathcal{G}_i) \quad (m < i \le n)$$

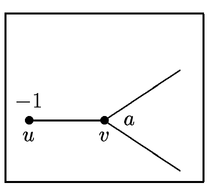
which allow us to identify every set $\mathrm{Vtx}(\mathcal{G}_j)$ with a subset of $\mathrm{Vtx}(\mathcal{G}_m)$; so the statement of condition (3) makes sense.

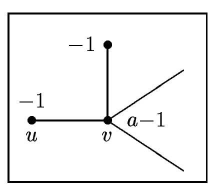
Proof of 3.2.1. The graph $\mathcal{G}_n$ is the blowing-down of $\mathcal{G}_{n-1}$ at some vertex u; since this decreases the degree of v, $\{u\}$ is in fact a branch of $\mathcal{G}_{n-1}$ at v. We proceed by induction on n. The case $n = 2$ is either as in (1) or as in (2):

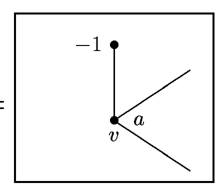
(1) $\mathcal{G}_0 =$ 

$\mathcal{G}_1 =$ 

$\mathcal{G}_2 =$ 

(2) $\mathcal{G}_0 =$ 

$\mathcal{G}_1 =$ 

$\mathcal{G}_2 =$ 

In (1), $\mathcal{G}_2 = \mathcal{G}_0$; in (2), $\mathcal{G}_2$ is isomorphic to $\mathcal{G}_0$; so in all cases $\mathcal{G}_0$ can be transformed into $\mathcal{G}_2$ in zero steps. Hence, the case $n = 2$ is true.

Let $n > 2$ and assume that the result is true for smaller values of n. Let $\mathcal{B}$ be the branch of $\mathcal{G}_m$ at v such that $u \in \mathrm{Vtx}(\mathcal{B})$. Define vertices $e_1, \dots, e_m$ by writing $\mathrm{Vtx}(\mathcal{G}_i) = \mathrm{Vtx}(\mathcal{G}_{i-1}) \cup \{e_i\}$ for $1 \le i \le m$. Consider the set $E = \{e_1, \dots, e_m\} \cap \mathrm{Vtx}(\mathcal{B})$.

If $E \neq \varnothing$ then there exists $e_j \in E$ which also satisfies $w(e_j, \mathcal{G}_m) = -1$ (namely, let $j = \max\{\, i \mid e_i \in E \,\}$). Since every vertex of $\mathcal{B}$ disappears in the blowing-down process from $\mathcal{G}_m$ to $\mathcal{G}_n$, we may consider k such that $m < k \le n$ and such that $\mathcal{G}_k$ is the blowing-down of $\mathcal{G}_{k-1}$ at e_j. Because $w(e_j, \mathcal{G}_m) = -1$, it follows that

$$w(e_j, \mathcal{G}_i) = -1 \textit{ for all } i \textit{ such that } j \le i < k. \tag{3}$$

Since e_j is created by a blowing-up and later deleted by a blowing-down, and since (3) holds, it follows that these two steps, the creation and deletion of e_j, can be omitted. So $\mathcal{G}_0$ can be transformed into $\mathcal{G}_n$ in $n-2$ steps and we are done.

So we may assume that $E = \varnothing$. Since $\operatorname{Vtx}(\mathcal{G}_m) = \operatorname{Vtx}(\mathcal{G}_0) \cup \{e_1, \dots, e_m\}$, this implies that $\operatorname{Vtx}(\mathcal{B}) \subset \operatorname{Vtx}(\mathcal{G}_0)$, i.e., that $\mathcal{B}$ is present in $\mathcal{G}_0$ and is "ready to be shrunk". So we may reorder the blowings-up and blowings-down in $(\mathcal{G}_0, \dots, \mathcal{G}_n)$ in such a way that (i) we first perform a sequence of $p = |\operatorname{Vtx}(\mathcal{B})|$ blowings-down, at the end of which $\mathcal{B}$ has disappeared; (ii) then we perform a sequence of m blowings-up, corresponding exactly to the operations performed in $(\mathcal{G}_0, \dots, \mathcal{G}_m)$; (iii) then we perform $n-m-p$ blowings-down so as to obtain $\mathcal{G}_n$ at the end. In other words, there exists a sequence $(\mathcal{G}'_0, \dots, \mathcal{G}'_n)$ such that $\mathcal{G}'_0 = \mathcal{G}_0$, $\mathcal{G}'_n = \mathcal{G}_n$ and the following hold (where $p = |\operatorname{Vtx}(\mathcal{B})|$):

- $\mathcal{G}'_i$ is a blowing-down of $\mathcal{G}'_{i-1}$ at a vertex of $\mathcal{B}$ for all i such that $0 < i \le p$;
- $\mathcal{G}'_{p+1}$ is the blowing-up of $\mathcal{G}'_p$ at v;
- $\mathcal{G}'_i$ is a blowing-up or a blowing-down of $\mathcal{G}'_{i-1}$ for all i such that $p+1 < i \le n$.

Since the last vertex of $\mathcal{B}$ disappears in the blowing-down which transforms $\mathcal{G}'_{p-1}$ into $\mathcal{G}'_p$, it follows that $\mathcal{G}'_{p-1}$ is the blowing-up of $\mathcal{G}'_p$ at v; so $\mathcal{G}'_{p-1} \cong \mathcal{G}'_{p+1}$ and $\mathcal{G}_0$ can be transformed into $\mathcal{G}_n$ in $n-2$ steps. So we are done. □

Proof of 3.2. Let $\mathcal{G} \sim \mathcal{G}'$ be minimal weighted graphs. Then there exist sequences of blowings-up and blowings-down which transform $\mathcal{G}$ into $\mathcal{G}'$. Among all such sequences, choose one

$$s = (\mathcal{G}_0, \dots, \mathcal{G}_n) \qquad (\text{where } \mathcal{G}_0 = \mathcal{G} \text{ and } \mathcal{G}_n = \mathcal{G}')$$

of minimal length, i.e., it is impossible to transform $\mathcal{G}$ into $\mathcal{G}'$ in fewer than n steps. We may assume that $n > 0$, otherwise the assertion holds trivially; since $\mathcal{G}$ and $\mathcal{G}'$ are minimal, it follows that s contains both blowings-up and blowings-down. As is well-known, one may arrange (without changing the number of steps) that all blowings-up are performed before the blowings-down, i.e., for some m such that $0 < m < n$ we have: *$\mathcal{G}_i$ is a blowing-up (resp. blowing-down) of $\mathcal{G}_{i-1}$ for all i such that $0 < i \le m$ (resp. $m < i \le n$).* As explained before the proof of 3.2.1, we may regard the sets $\operatorname{Vtx}(\mathcal{G}_i)$ as subsets of $\operatorname{Vtx}(\mathcal{G}_m)$. Define vertices $e_1, \dots, e_m$ by writing $\operatorname{Vtx}(\mathcal{G}_i) = \operatorname{Vtx}(\mathcal{G}_{i-1}) \cup \{e_i\}$ for $1 \le i \le m$.

We claim that every blowing-up and blowing-down in s is strict. Suppose the contrary. Then we may assume that s contains a blowing-up which is not strict (if not, interchange $\mathcal{G}$ and $\mathcal{G}'$ and work with $s = (\mathcal{G}_n, \dots, \mathcal{G}_0)$ instead). Let $\mathcal{G}_j$ be a non strict blowing-up of $\mathcal{G}_{j-1}$ (where $j \le m$).

If $\mathcal{G}_j$ is a free blowing-up of $\mathcal{G}_{j-1}$ then consider the connected component $\mathcal{B}$ of $\mathcal{G}_m$ containing e_j. Then $\mathcal{B} \sim \boldsymbol{\varnothing}$ and $\operatorname{Vtx}(\mathcal{B}) \subseteq \{e_j, e_{j+1}, \dots, e_m\}$. Clearly, there exists $e_k \in \operatorname{Vtx}(\mathcal{B})$ satisfying $w(e_k, \mathcal{G}_m) = -1$. By minimality of n, one sees that e_k must still be present in $\mathcal{G}_n$, otherwise the blowing-up which creates e_k and the blowing-down at e_k are two operations which could be omitted from s; so $\mathrm{Vtx}(\mathcal{B}) \cap \mathrm{Vtx}(\mathcal{G}_n) \neq \varnothing$. On the other hand, $\mathcal{B} \sim \boldsymbol{\varnothing}$ and the fact that $\mathcal{B}$ is a connected component of $\mathcal{G}_m$ imply that all vertices of $\mathcal{B}$ must disappear in the course of the blowing-down process from $\mathcal{G}_m$ to $\mathcal{G}_n$ (for $\mathcal{G}_n$ is a minimal weighted graph); so $\mathrm{Vtx}(\mathcal{B})$ is disjoint from $\mathrm{Vtx}(\mathcal{G}_n)$, a contradiction.

So $\mathcal{G}_j$ must be the blowing-up of $\mathcal{G}_{j-1}$ at some vertex v satisfying $\deg(v, \mathcal{G}_{j-1}) \geq 2$. Consider the branch $\mathcal{B}$ of $\mathcal{G}_m$ at v such that $e_j \in \mathrm{Vtx}(\mathcal{B})$. Again, we have $\mathcal{B} \sim \boldsymbol{\varnothing}$ and $\mathrm{Vtx}(\mathcal{B}) \subseteq \{e_j, e_{j+1}, \dots, e_m\}$; also, the set $E = \big\{\, e \in \mathrm{Vtx}(\mathcal{B}) \mid w(e, \mathcal{G}_m) = -1 \,\big\}$ is not empty and each $e \in E$ must still be present in $\mathcal{G}_n$, by minimality of n (see the previous paragraph); so $\varnothing \neq E \subseteq \mathrm{Vtx}(\mathcal{G}_n)$. Since $\mathcal{G}_n$ is minimal, $w(e, \mathcal{G}_n) \neq -1$ for all $e \in E$; in particular, the weight of some vertex of $\mathcal{B}$ is increased by the blowing-down process. It follows that some vertex of $\{v\} \cup \mathcal{B}$ disappears in the blowing-down; consequently, it makes sense to consider the least integer k such that $m < k \leq n$ and: *$\mathcal{G}_k$ is the blowing-down of $\mathcal{G}_{k-1}$ at some vertex u of $\{v\} \cup \mathcal{B}$.* From the minimality of k it follows that $w(x, \mathcal{G}_m) = w(x, \mathcal{G}_{k-1})$ for all $x \in \mathrm{Vtx}(\mathcal{B})$; so, if $u \in \mathrm{Vtx}(\mathcal{B})$, we must have $-1 = w(u, \mathcal{G}_{k-1}) = w(u, \mathcal{G}_m)$, so $u \in E$, which is absurd (because $u \notin \mathrm{Vtx}(\mathcal{G}_n)$ and $E \subseteq \mathrm{Vtx}(\mathcal{G}_n)$). This shows that $\mathcal{G}_k$ is the blowing-down of $\mathcal{G}_{k-1}$ at v. In particular $\deg(v, \mathcal{G}_{k-1}) \leq 2 < \deg(v, \mathcal{G}_m)$ so there exists an integer ℓ such that $m < \ell \leq k$ and $\deg(v, \mathcal{G}_{\ell-1}) > \deg(v, \mathcal{G}_\ell)$. Then $(\mathcal{G}_{j-1}, \mathcal{G}_j, \dots, \mathcal{G}_\ell)$ satisfies the hypothesis of 3.2.1 and consequently there exists a sequence of blowings-up and blowings-down which transforms $\mathcal{G}_{j-1}$ into $\mathcal{G}_\ell$ in fewer than $\ell - j + 1$ operations. It follows that $\mathcal{G}_0$ can be transformed into $\mathcal{G}_n$ in fewer than n operations, which is a contradiction. □

Result 3.2 has several consequences; we begin with:

3.3. **Corollary.** *Let $\mathcal{L}$ and $\mathcal{L}'$ be equivalent linear chains. Then there exists a sequence of blowings-up and blowings-down which transforms $\mathcal{L}$ into $\mathcal{L}'$ and which has the additional property that every graph which occurs in the sequence is itself a linear chain.*

Proof. There exists a sequence of blowings-down which transforms $\mathcal{L}$ into a minimal weighted graph $\mathcal{M}$; then $\mathcal{M}$ and every graph which occurs in this sequence is a linear chain. By this remark (also applied to $\mathcal{L}'$), we may assume that both $\mathcal{L}$ and $\mathcal{L}'$ are minimal. Then 3.2 implies that there exists a sequence of strict blowings-up and strict blowings-down which transforms $\mathcal{L}$ into $\mathcal{L}'$; this sequence has the desired additional property, because any weighted graph strictly equivalent to a linear chain is itself a linear chain. □

See 2.4 and 2.8 for the notion of skeletal map.

3.4. **Definition.** Given a weighted forest $\mathcal{G}$ and a strict blowing-down $\mathcal{G}'$ of $\mathcal{G}$, we shall now define a skeletal map $\pi : \mathcal{G} \dashrightarrow \mathcal{G}'$, which we call the *blowing-down map.*

Say that $\mathcal{G}'$ is the blowing-down of $\mathcal{G}$ at e, so $\mathrm{Vtx}(\mathcal{G}) = \{e\} \cup \mathrm{Vtx}(\mathcal{G}')$. We define the set map $\pi : \mathrm{Vtx}_{\neq 2}(\mathcal{G}) \to \mathrm{Vtx}_{\neq 2}(\mathcal{G}')$ as follows.

- If $\deg(e, \mathcal{G}) = 2$ then $\mathrm{Vtx}_{\neq 2}(\mathcal{G}) = \mathrm{Vtx}_{\neq 2}(\mathcal{G}')$ and we let π be the identity map.
- If $\deg(e, \mathcal{G}) = 1$ then let $v \in \mathrm{Vtx}(\mathcal{G})$ be the unique neighbor of e in $\mathcal{G}$. Since the blowing-down is strict, we have $\deg(v, \mathcal{G}) < 3$; so $v \in \mathrm{Vtx}_{<2}(\mathcal{G}')$, which allows us to define $\pi(e) = v$. In order to define π on $\mathrm{Vtx}_{\neq 2}(\mathcal{G}) \setminus \{e\}$, we note:
$$\mathrm{Vtx}_{\neq 2}(\mathcal{G}) \setminus \{e\} = \begin{cases} \mathrm{Vtx}_{\neq 2}(\mathcal{G}'), & \text{if } \deg(v, \mathcal{G}) = 1 \\ \mathrm{Vtx}_{\neq 2}(\mathcal{G}') \setminus \{v\}, & \text{if } \deg(v, \mathcal{G}) \neq 1 \end{cases}$$
so it makes sense to define $\pi(x) = x$ for all $x \in \mathrm{Vtx}_{\neq 2}(\mathcal{G}) \setminus \{e\}$.

One can verify that π is a skeletal map from $\mathcal{G}$ to $\mathcal{G}'$.

3.5. **Lemma.** *Strictly equivalent weighted forests have isomorphic skeletons.*

Proof. It suffices to verify that if $\mathcal{G}$ is a weighted forest and $\mathcal{G}'$ is a strict blowing-down of $\mathcal{G}$, then $\mathcal{G}$ and $\mathcal{G}'$ have the same skeleton. Consider the skeleton S of $\mathcal{G}$; then (by definition) there exists a skeletal map $\sigma : S \dashrightarrow \mathcal{G}$. Composing σ with the blowing-down map $\pi : \mathcal{G} \dashrightarrow \mathcal{G}'$ gives a skeletal map $S \dashrightarrow \mathcal{G}'$, so S is the skeleton of $\mathcal{G}'$. □

Together with 3.2, this gives:

3.6. **Corollary.** *Let $\mathcal{C}$ be the equivalence class of some weighted forest. Then any two minimal elements of $\mathcal{C}$ have isomorphic skeletons.*

The special case of 3.6 where $\mathcal{C}$ is the equivalence class of a linear chain is the following well-known fact:

Any minimal weighted graph equivalent to a linear chain is a linear chain.

Pseudo-minimal forests

3.7. **Definition.** Given a weighted forest $\mathcal{G}$, we define $P(\mathcal{G}) = P(G)$ where G is the underlying graph of $\mathcal{G}$ (see 2.2). Given $\gamma = (v_0, \dots, v_n) \in P(\mathcal{G})$, let $i_1 < \cdots < i_p$ be the elements of the set $\{ i \mid \deg(v_i, \mathcal{G}) < 3 \}$. Then define
$$W_{\mathcal{G}}(\gamma) = \big(w(v_{i_1}, \mathcal{G}), \dots, w(v_{i_p}, \mathcal{G})\big) \in \mathbb{Z}^*$$
(see 1.1 for the notation $\mathbb{Z}^*$). This defines a set map $W_{\mathcal{G}} : P(\mathcal{G}) \to \mathbb{Z}^*$.

3.8. **Lemma and Definition.** *For a weighted forest $\mathcal{G}$, the following conditions are equivalent:*

(1) *Every $\gamma \in P(\mathcal{G})$ satisfying $W_{\mathcal{G}}(\gamma) \sim \varnothing$ is of type $(+,+)$;*
(2) *$\mathcal{G}$ is strictly equivalent to a minimal weighted forest.*

We call $\mathcal{G}$ a *pseudo-minimal forest* if conditions (1), (2) hold.

Proof. Let us say (only in this proof) that a weighted forest $\mathcal{G}$ is 1-regular (resp. 2-regular) if it satisfies condition (1) (resp. condition (2)). We first show that if weighted forests $\mathcal{G}$ and $\mathcal{G}'$ are strictly equivalent, and if one of them is 1-regular, then both are 1-regular. We may assume that $\mathcal{G}'$ is a strict blowing-down of $\mathcal{G}$; let $\pi : \mathcal{G} \dashrightarrow \mathcal{G}'$ be the blowing-down map defined in 3.4. Recall that $\vec{\pi} : P(\mathcal{G}) \to P(\mathcal{G}')$ is surjective and (see 2.5) preserves type. Also, it is clear from the definition of π that, for every $\gamma \in P(\mathcal{G})$, the sequences $W_{\mathcal{G}}(\gamma)$ and $W_{\mathcal{G}'}\big(\vec{\pi}(\gamma)\big)$ are equivalent. It follows that $\mathcal{G}$ is 1-regular if and only if $\mathcal{G}'$ is 1-regular.

Since every minimal weighted forest is 1-regular, the above paragraph implies that every 2-regular forest is 1-regular.

Conversely, consider a 1-regular forest $\mathcal{G}$. Note that 1-regularity implies that if $\mathcal{G}'$ is any blowing-down of $\mathcal{G}$, then in fact $\mathcal{G}'$ is a strict blowing-down of $\mathcal{G}$. Then, by the first paragraph, $\mathcal{G}'$ is 1-regular. Reiterating this argument shows that $\mathcal{G}$ is strictly equivalent to a minimal forest, i.e., $\mathcal{G}$ is 2-regular. □

3.9. **Corollary.** *For pseudo-minimal forests, equivalence implies strict equivalence. Consequently, equivalent pseudo-minimal forests have isomorphic skeletons.*

Proof. If $\mathcal{G}$ and $\mathcal{G}'$ are pseudo-minimal forests then they are strictly equivalent to minimal forests $\mathcal{M}$ and $\mathcal{M}'$ respectively. If $\mathcal{G} \sim \mathcal{G}'$, then $\mathcal{M} \sim \mathcal{M}'$; so $\mathcal{M}$ is strictly equivalent to $\mathcal{M}'$ by 3.2, and consequently $\mathcal{G}$ is strictly equivalent to $\mathcal{G}'$. The last assertion follows from 3.5. □

The next statement is simply a reformulation of 3.9:

3.10. **Corollary.** *Let $\mathcal{G}$ and $\mathcal{G}'$ be equivalent weighted forests. If $\mathcal{G}$ and $\mathcal{G}'$ are minimal, or more generally if they are pseudo-minimal, then there exists a sequence of operations which transforms one into the other and such that:*

(1) *Every graph which occurs in the sequence is a pseudo-minimal forest*
(2) *every operation in the sequence is a strict blowing-up or a strict blowing-down.*

4. Finite sequences of integers

We consider $\mathbb{Z}^*$ and $\mathcal{N}^*$, where $\mathcal{N} = \big\{\, x \in \mathbb{Z} \mid x < -1 \,\big\}$ (see 1.1 for the notation E^*, where E is a set). As indicated in 1.3, given $A \in \mathbb{Z}^*$ we write $[A]$ for the corresponding linear chain.

This section classifies elements of $\mathbb{Z}^*$ up to equivalence. From this, a classification of linear chains will be obtained, in the next section, by recalling that $[A] = [A^-]$ for all $A \in \mathbb{Z}^*$.

The material up to 4.11 is well known when stated for linear chains. The main results of the section are 4.23 (complemented by 4.23.2), 4.24 and 4.31.1 (complemented by 4.30).

4.1. **Notation.** For each $i \in \{1, \dots, r\}$, let A_i be either an integer or an element of $\mathbb{Z}^*$. We write $(A_1, \dots, A_r)$ for the concatenation of $A_1, \dots, A_r$; that is, $(A_1, \dots, A_r) \in \mathbb{Z}^*$ is a single sequence. Also, we will use superscripts to indicate repetitions. For instance, if $A = (0^3, -5, -1) \in \mathbb{Z}^*$ and $B = (-2^3, 3, -2) \in \mathbb{Z}^*$ then

$$(A, -2, B) = (0^3, -5, -1, -2, -2^3, 3, -2) = (0, 0, 0, -5, -1, -2, -2, -2, -2, 3, -2).$$

Superscripts occurring in sequences (or linear chains) should always be interpreted in this way, never as exponents.

4.2. **Definition.** If $X = (x_1, \dots, x_n) \in \mathbb{Z}^*$ and $X \neq \varnothing$, then any of the following sequences $X' \in \mathbb{Z}^*$ is called a *blowing-up* of X:

- $X' = (-1, x_1 - 1, x_2, \dots, x_n)$;
- $X' = (x_1, \dots, x_{i-1}, x_i - 1, -1, x_{i+1} - 1, x_{i+2}, \dots, x_n)$ (where $1 \le i < n$);
- $X' = (x_1, \dots, x_{n-1}, x_n - 1, -1)$.

Moreover, we regard the one-term sequence (-1) as a blowing-up of the empty sequence $\varnothing$. If X' is a blowing-up of X, we also say that X is a blowing-down of X'. Two elements of $\mathbb{Z}^*$ are said to be *equivalent* if one can be obtained from the other by a finite sequence of blowings-up and blowings-down. This defines an equivalence relation "$\sim$" on the set $\mathbb{Z}^*$. We also consider the partial order relation "$\le$" on the set $\mathbb{Z}^*$ which is generated by the condition:

$$X \le X' \text{ whenever } X' \text{ is a blowing-up of } X.$$

Thus a minimal element of $\mathbb{Z}^*$ is a sequence which cannot be blown-down, i.e., an element of $(\mathbb{Z} \setminus \{-1\})^*$.

4.3. **Lemma.** *Given $X, Y \in \mathbb{Z}^*$,*

(1) $X \sim Y \iff X^- \sim Y^-$
(2) $[X] \sim [Y] \iff X \sim Y$ *or* $X \sim Y^-$.

Proof. The only nontrivial claim is implication "$\Rightarrow$" of assertion (2), and this easily follows from 3.3. □

Refer to 1.8 and 1.10 for the following:

4.4. **Definition.** Given $X \in \mathbb{Z}^*$, we define $\det(X) = \det\big([X]\big)$ and $\|X\| = \|[X]\|$.

4.5. **Lemma.** *If $X, Y \in \mathbb{Z}^*$ satisfy $X \sim Y$, then* $\det(X) = \det(Y)$ *and* $\|X\| = \|Y\|$.

Proof. Follows from 4.3, 1.9 and 1.11. □

Fact 4.5 allows us to define $\det(\mathcal{C})$ and $\|\mathcal{C}\|$ for any equivalence class $\mathcal{C} \subset \mathbb{Z}^*$ (the definitions are the obvious ones).

4.6. **Notation.** Given $X = (x_1, \dots, x_n) \in \mathbb{Z}^*$, define:

$$\det_i(X) = \begin{cases} \det\big(x_{i+1}, \dots, x_n\big), & \text{if } 0 \le i < n, \\ 1, & \text{if } i = n, \\ 0, & \text{if } i > n; \end{cases}$$

$$\det_*(X) = \begin{cases} \det(x_2, \dots, x_{n-1}), & \text{if } n > 2, \\ 1, & \text{if } n = 2, \\ 0, & \text{if } n < 2. \end{cases}$$

In particular, note that $\det_0(X) = \det(X)$. The sequence X determines the ordered pair

$$\operatorname{Sub}(X) = \big(\det_1(X), \det_1(X^-)\big)$$

which is an element of the $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z}$. This gives in particular $\operatorname{Sub}(\varnothing) = (0,0)$ and if $a \in \mathbb{Z}$, $\operatorname{Sub}\big((a)\big) = (1,1)$. Finally, let $d = \det(X)$ and define the pair

$$\overline{\operatorname{Sub}}(X) = \big(\pi(\det_1(X)), \pi(\det_1(X^-))\big) \in \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z},$$

where $\pi : \mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ is the canonical epimorphism and where we regard $\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ as a $\mathbb{Z}$-module.

Facts 4.7–4.11 are, in one form or another, contained in [2]. We omit some proofs.

4.7. **Lemma.** *If $X = (x_1, \dots, x_n) \in \mathbb{Z}^*$ then:*

$$\det_i(X) = (-x_{i+1}) \det_{i+1}(X) - \det_{i+2}(X) \qquad (0 \le i < n).$$

In particular, $\det X = (-x_1)\det_1(X) - \det_2(X)$.

4.8. **Lemma.** *The assignment $X \mapsto (\det(X), \det_1(X))$ is a well-defined bijection:*

$$\mathcal{N}^* \longrightarrow \big\{ (r_0, r_1) \in \mathbb{N}^2 \ \big| \ 0 \le r_1 < r_0 \text{ and } \gcd(r_0, r_1) = 1 \big\}.$$

4.9. **Lemma.** *If $X \in \mathbb{Z}^*$, $d = \det(X)$ and $(x, y) = \mathrm{Sub}(X)$, then $xy \equiv 1 \pmod d$.*

4.10. **Lemma.** *Suppose that $A, B \in \mathbb{Z}^*$ satisfy $A \sim B$ and let $d = \det(A) = \det(B)$. Then there exists $(x, y) \in \mathbb{Z}^2$ such that*

$$\mathrm{Sub}(A) = \mathrm{Sub}(B) + d(x, y). \tag{4}$$

Proof. Note that $\det(A) = \det(B)$ by 4.5. Since $A \sim B$, performing a certain sequence of blowings-up and blowings-down on A produces B; if the same sequence of operations is performed on $(0, A)$ then (obviously) we obtain (x, B) for some $x \in \mathbb{Z}$, which shows that $(0, A) \sim (x, B)$. By the same argument, $(A, 0) \sim (B, y)$ for some $y \in \mathbb{Z}$. By 4.7 we have $\det(0, A) = -\det_1(A)$ and $\det(x, B) = -xd - \det_1(B)$; since $(0, A) \sim (x, B)$ implies $\det(0, A) = \det(x, B)$, we obtain

$$\det_1(A) = \det_1(B) + dx.$$

Similarly, we have $(0, A^-) = (A, 0)^- \sim (B, y)^- = (y, B^-)$, so $\det(0, A^-) = \det(y, B^-)$ and consequently $\det_1(A^-) = \det_1(B^-) + dy$. So (x, y) satisfies (4). □

4.11. **Corollary.** *If $A, B \in \mathbb{Z}^*$ and $A \sim B$, then $\overline{\mathrm{Sub}}(A) = \overline{\mathrm{Sub}}(B)$.*

Proof. Obvious consequence of 4.10. □

Classification of sequences up to equivalence

Sequences of the form $(0^{2i}, A)$ (see 4.1 for notations) play an important role in the classification. We need the following facts.

4.12. **Lemma.** *Let $i \in \mathbb{N}$ and $A \in \mathbb{Z}^*$.*

(1) $\det(0^{2i}, A) = (-1)^i \det(A)$
(2) $\mathrm{Sub}(0^{2i}, A) = (-1)^i \,\mathrm{Sub}(A)$.

Proof. We may assume that $i > 0$, then 4.7 gives

$$\det(0^{2i}, A) = 0 \det_1(0^{2i}, A) - \det_2(0^{2i}, A) = -\det(0^{2i-2}, A)$$

and assertion (1) follows by induction. We also have:

$$\begin{aligned} \det_1(0^{2i}, A) = \det(0^{2i-2}, 0, A) &\overset{(1)}{=} (-1)^{i-1} \det(0, A) \\ &\overset{4.7}{=} (-1)^{i-1}(0 \det(A) - \det_1(A)) = (-1)^i \det_1(A) \end{aligned} \tag{5}$$

so, to prove (2), there remains only to show that

$$\det\nolimits_1\big((0^{2i},A)^-\big) = (-1)^i \det\nolimits_1(A^-). \tag{6}$$

If $A=\varnothing$ then (6) reads $\det(0^{2i-1})=0$, which is true by assertion (1). So we may assume that $A=(a_1,\dots,a_n)$ with $n\geq 1$, in which case

$$\begin{aligned}\det\nolimits_1\big((0^{2i},A)^-\big) &= \det(a_{n-1},\dots,a_1,0^{2i}) = \det(0^{2i},a_1,\dots,a_{n-1})\\ &\overset{(1)}{=} (-1)^i\det(a_1,\dots,a_{n-1}) = (-1)^i\det(a_{n-1},\dots,a_1) = (-1)^i\det\nolimits_1(A^-).\end{aligned}$$

So (6) holds and assertion (2) follows from (5) and (6). $\square$

4.13. **Lemma.** *If $i\in\mathbb{N}$ and $A\in\mathbb{Z}^*$, then $\|(0^{2i},A)\| = i+\|A\|$.*

Proof. This is an exercise in diagonalization. It suffices to prove that $\|(0,0,A)\| = 1+\|A\|$ for every $A\in\mathbb{Z}^*$. This is obvious if $A=\varnothing$, so assume that $A\neq\varnothing$ and write $A=(a_1,\dots,a_n)$. Consider the linear chain

$$\mathcal{L} = [0,0,A] = \overset{0}{\underset{u_1}{\bullet}} \!\!-\!\!-\!\!-\!\! \overset{0}{\underset{u_2}{\bullet}} \!\!-\!\!-\!\!-\!\! \overset{a_1}{\underset{v_1}{\bullet}} \!\!-\!\!-\!\!-\; \cdots \;-\!\!-\!\!-\!\! \overset{a_n}{\underset{v_n}{\bullet}}$$

and let V be the real vector space with basis $\mathrm{Vtx}(\mathcal{L})$. Then the matrix representing $B_{\mathcal{L}}$ with respect to the basis $(u_1,u_2,v_1-u_1,v_2,\dots,v_n)$ of V is:

$$\left(\begin{array}{cc|ccc} 0 & 1 & 0 & \cdots & 0\\ 1 & 0 & 0 & \cdots & 0\\ \hline 0 & 0 & & & \\ \vdots & \vdots & & M & \\ 0 & 0 & & & \end{array}\right) \tag{7}$$

where M is the $n\times n$ matrix given by $M_{ii}=a_i$, $M_{ij}=1$ if $|i-j|=1$ and $M_{ij}=0$ if $|i-j|>1$, that is, M is the matrix representing the intersection form of the linear chain $[A]$. Now $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ can be diagonalized to $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ and we conclude that a diagonal matrix congruent to (7) has $1+\|A\|$ nonnegative entries on its main diagonal, i.e., $\|\mathcal{L}\| = 1+\|A\|$. $\square$

4.14. **Lemma.** *Let $a,b,x\in\mathbb{Z}$ and $A,B\in\mathbb{Z}^*$. Then*

$$(A,a,0,b,B)\sim(A,a-x,0,b+x,B)\sim(A,0,0,a+b,B).$$

Proof. $(A,a,0,b,B)\sim(A,a-1,-1,-1,b,B)\sim(A,a-1,0,b+1,B)$, from which the result follows. $\square$

4.15. **Lemma.** *Let $n\in\mathbb{N}$ and $A,B,C\in\mathbb{Z}^*$. Then $(A,B,0^{2n},C)\sim(A,0^{2n},B,C)$.*

Proof. If $A,C\in\mathbb{Z}^*$ and $b\in\mathbb{Z}$ then by 4.14

$$(A,b,0,0,C)\sim(A,b-b,0,0+b,C)=(A,0,0,b,C),$$

from which the result follows. $\square$

4.16. **Lemma.** *Let $n\in\mathbb{N}$, $x,y\in\mathbb{Z}$ and $A\in\mathbb{Z}^*$. Then $(0^{2n+1},x,A)\sim(0^{2n+1},y,A)$.*

Proof. We first consider the case $n = 0$: $(0, x, A) \sim (-1, -1, x-1, A) \sim (0, x-1, A)$, from which we deduce $(0, x, A) \sim (0, y, A)$. Now the general case:

$$(0^{2n+1}, x, A) \overset{4.15}{\sim} (0, x, 0^{2n}, A) \overset{(n=0)}{\sim} (0, y, 0^{2n}, A) \overset{4.15}{\sim} (0^{2n+1}, y, A).$$

□

4.17. **Lemma.** *Let $n \in \mathbb{N}$ and $A, B \in \mathbb{Z}^*$. Then:*

$$A \sim B \quad \Longrightarrow \quad (0^{2n}, A) \sim (0^{2n}, B).$$

Proof. We may assume that $n \geq 1$. If $A \sim B$ then performing a certain sequence of blowings-up and blowings-down on A produces B; if the same sequence of operations is performed on $(0^{2n}, A) = (0^{2n-1}, 0, A)$, then we obtain $(0^{2n-1}, x, B)$ for some $x \in \mathbb{Z}$, i.e., only the rightmost zero in 0^{2n} is affected. So

$$(0^{2n}, A) \sim (0^{2n-1}, x, B) \overset{4.16}{\sim} (0^{2n-1}, 0, B) = (0^{2n}, B).$$

□

4.18. **Definition.** Let $B = (b_1, \dots, b_n) \in \mathbb{Z}^*$.

(1) Given $x \in \mathbb{Z}$, define ${}_xB = (0, x, B) = (0, x, b_1, \dots, b_n) \in \mathbb{Z}^*$ and $B_x = (B, x, 0) = (b_1, \dots, b_n, x, 0) \in \mathbb{Z}^*$.
(2) Suppose that $B \neq \varnothing$. Given $i \in \{1, \dots, n\}$ and $x, y \in \mathbb{Z}$ such that $x + y = b_i$, define $B_{(i;x,y)} = (b_1, \dots, b_{i-1}, x, 0, y, b_{i+1}, \dots, b_n) \in \mathbb{Z}^*$.

4.19. **Definition.** Given a minimal element $M = (m_1, \dots, m_k)$ of $\mathbb{Z}^*$, let $M^\oplus$ be the set of sequences $Z \in \mathbb{Z}^*$ which can be constructed in one of the following ways.

(1) Pick $x \in \mathbb{Z}$ and let Z be the unique minimal sequence such that $Z \leq {}_xM$.
(2) Pick $x \in \mathbb{Z}$ and let Z be the unique minimal sequence such that $Z \leq M_x$.
(3) Assuming that $M \neq \varnothing$, pick $j \in \{1, \dots, k\}$ and $x, y \in \mathbb{Z}$ such that $x + y = m_j$ and let Z be the unique minimal sequence such that $Z \leq M_{(j;x,y)}$.
(4) Pick $M' = (\mu_1, \dots, \mu_\ell)$ such that $M' \geq M$ and exactly one term μ_j is equal to -1; pick $x, y \in \mathbb{Z} \setminus \{-1\}$ such that $x + y = -1$ and let $Z = M'_{(j;x,y)}$.

Note that each element of $M^\oplus$ is a minimal element of $\mathbb{Z}^*$.

4.20. **Lemma.** *If M is a minimal element of $\mathbb{Z}^*$ and $Z \in M^\oplus$ then $Z \sim (0, 0, M)$. Moreover,* $\det Z = -\det M$ *and* $\|Z\| = \|M\| + 1$.

Proof. By definition 4.19 of $M^\oplus$, one of the following holds:

$$Z \leq {}_xM, \quad Z \leq M_x, \quad Z \leq M_{(j;x,y)} \quad \text{or} \quad Z = M'_{(j;x,y)} \text{ where } M' \sim M.$$

Consequently, one of the following holds:

$$Z \sim {}_xM, \quad Z \sim M_x \quad \text{or} \quad Z \sim M'_{(j;x,y)} \text{ where } M' \sim M.$$

By 4.16, ${}_xM = (0, x, M) \sim (0, 0, M)$. Since $X \sim Y$ implies $X^- \sim Y^-$, we also have $M_x = \big({}_x(M^-)\big)^- \sim (0, 0, M^-)^- = (M, 0, 0) \sim (0, 0, M)$ by 4.15.

Let $M' = (b_1, \dots, b_m)$ be any nonempty sequence equivalent to M and let $j \in \{1, \dots, m\}$ and $x, y \in \mathbb{Z}$ be such that $x + y = b_j$; then

$$M'_{(j;x,y)} = (b_1, \dots, b_{j-1}, x, 0, y, b_{j+1}, \dots, b_m) \overset{4.14}{\sim} (b_1, \dots, b_{j-1}, 0, 0, x + y, b_{j+1}, \dots, b_m)$$
$$= (b_1, \dots, b_{j-1}, 0, 0, b_j, b_{j+1}, \dots, b_m) \overset{4.15}{\sim} (0, 0, M') \overset{4.17}{\sim} (0, 0, M).$$

Thus $Z \sim (0, 0, M)$ whenever $Z \in M^{\oplus}$. By 4.12 and 4.13 we get $\det Z = -\det M$ and $\|Z\| = \|M\| + 1$. $\square$

4.21. **Proposition.** *Let Z be a minimal element of $\mathbb{Z}^*$ such that $\|Z\| > 0$ and $Z \neq (0)$. Then $Z \in M^{\oplus}$ for some minimal element M of $\mathbb{Z}^*$.*

Proof. Assume that $Z = (z_1, \dots, z_n)$ is minimal, $\|Z\| > 0$ and $Z \neq (0)$. In particular, $\|Z\| > 0$ implies that $z_i \geq -1$ for some i; so by minimality of Z there exists i such that $z_i \geq 0$. If $z_i = 0$ for some i, we distinguish three cases:

(i) If $z_1 = 0$ then, since $Z \neq (0)$, we have $Z = (0, x, M) = {}_xM$ for some $M \in \mathbb{Z}^*$ and $x \in \mathbb{Z}$; then M is minimal and $Z \in M^{\oplus}$.

(ii) If $z_n = 0$ then, similarly, $Z = M_x$ for some $M \in \mathbb{Z}^*$ and $x \in \mathbb{Z}$; then M is minimal and $Z \in M^{\oplus}$.

(iii) If $z_i = 0$ for some i such that $1 < i < n$, then $Z = (z_1, \dots, z_{i-1}, 0, z_{i+1}, \dots, z_n) = B_{(i-1; z_{i-1}, z_{i+1})}$ where $B = (z_1, \dots, z_{i-2}, z_{i-1} + z_{i+1}, z_{i+2}, \dots, z_n)$. If B is minimal then $Z \in M^{\oplus}$ where $M = B$. If B is not minimal then its $(i-1)$-th term $(z_{i-1} + z_{i+1})$ is the only one which is equal to -1; we have $B \geq M$ for some minimal M, then $Z \in M^{\oplus}$.

From now-on, assume that $z_j \neq 0$ for all $j \in \{1, \dots, n\}$. Then $z_i > 0$ for some i and we have four cases:

(iv) If $Z = (p)$ where $p > 0$, then $Z \leq (0, -1, -2^{p-1}) = {}_{-1}M$ where $M = (-2^{p-1})$ is minimal; then $Z \in M^{\oplus}$.

(v) If $z_1 > 0$ and $n > 1$ then $Z \leq (0, -1, -2^{z_1 - 1}, z_2 - 1, z_3, \dots, z_n) = {}_{-1}M$ where $M = (-2^{z_1 - 1}, z_2 - 1, z_3, \dots, z_n)$ is minimal; then $Z \in M^{\oplus}$.

(vi) If $z_n > 0$ and $n > 1$ then $Z \leq (z_1, \dots, z_{n-2}, z_{n-1} - 1, -2^{z_n - 1}, -1, 0) = M_{-1}$ where $M = (z_1, \dots, z_{n-2}, z_{n-1} - 1, -2^{z_n - 1})$ is minimal; then $Z \in M^{\oplus}$.

(vii) If $z_i > 0$ and $1 < i < n$ then $Z \leq (z_1, \dots, z_{i-1}, 0, -1, -2^{z_i - 1}, z_{i+1} - 1, z_{i+2}, \dots, z_n) = M_{(i-1; z_{i-1}, -1)}$, where $M = (z_1, \dots, z_{i-2}, z_{i-1} - 1, -2^{z_i - 1}, z_{i+1} - 1, z_{i+2}, \dots, z_n)$ is minimal; then $Z \in M^{\oplus}$.

$\square$

4.22. **Definition.** An element C of $\mathbb{Z}^*$ is a *canonical sequence* if it has the form

$C = (0^r, A)$, where $r \in \mathbb{N}$, $A \in \mathbb{N}^*$ and if $A \neq \varnothing$ then r is even.

Then one has the fundamental result:

4.23. **Theorem.** *Each element of $\mathbb{Z}^*$ is equivalent to a unique canonical sequence.*

The proof of 4.23 consists of 4.23.1 and 4.23.2, below.

4.23.1. *Every element of $\mathbb{Z}^*$ is equivalent to a canonical sequence.*

Proof. It suffices to show that every minimal element Z of $\mathbb{Z}^*$ is equivalent to a canonical sequence. We proceed by induction on $\|Z\|$. If $\|Z\| = 0$ then $Z \in \mathcal{N}^*$, so Z itself is canonical. If $\|Z\| > 0$ then, by 4.21, either $Z = (0)$ or $Z \in M^\oplus$ for some minimal element M of $\mathbb{Z}^*$. In the first case, Z is canonical and we are done. In the second case, 4.20 gives $\|M\| < \|Z\|$ so we may assume by induction that M is equivalent to a canonical sequence C; then $Z \sim (0,0,M) \sim (0,0,C)$ by 4.20 and 4.17, and clearly $(0,0,C)$ is canonical. □

4.23.2. *Let $L \in \mathbb{Z}^*$, let $n = \|L\|$ and let d be the absolute value of* $\det(L)$.
If $(0^r, A)$ (where $r \in \mathbb{N}$ and $A \in \mathcal{N}^*$) is a canonical sequence equivalent to L, then:

(a) *If $d = 0$ then $r = 2n-1$ and $A = \varnothing$.*
(b) *If $d \neq 0$ then $r = 2n$ and A is the unique element of $\mathcal{N}^*$ which satisfies:*

$$\det(A) = d \quad \textit{and} \quad \overline{\mathrm{Sub}}(A) = (-1)^n\,\overline{\mathrm{Sub}}(L).$$

In particular, r and A are uniquely determined by L.

Proof. The claim that r and A are uniquely determined by L is obvious in case (a), and follows from 4.8 in case (b). Consider any canonical sequence $(0^r, A)$ equivalent to L; we have $r \in \mathbb{N}$, $A \in \mathcal{N}^*$, and if $A \neq \varnothing$ then r is even. To prove (a) and (b), it suffices to show:

(a′) *If r is odd then $d = 0$ and $r = 2n-1$.*
(b′) *If r is even then* $\det(A) = d \neq 0$, $r = 2n$ *and* $\overline{\mathrm{Sub}}(A) = (-1)^n\,\overline{\mathrm{Sub}}(L)$.

If r is odd then $A = \varnothing$; writing $r = 2i+1$, we get $\pm d = \det(L) = \det(0^{2i+1}) = \det(0^{2i}, 0) = (-1)^i \det(0) = 0$ by 4.12 and $n = \|L\| = \|(0^{2i+1})\| = i + \|(0)\| = i+1$ by 4.13. This proves (a′).

If r is even then 4.13 gives $n = \|L\| = \|(0^r, A)\| = \frac{r}{2} + \|A\| = \frac{r}{2}$, so $r = 2n$. Then 4.12 gives $\pm d = \det(L) = \det(0^{2n}, A) = (-1)^n \det(A)$; since $\det(A) > 0$ by 4.8, we obtain $\det(A) = d \neq 0$. Since $(0^{2n}, A) \sim L$, 4.10 implies that there exist $(u,v) \in \mathbb{Z}^2$ such that $\mathrm{Sub}(0^{2n}, A) = \mathrm{Sub}(L) + d(u,v)$. On the other hand, 4.12 gives $\mathrm{Sub}(A) = (-1)^n \mathrm{Sub}(0^{2n}, A)$, so

$$\mathrm{Sub}(A) = (-1)^n\big(\mathrm{Sub}(L) + d(u,v)\big).$$

It follows that $\overline{\mathrm{Sub}}(A) = (-1)^n\,\overline{\mathrm{Sub}}(L)$ and that (b′) is true. □

4.24. **Corollary.** *For $L, L' \in \mathbb{Z}^*$, the following are equivalent:*

(1) $L \sim L'$
(2) $\|L\| = \|L'\|$, $\det(L) = \det(L')$ *and* $\overline{\mathrm{Sub}}(L) = \overline{\mathrm{Sub}}(L')$.

Proof. Immediate consequence of 4.23.2. □

Remark. One can state some variants of 4.24, for instance:

- Suppose that $L, L' \in \mathbb{Z}^*$ satisfy $\det(L) = 0 = \det(L')$. Then
$$L \sim L' \iff \|L\| = \|L'\|.$$
- Suppose that $L, L' \in \mathbb{Z}^*$ satisfy $\det(L) = d = \det(L')$ and $\|L\| = \|L'\|$. Then
$$L \sim L' \iff \det_1(L) \equiv \det_1(L') \pmod{d}.$$

4.25. **Definition.** Let $C = (0^r, A) \in \mathbb{Z}^*$ be a canonical sequence (where $r \in \mathbb{N}$ and $A \in \mathcal{N}^*$). The *transpose* C^t of C is defined by $C^t = (0^r, A^-)$. Note that C^t is a canonical sequence.

4.26. **Lemma.** *Let $X \in \mathbb{Z}^*$. If C is the unique canonical sequence equivalent to X, then C^t is the unique canonical sequence equivalent to X^-.*

Proof. Since $X^- \sim C^-$, it suffices to show that $C^- \sim C^t$. Write $C = (0^r, A)$ with $A \in \mathcal{N}^*$. If r is odd then $A = \varnothing$ and the result holds trivially. Assume that r is even, then:
$$C^- = (0^r, A)^- = (A^-, 0^r) \overset{4.15}{\sim} (0^r, A^-) = C^t.$$
□

Further results on the classification of sequences

We write $\mathcal{C} \in \mathbb{Z}^*/{\sim}$ to indicate that $\mathcal{C} \subset \mathbb{Z}^*$ is an equivalence class of sequences. Result 4.31.1, below, gives a surprisingly simple description of the set $\mathbb{Z}^*/{\sim}$.

Given $\mathcal{C} \in \mathbb{Z}^*/{\sim}$, let $\min \mathcal{C} = \big\{ M \in \mathcal{C} \mid M \text{ is a minimal element of } \mathbb{Z}^* \big\}$ denote the set of minimal elements of $\mathcal{C}$ (see 4.2 for the notion of minimal sequence).

4.27. **Lemma.** *Suppose that M_1, M_2 are minimal elements of $\mathbb{Z}^*$ and $X_i \in M_i^{\oplus}$ ($i = 1, 2$). Then*
$$X_1 \sim X_2 \iff M_1 \sim M_2.$$

Proof. For each i we have $X_i \sim (0, 0, M_i)$ by 4.20, so it suffices to prove:
$$(0, 0, A_1) \sim (0, 0, A_2) \iff A_1 \sim A_2, \quad \textit{for all } A_1, A_2 \in \mathbb{Z}^*.$$
If $A_1 \sim A_2$ then $(0, 0, A_1) \sim (0, 0, A_2)$ by 4.17. For the converse, note that 4.12 and 4.13 give $\det(0, 0, A_i) = -\det A_i$, $\operatorname{Sub}(0, 0, A_i) = -\operatorname{Sub} A_i$ and $\|(0, 0, A_i)\| = 1 + \|A_i\|$ (for $i = 1, 2$), so if $(0, 0, A_1) \sim (0, 0, A_2)$ we obtain $A_1 \sim A_2$ by 4.24. □

4.28. **Definition.** For each $\mathcal{C} \in \mathbb{Z}^*/{\sim}$ we define an element $\mathcal{C}^{\oplus}$ of $\mathbb{Z}^*/{\sim}$ as follows: Pick any minimal element M of $\mathcal{C}$, pick any $X \in M^{\oplus}$ and let $\mathcal{C}^{\oplus}$ be the class of X. By 4.27, $\mathcal{C}^{\oplus}$ is well-defined and:

$$\mathcal{C} \longmapsto \mathcal{C}^{\oplus} \quad \text{is an injective map from } \mathbb{Z}^*/{\sim} \text{ to itself.} \tag{8}$$

We call $\mathcal{C}^{\oplus}$ the *successor* of $\mathcal{C}$. If $\mathcal{C} = \mathcal{C}_1^{\oplus}$ for some $\mathcal{C}_1$ then $\mathcal{C}_1$ is unique by (8); in this case we say that "$\mathcal{C}$ has a predecessor" and we call $\mathcal{C}_1$ the *predecessor* of $\mathcal{C}$. If $\mathcal{C} \in \mathbb{Z}^*/{\sim}$ then 4.20 gives

$$\det\left(\mathcal{C}^{\oplus}\right) = -\det \mathcal{C} \quad \text{and} \quad \|\mathcal{C}^{\oplus}\| = 1 + \|\mathcal{C}\|. \tag{9}$$

4.29. **Lemma and Definition.** *For an element* $\mathcal{C}$ *of* $\mathbb{Z}^*/\!\sim$, *the following are equivalent:*

(1) $\min \mathcal{C}$ *is a singleton*
(2) $\min \mathcal{C}$ *is a finite set*
(3) $r \leq 1$, *where* r *is defined by the condition: The unique canonical sequence belonging to* $\mathcal{C}$ *is* $(0^r, A)$, *where* $r \in \mathbb{N}$ *and* $A \in \mathcal{N}^*$.
(4) $\mathcal{C}$ *does not have a predecessor.*

If $\mathcal{C}$ satisfies these conditions then we call it a *prime class.*

Proof. Note that $\neg(2) \Rightarrow \neg(1)$ is trivial; we prove $\neg(1) \Rightarrow \neg(4) \Rightarrow \neg(3) \Rightarrow \neg(2)$.

If $\|\mathcal{C}\| = 0$ then the canonical element of $\mathcal{C}$ is a sequence $X \in \mathcal{N}^*$; clearly, X is then the unique minimal element of $\mathcal{C}$, so the condition $\|\mathcal{C}\| = 0$ implies (1).

Hence, if (1) is false then $\|\mathcal{C}\| > 0$; since $\min \mathcal{C}$ has more than one element, we may pick a minimal $X \in \mathcal{C}$ such that $X \neq (0)$; then 4.21 gives $X \in M^{\oplus}$ for some minimal element M of $\mathbb{Z}^*$. Thus (4) is false.

If (4) is false then we may consider the unique canonical sequence C which belongs to the predecessor of $\mathcal{C}$. Then $(0, 0, C)$ is the unique canonical sequence belonging to $\mathcal{C}$, so (3) is false.

If (3) is false then the canonical element $(0^r, A)$ of $\mathcal{C}$ satisfies $r \geq 2$. By 4.16, $(0, x, 0^{r-2}, A) \in \min \mathcal{C}$ for every $x \in \mathbb{Z} \setminus \{-1\}$, so (2) is false. □

4.30. **Corollary.** *The set of prime classes is* $\{\mathcal{C}_0\} \cup \{\, \mathcal{C}_X \mid X \in \mathcal{N}^* \,\}$, *where* $\mathcal{C}_0$ *denotes the equivalence class of the sequence* (0) *and, for each* $X \in \mathcal{N}^*$, $\mathcal{C}_X$ *is the class of* X.

Proof. Follows immediately from condition (3) of 4.29. □

4.31. Given $\mathcal{C}, \mathcal{C}' \in \mathbb{Z}^*/\!\sim$, write $\mathcal{C} \preceq \mathcal{C}'$ to indicate that there exists a sequence $\mathcal{C}_0, \dots, \mathcal{C}_n$ in $\mathbb{Z}^*/\!\sim$ satisfying $n \in \mathbb{N}$, $\mathcal{C}_0 = \mathcal{C}$, $\mathcal{C}_n = \mathcal{C}'$ and $\mathcal{C}_{i+1} = \mathcal{C}_i^{\oplus}$ for all i such that $0 \leq i < n$. Then $\preceq$ is a partial order on the set $\mathbb{Z}^*/\!\sim$ such that the minimal elements are precisely the prime classes. Given $\mathcal{C} \in \mathbb{Z}^*/\!\sim$, define the interval

$$[\mathcal{C}, \infty) = \{\, \mathcal{C}' \in \mathbb{Z}^*/\!\sim \;\mid\; \mathcal{C} \preceq \mathcal{C}' \,\}.$$

The following is now clear:

4.31.1. **Proposition.** *Let* $P = \{\, [\mathcal{C}, \infty) \mid \mathcal{C} \text{ is a prime class} \,\}$.

(1) *The set* P *is a partition of* $\mathbb{Z}^*/\!\sim$.
(2) *If* $I \in P$ *then* $(I, \preceq)$ *is isomorphic to* $(\mathbb{N}, \leq)$ *as a partially ordered set.*
(3) *If* I, I' *are distinct elements of* P, *and if* $\mathcal{C} \in I$ *and* $\mathcal{C}' \in I'$, *then* $\mathcal{C}$ *and* $\mathcal{C}'$ *are not comparable w.r.t.* $\preceq$.

Note that 4.30 and 4.31.1 completely describe the partially ordered set $\big(\mathbb{Z}^*/\!\sim, \preceq\big)$.

5. Classification of linear chains

It is clear that every result about sequences gives rise to a result about linear chains. This section reformulates 4.23 and 4.24 in terms of linear chains but leaves it to the reader to translate the other results of Section 4 (in particular 4.23.2, the remark after 4.24, 4.29–4.31.1).

5.1. **Definition.** By a *canonical chain*, we mean a linear chain of the form $[L]$ where $L \in \mathbb{Z}^*$ is a canonical sequence. The *transpose* $\mathcal{L}^t$ of a canonical chain $\mathcal{L}$ is defined by:

$$\mathcal{L}^t = [L^t]$$

where $L \in \mathbb{Z}^*$ is a canonical sequence satisfying $\mathcal{L} = [L]$ and where L^t was defined in 4.25. Note that $\mathcal{L}^t$ is a canonical chain.

Remark. The linear chain $\mathcal{L}^t$ is well-defined even when L is not uniquely determined by $\mathcal{L}$ (i.e., when L and L^- are canonical and distinct). Indeed, this only happens when $L \in \mathbb{N}^*$ and in that case we have $[L^t] = [(L^-)^t] = \mathcal{L}$.

Concretely, a linear chain is canonical if it is $[0^r]$ with r odd or if it has the form:

$$\underbrace{\overset{0}{\bullet} - \cdots - \overset{0}{\bullet}}_{r \text{ vertices}} - \overset{a_1}{\bullet} - \cdots - \overset{a_n}{\bullet} \qquad (r \geq 0 \text{ is even, } n \geq 0 \text{ and } \forall_i \ a_i \leq -2). \tag{10}$$

The transpose of $[0^r]$ is the same graph $[0^r]$ and the transpose of (10) is:

$$\underbrace{\overset{0}{\bullet} - \cdots - \overset{0}{\bullet}}_{r \text{ vertices}} - \overset{a_n}{\bullet} - \cdots - \overset{a_1}{\bullet}.$$

As a corollary to the classification of sequences, we obtain:

5.2. **Theorem.** *Every linear chain is equivalent to a canonical chain. Moreover, if $\mathcal{L}$ and $\mathcal{L}'$ are canonical chains then*

$$\mathcal{L} \sim \mathcal{L}' \iff \mathcal{L}' \in \{\mathcal{L}, \mathcal{L}^t\}.$$

Proof. In view of 4.3, this is a corollary to 4.23 and 4.26. □

For the next result, we need:

5.3. **Definition.** Let $\mathcal{L}$ be a linear chain. Define a subset $\mathrm{Sub}(\mathcal{L})$ of $\mathbb{Z}$ as follows: Choose $L \in \mathbb{Z}^*$ such that $\mathcal{L} = [L]$, let $(x, y) = \mathrm{Sub}(L) \in \mathbb{Z} \times \mathbb{Z}$ and set

$$\mathrm{Sub}(\mathcal{L}) = \{x, y\}.$$

We also define the subset $\overline{\mathrm{Sub}}(\mathcal{L})$ of $\mathbb{Z}/d\mathbb{Z}$, where $d = \det(\mathcal{L})$, by taking the image of $\mathrm{Sub}(\mathcal{L})$ via the canonical epimorphism $\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$.

5.4. **Corollary.** *For linear chains $\mathcal{L}$ and $\mathcal{L}'$, the following are equivalent:*

(1) $\mathcal{L} \sim \mathcal{L}'$
(2) $\|\mathcal{L}\| = \|\mathcal{L}'\|$, $\det \mathcal{L} = \det \mathcal{L}'$ *and* $\overline{\mathrm{Sub}}(\mathcal{L}) \cap \overline{\mathrm{Sub}}(\mathcal{L}') \neq \varnothing$.

Proof. Follows immediately from 4.24. Note that the condition $\overline{\mathrm{Sub}}(\mathcal{L}) \cap \overline{\mathrm{Sub}}(\mathcal{L}') \neq \varnothing$ is equivalent to $\overline{\mathrm{Sub}}(\mathcal{L}) = \overline{\mathrm{Sub}}(\mathcal{L}')$ by 4.9. □

6. τ-EQUIVALENCE OF SEQUENCES

6.1. **Definition.** The sentence *"τ is a type"* means that τ is one of the four symbols $(-,-)$, $(+,-)$, $(-,+)$, $(+,+)$. For each type τ, we define a subset $\mathbb{Z}^*_\tau$ of $\mathbb{Z}^*$ and an equivalence relation $\overset{\tau}{\sim}$ on $\mathbb{Z}^*_\tau$. The sets $\mathbb{Z}^*_\tau$ are defined by:

$$\mathbb{Z}^*_{(-,-)} = \mathbb{Z}^*$$
$$\mathbb{Z}^*_{(+,-)} = \mathbb{Z}^*_{(-,+)} = \big\{ (x_1, \dots, x_n) \in \mathbb{Z}^* \mid n \geq 1 \big\} = \mathbb{Z}^* \setminus \{\varnothing\}$$
$$\mathbb{Z}^*_{(+,+)} = \big\{ (x_1, \dots, x_n) \in \mathbb{Z}^* \mid n \geq 2 \big\}.$$

Let $X = (x_1, \dots, x_n) \in \mathbb{Z}^*$.

(1) By a $(-,-)$-*blowing-up* of X, we mean a blowing-up of X in the sense of 4.2.

(2) If $X \in \mathbb{Z}^*_{(+,-)}$, then any of the following is called a $(+,-)$-*blowing-up* of X:
- $X' = (x_1, \dots, x_{i-1}, x_i - 1, -1, x_{i+1} - 1, x_{i+2}, \dots, x_n)$, for some i such that $1 \leq i < n$;
- $X' = (x_1, \dots, x_{n-1}, x_n - 1, -1)$.

(3) If $X \in \mathbb{Z}^*_{(-,+)}$, then any of the following is called a $(-,+)$-*blowing-up* of X:
- $X' = (x_1, \dots, x_{i-1}, x_i - 1, -1, x_{i+1} - 1, x_{i+2}, \dots, x_n)$, for some i such that $1 \leq i < n$;
- $X' = (-1, x_1 - 1, x_2, \dots, x_n)$.

(4) If $X \in \mathbb{Z}^*_{(+,+)}$, then any of the following is called a $(+,+)$-*blowing-up* of X:
- $X' = (x_1, \dots, x_{i-1}, x_i - 1, -1, x_{i+1} - 1, x_{i+2}, \dots, x_n)$, for some i such that $1 \leq i < n$.

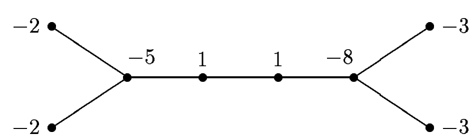
Note that if $X \in \mathbb{Z}^*_\tau$ and X' is a τ-blowing-up of X then $X' \in \mathbb{Z}^*_\tau$; in this situation, we also say that X is a τ-*blowing-down* of X'. Two elements of $\mathbb{Z}^*_\tau$ are τ-*equivalent* if one can be obtained from the other by a finite sequence of τ-blowings-up and τ-blowings-down. We write $X \overset{\tau}{\sim} X'$ for τ-equivalence.

Remark. The theory of $(-,-)$-equivalence is exactly the content of section 4. In fact we will rarely use the notation $X \overset{(-,-)}{\sim} X'$, since it simply means $X \sim X'$.

Before developing the theory of τ-equivalence, let us explain how it will be used.

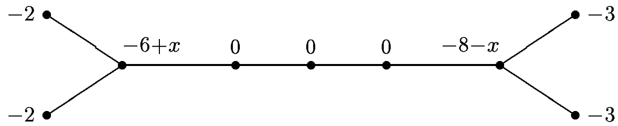
6.2. **Example.** Consider the weighted tree

$\mathcal{G}$:



Consider the path γ in $\mathcal{G}$ which goes from the vertex of weight -5 to that of weight -8; note that $\gamma \in P(\mathcal{G})$ is of type $(+,+)$. One can see (e.g. by 6.8, below) that $(-5, 1, 1, -8) \overset{(+,+)}{\sim} (-6 + x, 0, 0, 0, -8 - x)$ for any $x \in \mathbb{Z}$. Then it follows that $\mathcal{G} \sim \mathcal{G}'$, where

$\mathcal{G}'$:



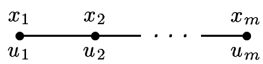
Indeed, the definition of $(+,+)$-equivalence takes into account that a graph cannot be blown-down at a vertex of degree greater than two.

More generally, we state the following (trivial) fact:

6.3. *Let $\mathcal{G}$ be a weighted forest, let $\gamma = (u_1, \dots, u_m) \in P(\mathcal{G})$ and let τ be the type of γ. Write $x_i = w(u_i, \mathcal{G})$ for $i = 1, \dots, m$ and suppose that $(y_1, \dots, y_n) \in \mathbb{Z}^*_\tau$ satisfies*

$$(x_1, \dots, x_m) \overset{\tau}{\sim} (y_1, \dots, y_n).$$

Consider the weighted forest $\mathcal{G}'$ obtained from $\mathcal{G}$ by replacing



by

and leaving the rest of the graph unchanged. Then $\mathcal{G} \sim \mathcal{G}'$.

We shall now prove some properties of τ-equivalence.

Remark. If $X, X' \in \mathbb{Z}^*_{(+,-)} = \mathbb{Z}^*_{(-,+)}$ then $X \overset{(+,-)}{\sim} X' \iff X^- \overset{(-,+)}{\sim} (X')^-$. So any result regarding $(+,-)$-equivalence gives rise to a result on $(-,+)$-equivalence, and vice-versa.

Given $X, Y \in \mathbb{Z}^*$, it is obvious that $X \overset{(+,+)}{\sim} Y$ implies both $X \overset{(+,-)}{\sim} Y$ and $X \overset{(-,+)}{\sim} Y$, and that "$X \overset{(+,-)}{\sim} Y$ or $X \overset{(-,+)}{\sim} Y$" implies $X \sim Y$. Here is a slightly less obvious fact:

6.4. **Lemma.** *Let $X, Y \in \mathbb{Z}^*$ and $a, b, \alpha, \beta \in \mathbb{Z}$.*

(1) *If $(a, X, b) \overset{(+,+)}{\sim} (\alpha, Y, \beta)$ then $(a, X) \overset{(+,-)}{\sim} (\alpha, Y)$ and $(X, b) \overset{(-,+)}{\sim} (Y, \beta)$.*
(2) *If $(a, X) \overset{(+,-)}{\sim} (\alpha, Y)$ then $X \sim Y$.*
(3) *If $(X, b) \overset{(-,+)}{\sim} (Y, \beta)$ then $X \sim Y$.*

Proof. To prove (1), it suffices to consider the case where (α, Y, β) is a $(+,+)$-blowing-up of (a, X, b); but the assertion is easily verified in this case, so (1) is true. Similar arguments prove (2) and (3). $\square$

6.5. **Lemma.** *Let $X, Y \in \mathbb{Z}^*$ be such that $X \sim Y$. Then:*

$$\forall_{a,b \in \mathbb{Z}}\ \exists_{\alpha,\beta \in \mathbb{Z}}\ (a, X, b) \overset{(+,+)}{\sim} (\alpha, Y, \beta),$$

$$\forall_{a \in \mathbb{Z}}\ \exists_{\alpha \in \mathbb{Z}}\ (a, X) \overset{(+,-)}{\sim} (\alpha, Y) \quad \textit{and} \quad \forall_{b \in \mathbb{Z}}\ \exists_{\beta \in \mathbb{Z}}\ (X, b) \overset{(-,+)}{\sim} (Y, \beta).$$

Proof. Let $a, b \in \mathbb{Z}$. Since $X \sim Y$, there exists a sequence s of blowings-up and blowings-down of sequences which transforms X into Y. If we now replace X by (a, X, b) and perform the "same" sequence s of operations, we get a sequence s' of $(+,+)$-blowings-up and $(+,+)$-blowings-down which transforms (a, X, b) into (α, Y, β), for suitable $\alpha, \beta \in \mathbb{Z}$ (this claim is obvious in the case where s consists of a single blowing-up or a single blowing-down, and we may reduce to that case). Thus $(a, X, b) \overset{(+,+)}{\sim} (\alpha, Y, \beta)$. The same argument shows that $(a, X) \overset{(+,-)}{\sim} (\alpha, Y)$ for some α and $(X, b) \overset{(-,+)}{\sim} (Y, \beta)$ for some β. $\square$

6.6. **Lemma.** *Let $X, Y \in \mathbb{Z}^*$ and $a, b, \alpha, \beta \in \mathbb{Z}$. Then:*

(1) $(a, X, b) \overset{(+,+)}{\sim} (\alpha, Y, \beta) \implies \forall_{i,j \in \mathbb{Z}}\ (a+i, X, b+j) \overset{(+,+)}{\sim} (\alpha+i, Y, \beta+j)$
(2) $(a, X) \overset{(+,-)}{\sim} (\alpha, Y) \implies \forall_{i \in \mathbb{Z}}\ (a+i, X) \overset{(+,-)}{\sim} (\alpha+i, Y)$
(3) $(X, b) \overset{(-,+)}{\sim} (Y, \beta) \implies \forall_{j \in \mathbb{Z}}\ (X, b+j) \overset{(-,+)}{\sim} (Y, \beta+j)$

Proof. Suppose that $(a, X, b) \overset{(+,+)}{\sim} (\alpha, Y, \beta)$. Then there exists a sequence of $(+,+)$-blowings-up and $(+,+)$-blowings-down which transforms (a, X, b) into (α, Y, β); clearly, the same sequence of operations applied to $(a+i, X, b+j)$ yields $(\alpha+i, Y, \beta+j)$ (this claim is obvious in the case of a single $(+,+)$-blowing-up or a single $(+,+)$-blowing-down, and we may reduce to that case). This proves (1) and the other assertions are proved by the same argument. $\square$

For the next result, we need:

6.7. **Definition.** Let $X, Y \in \mathbb{Z}^*$ be equivalent sequences, let $d = \det(X) = \det(Y)$ and $n = \|X\| = \|Y\|$. Define the integer $\delta(X, Y)$ by:

$$\delta(X,Y) = \begin{cases} \frac{1}{d}\big(\det_1(X) - \det_1(Y)\big), & \text{if } d \neq 0; \\ (-1)^{n-1}\big(\det_*(X) - \det_*(Y)\big), & \text{if } d = 0. \end{cases}$$

6.8. **Proposition.** *Let $X, Y \in \mathbb{Z}^*$ be such that $X \sim Y$ and let $a, b, \alpha, \beta \in \mathbb{Z}$.*

(1) *If* $\det(X) \neq 0$ *then*
  (a) $(a, X, b) \overset{(+,+)}{\sim} (\alpha, Y, \beta) \iff \alpha = a + \delta(X, Y)$ *and* $\beta = b + \delta(X^-, Y^-)$
  (b) $(a, X) \overset{(+,-)}{\sim} (\alpha, Y) \iff \alpha = a + \delta(X, Y)$
  (c) $(X, b) \overset{(-,+)}{\sim} (Y, \beta) \iff \beta = b + \delta(X^-, Y^-)$.
(2) *If* $\det(X) = 0$ *then*
  (a) $(a, X, b) \overset{(+,+)}{\sim} (\alpha, Y, \beta) \iff \alpha + \beta = a + b + \delta(X, Y)$
  (b) *For all $i, j \in \mathbb{Z}$, $(i, X) \overset{(+,-)}{\sim} (j, Y)$ and $(X, i) \overset{(-,+)}{\sim} (Y, j)$.*

Proof. Throughout, we fix $X, Y \in \mathbb{Z}^*$ such that $X \sim Y$. Let $d = \det(X)$.

Assume that $d \neq 0$.

If $a, \alpha \in \mathbb{Z}$ are such that $(a, X) \overset{(+,-)}{\sim} (\alpha, Y)$ then $(a, X) \sim (\alpha, Y)$ and consequently $\det(a, X) = \det(\alpha, Y)$; then 4.7 gives

$$(-a)d - \det_1(X) = \det(a, X) = \det(\alpha, Y) = (-\alpha)d - \det_1(Y),$$

from which $\alpha = a + \delta(X, Y)$ follows. This proves implication "$\Rightarrow$" of assertion (1b). Conversely, let $a \in \mathbb{Z}$ and $\alpha = a + \delta(X, Y)$. By 6.5, $(a, X) \overset{(+,-)}{\sim} (\alpha_1, Y)$ for some α_1, and by "$\Rightarrow$" of (1b) we have $\alpha_1 = a + \delta(X, Y) = \alpha$. So assertion (1b) is true.

If $b, \beta \in \mathbb{Z}$ then $(X, b) \overset{(-,+)}{\sim} (Y, \beta) \Leftrightarrow (X, b)^- \overset{(+,-)}{\sim} (Y, \beta)^- \Leftrightarrow (b, X^-) \overset{(+,-)}{\sim} (\beta, Y^-)$; so (1b) implies (1c).

If $a, b, \alpha, \beta \in \mathbb{Z}$ are such that $(a, X, b) \overset{(+,+)}{\sim} (\alpha, Y, \beta)$ then, by 6.4, both $(a, X) \overset{(+,-)}{\sim} (\alpha, Y)$ and $(X, b) \overset{(-,+)}{\sim} (Y, \beta)$ hold; then (1b) and (1c) imply that $\alpha = a + \delta(X, Y)$ and $\beta = b + \delta(X^-, Y^-)$, so implication "$\Rightarrow$" of (1a) is proved. The converse follows from "$\Rightarrow$" and the fact (6.5) that $(a, X, b) \overset{(+,+)}{\sim} (\alpha_1, Y, \beta_1)$ for some α_1, β_1 (see the proof of (1b)). This completes the proof of assertion (1).

To prove assertion (2), assume that $\det(X) = 0$. Let $n = \|X\| = \|Y\|$; then 4.23.2 implies that $n \geq 1$ and that $X \sim Z \sim Y$, where $Z = (0^{2n-1}) \in \mathbb{Z}^*$. If $a, b \in \mathbb{Z}$ then

$$\begin{aligned}(a, Z, b) = (a, 0^{2n-1}, b) \overset{(+,+)}{\sim} (a-1, -1, -1, 0^{2n-2}, b) &\overset{4.15}{\sim} (a-1, 0^{2n-2}, -1, -1, b) \\ &\overset{(+,+)}{\sim} (a-1, 0^{2n-2}, 0, b+1) = (a-1, Z, b+1)\end{aligned}$$

and it is easily verified that the equivalence given by 4.15 is actually a $(+,+)$-equivalence. So $(a, Z, b) \overset{(+,+)}{\sim} (a-1, Z, b+1)$ and consequently:

$$\forall_{a,b,\alpha,\beta \in \mathbb{Z}} \quad a + b = \alpha + \beta \implies (a, Z, b) \overset{(+,+)}{\sim} (\alpha, Z, \beta). \tag{11}$$

By 6.4, $(a, Z, b) \overset{(+,+)}{\sim} (\alpha, Z, \beta)$ implies both $(a, Z) \overset{(+,-)}{\sim} (\alpha, Z)$ and $(Z, b) \overset{(-,+)}{\sim} (Z, \beta)$. So (11) implies:

$$\forall_{i,j \in \mathbb{Z}} \quad (i, Z) \overset{(+,-)}{\sim} (j, Z) \text{ and } (Z, i) \overset{(-,+)}{\sim} (Z, j). \tag{12}$$

By 6.5 we have $(0, X) \overset{(+,-)}{\sim} (r, Z)$ and $(0, Y) \overset{(+,-)}{\sim} (s, Z)$ for suitable $r, s \in \mathbb{Z}$; so given any $i, j \in \mathbb{Z}$ we have

$$(i, X) \overset{(+,-)}{\sim} (r+i, Z) \overset{(+,-)}{\sim} (s+j, Z) \overset{(+,-)}{\sim} (j, Y)$$

by 6.6 and (12), and similarly $(X, i) \overset{(-,+)}{\sim} (Y, j)$, which proves assertion (2b).

There remains only to prove (2a). In view of 4.11, the condition $X \sim Z$ implies that $\det_1(X)$ is congruent to $\det_1(Z)$ modulo $\det(X)$; so $\det_1(X) = \det_1(Z) = \det(0^{2n-2}) = (-1)^{n-1}$ (by 4.12) and for the same reason we have:

$$\det_1(X) = \det_1(X^-) = \det_1(Y) = \det_1(Y^-) = (-1)^{n-1}.$$

Let $a, b, \alpha, \beta \in \mathbb{Z}$ be such that $(a, X, b) \overset{(+,+)}{\sim} (\alpha, Y, \beta)$. Then $(a, X, b) \sim (\alpha, Y, \beta)$ and consequently

$$\det(a, X, b) = \det(\alpha, Y, \beta). \tag{13}$$

Using 4.7 three times gives:

$$\det(a, X, b) = (-a)\det(X, b) - \det_1(X, b), \tag{14}$$

$$\det(X, b) = \det(b, X^-) = (-b)\det(X^-) - \det_1(X^-) = -\det_1(X^-) = (-1)^n, \tag{15}$$

$$\det_1(X, b) = \det(x_2, \dots, x_m, b) = (-b)\det_1(X) - \det_*(X) = (-1)^n b - \det_*(X) \tag{16}$$

where we wrote $X = (x_1, \dots, x_m)$. Substituting (15) and (16) into (14) gives $\det(a, X, b) = (-1)^{n-1}(a+b) + \det_*(X)$ and similarly we obtain $\det(\alpha, Y, \beta) = (-1)^{n-1}(\alpha+\beta) + \det_*(Y)$. So (13) gives $(-1)^{n-1}(a+b) + \det_*(X) = (-1)^{n-1}(\alpha+\beta) + \det_*(Y)$ and consequently

$$\alpha + \beta = (a+b) + (-1)^{n-1}(\det_*(X) - \det_*(Y)) = a + b + \delta(X, Y).$$

Conversely, let $a, b, \alpha, \beta \in \mathbb{Z}$ be such that $\alpha + \beta = a + b + \delta(X, Y)$; we show that $(a, X, b) \overset{(+,+)}{\sim} (\alpha, Y, \beta)$. By 6.5, we have $(a, X, b) \overset{(+,+)}{\sim} (\alpha_0, Y, \beta_0)$ for suitable $\alpha_0, \beta_0 \in \mathbb{Z}$; so the above paragraph implies that $\alpha_0 + \beta_0 = a + b + \delta(X, Y)$, hence $\alpha_0 + \beta_0 = \alpha + \beta$. By 6.5 we have $(0, Y, 0) \overset{(+,+)}{\sim} (r, Z, s)$ for suitable $r, s \in \mathbb{Z}$, so

$$(\alpha_0, Y, \beta_0) \overset{(+,+)}{\sim} (r + \alpha_0, Z, s + \beta_0) \overset{(+,+)}{\sim} (r + \alpha, Z, s + \beta) \overset{(+,+)}{\sim} (\alpha, Y, \beta)$$

by 6.6 and (11). Since $(a, X, b) \overset{(+,+)}{\sim} (\alpha_0, Y, \beta_0)$, we conclude that $(a, X, b) \overset{(+,+)}{\sim} (\alpha, Y, \beta)$ and the proof is complete. $\square$

7. Minimal sequences and linear chains

See 1.6 for the notion of minimal weighted graph and 4.2 for equivalence and minimality in $\mathbb{Z}^*$. Consider the following:

Problem 1. *List all minimal elements of a given equivalence class of weighted forests.*

In Section 9, it is shown that the above problem reduces to the special case of linear chains, or more precisely to:

Problem 2. *Given $X \in \mathbb{Z}^*$, list all minimal sequences equivalent to X.*

Apparently, very little is known about these problems. One notable exception is [3], which can be interpreted as solving Problem 2 for $X = (1)$.

This section begins by solving Problem 2 recursively (7.1); together with 7.5, and keeping in mind 4.31.1, this gives substantial information about Problem 2. Then we make use of those results to describe explicitly all minimal elements of certain classes $\mathcal{C} \in \mathbb{Z}^*/\!\sim$; in fact we can do this when $\mathcal{C}$ is either a prime class (4.29) or the successor of a prime class. Finally, we show (7.8) that the cases that we can describe explicitely are precisely those which arise in the study of algebraic surfaces.

The notations $\mathbb{Z}^*/\!\sim$ and $\min(\mathcal{C})$ are defined before 4.27.

7.1. **Proposition.** *If $\mathcal{C} \in \mathbb{Z}^*/\!\sim$ then* $\min\left(\mathcal{C}^\oplus\right) = \bigcup_{M \in \min \mathcal{C}} M^\oplus$.

Proof. The inclusion "$\supseteq$" is trivial by definition 4.28 of $\mathcal{C}^\oplus$. Consider $Z \in \min(\mathcal{C}^\oplus)$. Since $\mathcal{C}^\oplus$ has a predecessor (namely $\mathcal{C}$) but $\mathcal{C}_0$ doesn't by 4.30, we have $\mathcal{C}^\oplus \neq \mathcal{C}_0$ and hence $Z \neq (0)$; we also have $\|Z\| > 0$ by (9); so 4.21 gives $Z \in M^\oplus$ for some minimal element M of $\mathbb{Z}^*$. We have $M \in \mathcal{C}$ by uniqueness of the predecessor of $\mathcal{C}^\oplus$, so $Z \in \cup_{M \in \min \mathcal{C}} M^\oplus$. $\square$

In order to derive explicit results from 7.1, we need to describe the elements of $M^\oplus$ where M is a minimal element of $\mathbb{Z}^*$. In other words, we have to describe the sequences

M' which occur in part (4) of 4.19. Some preliminary work is needed. If $x \in \mathbb{R}$, let $\lceil x \rceil$ denote the least integer n such that $x \le n$.

7.2. **Lemma.** *Let* $\mathcal{M} = \big\{ (x_1, \dots, x_n) \in \mathbb{Z}^* \setminus \{\varnothing\} \mid x_1 \neq -1 \text{ and } \forall_{i>1}\, x_i \le -2 \big\}$ *and* $\mathcal{M}^- = \big\{ X^- \mid X \in \mathcal{M} \big\}$. *Then* $X \mapsto (\det X, \det_1 X)$ *is a bijection from* $\mathcal{M}$ *to*

$$\big\{ (r_0, r_1) \in \mathbb{Z}^2 \mid r_1 > 0,\ \gcd(r_0, r_1) = 1 \text{ and } \lceil \tfrac{r_0}{r_1} \rceil \neq 1 \big\} \tag{17}$$

and $X \mapsto (\det(X^-), \det_1(X^-))$ *is a bijection from* $\mathcal{M}^-$ *to* (17).

Proof. It is well known that $\gcd\big(\det(X), \det_1(X)\big) = 1$ holds for every $X \in \mathbb{Z}^*$. Consider an element $X = (-q, N)$ of $\mathbb{Z}^* \setminus \{\varnothing\}$, where $q \in \mathbb{Z}$ and $N \in \mathbb{Z}^*$. By 4.7,

$$\det(X) = q \det(N) - \det_1(N).$$

If $N \in \mathcal{N}^*$ then by 4.8 we have $0 \le \det_1(N) < \det(N)$, so $q = \lceil \frac{\det X}{\det N} \rceil$; thus $\mathcal{M}$ is mapped into the set (17). If (r_0, r_1) belongs to the set (17), there is a unique pair $(q, r_2) \in \mathbb{Z}^2$ such that $r_0 = q r_1 - r_2$ and $0 \le r_2 < r_1$; by 4.8, a unique $N \in \mathcal{N}^*$ satisfies $\det(N) = r_1$ and $\det_1(N) = r_2$; then $(-q, N) \in \mathcal{M}$ and this defines a map from the set (17) to $\mathcal{M}$. It is clear that the two maps are inverse of each other, so the first assertion is proved. The second assertion follows from the first. □

7.3. **Definition.** Let $Z, Z' \in \mathbb{Z}^*$. We say that Z can be $(+,-)$-*contracted* to Z' (resp. $(-,+)$-*contracted,* $(+,+)$-*contracted*) if there exists a sequence of blowings-down which transforms Z into Z' and such that no blowing-down is performed at the leftmost (resp. rightmost, leftmost or rightmost) term of a sequence. Observe that this condition is stronger than $Z \overset{\tau}{\sim} Z'$ (see 6.1), where $\tau = (+,-)$ (resp. $\tau = (-,+)$, $\tau = (+,+)$).

7.4. **Definition.** Given $\alpha, \beta \in \mathbb{Z}$, define the following subsets of $\mathbb{N}^3$:

$$P = \big\{ (n,p,c) \in \mathbb{N}^3 \mid 1 \le p \le c \text{ and } \gcd(c,p) = 1 \big\}$$
$${}^{\alpha}P = P^{\alpha} = \big\{ (n,p,c) \in \mathbb{N}^3 \mid 1 \le p \le c,\ \gcd(p,c) = 1 \text{ and } \lceil \tfrac{c}{nc+p} \rceil \neq \alpha + 1 \big\}$$
$${}^{\alpha}P^{\beta} = \big\{ (n,p,c) \in \mathbb{N}^3 \mid 1 \le p \le c,\ \gcd(p,c) = 1,\ \lceil \tfrac{c}{nc+p} \rceil \neq \alpha + 1 \text{ and } n \neq \beta \big\}$$

and the following subsets of $\mathbb{Z}^* \times \mathbb{Z}^*$:

$$E = \big\{ (X,Y) \in \mathcal{N}^* \times \mathcal{N}^* \mid (X,-1,Y) \sim \varnothing \big\}$$
$${}^{\alpha}E = \big\{ (X,Y) \in \mathcal{M} \times \mathcal{N}^* \mid (X,-1,Y) \text{ can be } (+,-)\text{-contracted to } (\alpha) \big\}$$
$$E^{\alpha} = \big\{ (X,Y) \in \mathcal{N}^* \times \mathcal{M}^- \mid (X,-1,Y) \text{ can be } (-,+)\text{-contracted to } (\alpha) \big\}$$
$${}^{\alpha}E^{\beta} = \big\{ (X,Y) \in \mathcal{M} \times \mathcal{M}^- \mid (X,-1,Y) \text{ can be } (+,+)\text{-contracted to } (\alpha,\beta) \big\}$$

where $\mathcal{M}$ and $\mathcal{M}^-$ are defined in 7.2. Then define four maps

(1) $f : P \to E$, $(n,p,c) \mapsto (X,Y)$,
(2) ${}^{\alpha}f : {}^{\alpha}P \to {}^{\alpha}E$, $(n,p,c) \mapsto (X,Y)$
(3) $f^{\alpha} : P^{\alpha} \to E^{\alpha}$, $(n,p,c) \mapsto (X,Y)$
(4) ${}^{\alpha}f^{\beta} : {}^{\alpha}P^{\beta} \to {}^{\alpha}E^{\beta}$, $(n,p,c) \mapsto (X,Y)$

by declaring in each case that (X,Y) is the unique pair of sequences satisfying:

(1′) $(X,Y) \in \mathcal{N}^* \times \mathcal{N}^*$ and:

$$\det(X) = nc+p \qquad\qquad \det(Y^-) = c$$
$$\det_1(X) \equiv -c \pmod{nc+p} \qquad\qquad \det_1(Y^-) = c-p$$

(2′) $(X,Y) \in \mathcal{M} \times \mathcal{N}^*$ and:

$$\det(X) = c - \alpha(nc+p) \qquad\qquad \det(Y^-) = c$$
$$\det_1(X) = nc+p \qquad\qquad \det_1(Y^-) = c-p$$

(3′) $(X,Y) \in \mathcal{N}^* \times \mathcal{M}^-$ and:

$$\det(X) = c \qquad\qquad \det(Y^-) = c - \alpha(nc+p)$$
$$\det_1(X) = c-p \qquad\qquad \det_1(Y^-) = nc+p$$

(4′) $(X,Y) \in \mathcal{M} \times \mathcal{M}^-$ and:

$$\det(X) = c - \alpha(nc+p) \qquad\qquad \det(Y^-) = (n-\beta)c+p$$
$$\det_1(X) = nc+p \qquad\qquad \det_1(Y^-) = c.$$

7.4.1. **Lemma.** *The maps f, ${}^\alpha f$, f^α and ${}^\alpha f^\beta$ are well-defined and bijective.*

Proof of 7.4.1. Let us first argue that if $Z \in \mathbb{Z}^*_{(+,+)}$ satisfies $Z \overset{(+,+)}{\sim} (\alpha,\beta)$ then Z can be $(+,+)$-contracted to (α,β). Write $Z = (a, Z_1, b)$ where $a,b \in \mathbb{Z}$ and $Z_1 \in \mathbb{Z}^*$. Then $Z_1 \sim \varnothing$ by 6.4 so, as is well-known, Z_1 "contracts" to $\varnothing$; it follows that Z can be $(+,+)$-contracted to (α',β'), for suitable $\alpha',\beta' \in \mathbb{Z}$. Then $Z \overset{(+,+)}{\sim} (\alpha',\beta')$ and hence $(\alpha',\varnothing,\beta') \overset{(+,+)}{\sim} (\alpha,\varnothing,\beta)$, so 6.8 gives $(\alpha',\beta') = (\alpha,\beta)$, which proves our claim. Similar remarks apply to $(+,-)$- and $(-,+)$-contraction, so the alternative definitions

$${}^\alpha E = \big\{\, (X,Y) \in \mathcal{M} \times \mathcal{N}^* \mid (X,-1,Y) \overset{(+,-)}{\sim} (\alpha) \,\big\}$$
$$E^\alpha = \big\{\, (X,Y) \in \mathcal{N}^* \times \mathcal{M}^- \mid (X,-1,Y) \overset{(-,+)}{\sim} (\alpha) \,\big\}$$
$${}^\alpha E^\beta = \big\{\, (X,Y) \in \mathcal{M} \times \mathcal{M}^- \mid (X,-1,Y) \overset{(+,+)}{\sim} (\alpha,\beta) \,\big\}$$

can be used if convenient.

By 7.2 and 4.8, four injective maps

$$P \to \mathcal{N}^* \times \mathcal{N}^*, \qquad {}^\alpha P \to \mathcal{M} \times \mathcal{N}^*, \qquad P^\alpha \to \mathcal{N}^* \times \mathcal{M}^- \quad \text{and} \quad {}^\alpha P^\beta \to \mathcal{M} \times \mathcal{M}^-$$

are defined by stipulations (1′–4′). The fact that the image of the second map is ${}^\alpha E$ can be derived from 3.23 of [1], or from the reader's favorite technique for handling linear chains. Then it immediately follows that the third map has image E^α (simply because $Z_1 \overset{(-,+)}{\sim} Z_2 \iff Z_1^- \overset{(+,-)}{\sim} Z_2^-$). Let us deduce that the fourth map has image ${}^\alpha E^\beta$ (the case of the first map is easier and is left to the reader). We begin with:

(18) *Let $(X,Y) \in \mathcal{M} \times \mathcal{M}^-$ and write $X = (a, X_1)$ and $Y = (Y_1, b)$, where $a,b \in \mathbb{Z}$ and $X_1, Y_1 \in \mathcal{N}^*$. Then $(X,Y) \in {}^\alpha E^\beta \iff (X,Y_1) \in {}^\alpha E$ and $(X_1,Y) \in E^\beta$.*

Indeed, if $(X,Y) \in {}^{\alpha}E^{\beta}$ then $(X,-1,Y) \overset{(+,+)}{\sim} (\alpha,\beta)$, so $(X,-1,Y_1) \overset{(+,-)}{\sim} (\alpha)$ and $(X_1,-1,Y) \overset{(-,+)}{\sim} (\beta)$ by 6.4, so $(X,Y_1) \in {}^{\alpha}E$ and $(X_1,Y) \in E^{\beta}$.

Conversely, suppose that $(X,Y_1) \in {}^{\alpha}E$ and $(X_1,Y) \in E^{\beta}$. Then

$$(X,-1,Y_1) \overset{(+,-)}{\sim} (\alpha) \tag{19}$$

and $(X_1,-1,Y) \overset{(-,+)}{\sim} (\beta)$. Applying 6.4 to (19) gives $(X_1,-1,Y_1) \sim \varnothing$, so 6.5 gives

$$(X,-1,Y) \overset{(+,+)}{\sim} (\alpha',\beta') \tag{20}$$

for suitable $\alpha',\beta' \in \mathbb{Z}$. Applying 6.4 to (20) gives $(X,-1,Y_1) \overset{(+,-)}{\sim} (\alpha')$, so by (19) we have $(\alpha) \overset{(+,-)}{\sim} (\alpha')$ and hence $\alpha = \alpha'$ by comparing determinants. Similarly, $\beta = \beta'$. So $(X,Y) \in {}^{\alpha}E^{\beta}$ and (18) is proved.

Let $(n,p,c) \in {}^{\alpha}P^{\beta}$ and define $(X,Y) \in \mathcal{M} \times \mathcal{M}^-$ by condition $(4')$. We show that $(X,Y) \in {}^{\alpha}E^{\beta}$. Write $X = (a,X_1)$ and $Y = (Y_1,b)$, with $a,b \in \mathbb{Z}$ and $X_1,Y_1 \in \mathcal{N}^*$. Note that $(4')$ implies

$$\det(Y^-) = (n-\beta+1)c - (c-p), \qquad \text{where } 0 \le c-p < c. \tag{21}$$

From $(4')$ we also have $\det(Y_1^-) = c$, so applying 4.7 to $Y^- = (b, Y_1^-)$ gives

$$\det(Y^-) = -bc - \det{}_1(Y_1^-), \qquad \text{where } 0 \le \det{}_1(Y_1^-) < c \tag{22}$$

(note that $0 \le \det_1(Y_1^-) < \det(Y_1^-) = c$ follows from $Y_1 \in \mathcal{N}^*$ and 4.8); comparing (21) and (22), we obtain $\det_1(Y_1^-) = c-p$. Together with $(4')$, this implies

$$\begin{aligned} \det(X) &= c - \alpha(nc+p) \qquad & \det(Y_1^-) &= c \\ \det{}_1(X) &= nc+p & \det{}_1(Y_1^-) &= c-p. \end{aligned}$$

Since $(n,p,c) \in {}^{\alpha}P$, these equations imply that $(X,Y_1) = {}^{\alpha}f(n,p,c)$, so $(X,Y_1) \in {}^{\alpha}E$.

Define $n' = \left\lceil \frac{c}{nc+p} \right\rceil - 1$, $c' = nc+p$ and $p' = c - n'c'$. Then one can verify that $(n',p',c') \in P^{\beta}$ and that $f^{\beta}(n',p',c') = (X_1,Y)$, so $(X_1,Y) \in E^{\beta}$. By (18), it follows that $(X,Y) \in {}^{\alpha}E^{\beta}$.

This shows that ${}^{\alpha}f^{\beta} : {}^{\alpha}P^{\beta} \to {}^{\alpha}E^{\beta}$ is a well-defined injective map. Surjectivity is also proved by using (18) and the fact that ${}^{\alpha}f$ and f^{β} are well-defined and bijective; that part is easier, and is left to the reader. $\square$

The next result lists all elements of $M^{\oplus}$, where M is any minimal element of $\mathbb{Z}^*$. Since E, ${}^{\alpha}E$, E^{α} and ${}^{\alpha}E^{\beta}$ are explicitely described by 7.4.1, this description of $M^{\oplus}$ is explicit.

7.5. **Proposition.** *The elements of $\varnothing^{\oplus}$ are:*

(i) (1)
(ii) $(0,x)$ *where* $x \in \mathbb{Z} \setminus \{-1\}$
(iii) $(x,0)$ *where* $x \in \mathbb{Z} \setminus \{-1\}$
(iv) $(X,x,0,y,Y)$, *where* $(X,Y) \in E$ *and* $x,y \in \mathbb{Z} \setminus \{-1\}$ *satisfy* $x+y=-1$.

If $M = (m_1,\dots,m_k) \neq \varnothing$ is a minimal element of $\mathbb{Z}^*$, the elements of $M^{\oplus}$ are:

(1) (a) $(0, x, m_1, \dots, m_k)$, *for all* $x \in \mathbb{Z} \setminus \{-1\}$
(b) *the unique minimal sequence obtained by blowing-down* $(0, -1, m_1, \dots, m_k)$
(2) (a) $(m_1, \dots, m_k, x, 0)$, *for all* $x \in \mathbb{Z} \setminus \{-1\}$
(b) *the unique minimal sequence obtained by blowing-down* $(m_1, \dots, m_k, -1, 0)$
(3) *For each* $j \in \{1, \dots, k\}$,
(a) $(m_1, \dots, m_{j-1}, x, 0, y, m_{j+1}, \dots, m_k)$, *for all* $x, y \in \mathbb{Z} \setminus \{-1\}$ *satisfying* $x + y = m_j$
(b) *the unique minimal sequence obtained by blowing-down*

$$(m_1, \dots, m_{j-1}, -1, 0, m_j + 1, m_{j+1}, \dots, m_k)$$

(c) *the unique minimal sequence obtained by blowing-down*

$$(m_1, \dots, m_{j-1}, m_j + 1, 0, -1, m_{j+1}, \dots, m_k)$$

(4) (a) $(X, x, 0, y, Y, m_2, \dots, m_k)$, *for all* $x, y \in \mathbb{Z} \setminus \{-1\}$ *satisfying* $x + y = -1$ *and all* $(X, Y) \in E^{m_1}$
(b) $(m_1, \dots, m_{i-1}, X, x, 0, y, Y, m_{i+2}, \dots, m_k)$, *for all* $x, y \in \mathbb{Z} \setminus \{-1\}$ *satisfying* $x + y = -1$, *all* $(X, Y) \in {}^{m_i}E^{m_{i+1}}$ *and all* i *such that* $1 \le i < k$
(c) $(m_1, \dots, m_{k-1}, X, x, 0, y, Y)$, *for all* $x, y \in \mathbb{Z} \setminus \{-1\}$ *satisfying* $x + y = -1$ *and all* $(X, Y) \in {}^{m_k}E$.

Proof. Follows from definitions 4.19 (of $M^{\oplus}$) and 7.4 (of E, ${}^{\alpha}E$, E^{α} and ${}^{\alpha}E^{\beta}$). $\square$

It is now clear that we can list the minimal elements of any class $\mathcal{C} \in \mathbb{Z}^*/\!\sim$ which is either a prime class or the successor of a prime class. Indeed, the problem is trivial if $\mathcal{C}$ is a prime class, and if $\mathcal{C}$ is the successor of a prime class $\mathcal{C}_1$ then 7.1 gives $\min(\mathcal{C}) = M^{\oplus}$ where M denotes the unique minimal element of $\mathcal{C}_1$; since the set $M^{\oplus}$ is described in 7.5, we obtain the desired list. We give two concrete examples of this process:

7.6. **Example.** Let $\mathcal{C}$ denote the equivalence class of the sequence (1). Then $\mathcal{C} = \mathcal{C}_{\varnothing}^{\oplus}$, where $\mathcal{C}_{\varnothing}$ is the equivalence class of the empty sequence $\varnothing$. We have $\min \mathcal{C} = \varnothing^{\oplus}$ by 7.1 so, by 7.5, the minimal elements of $\mathcal{C}$ are:

- (1)
- $(0, x)$ where $x \in \mathbb{Z} \setminus \{-1\}$
- $(x, 0)$ where $x \in \mathbb{Z} \setminus \{-1\}$
- $(X, x, 0, y, Y)$, where $(X, Y) \in E$ and $x, y \in \mathbb{Z} \setminus \{-1\}$ satisfy $x + y = -1$.

See 7.4.1 for an explicit description of E.

Remark. The result contained in 7.6 first appeared in [3] and was later reproved by several authors.

7.7. **Example.** Let $\mathcal{C} = \mathcal{C}_0^{\oplus}$, where $\mathcal{C}_0 \in \mathbb{Z}^*/\!\sim$ is the equivalence class of the sequence (0). By 4.30, (0) is the unique minimal element of $\mathcal{C}_0$; so 7.1 gives $\min \mathcal{C} = (0)^{\oplus}$ and, by 7.5, the complete list of minimal elements of $\mathcal{C}$ is:

- $(1, 1)$
- $(0, x, 0)$ where $x \in \mathbb{Z} \setminus \{-1\}$
- $(x, 0, -x)$ where $x \in \mathbb{Z} \setminus \{1, -1\}$

- $(X, x, 0, y, Y)$, where $(X, Y) \in E^0 \cup {}^0E$ and $x, y \in \mathbb{Z} \setminus \{-1\}$ satisfy $x + y = -1$.

See 7.4.1 for an explicit description of E^0 and ${}^0E$.

We leave it to the reader to reformulate the above facts (7.1–7.7) in terms of linear chains.

Geometric weighted graphs

If S is a smooth projective algebraic surface over an algebraically closed field, and if D is an SNC-divisor of S, then the pair (D, S) determines a weighted graph $\mathcal{G}(D, S)$ called *the dual graph of D in S* (see for instance [1], [6] or [7]). A weighted graph $\mathcal{G}$ is said to be *geometric* if it is isomorphic to $\mathcal{G}(D, S)$ for some pair (D, S), where we require that every irreducible component of D is a rational curve. The purpose of this subsection is to point out:

7.8. **Proposition.** *For a linear chain $\mathcal{L}$, the following conditions are equivalent:*

(1) *$\mathcal{L}$ is geometric*
(2) *$\|\mathcal{L}\| \leq 1$ or $\mathcal{L} \sim [0, 0, 0]$*
(3) *$\mathcal{L}$ is equivalent to one of the following:*

$$[0],\ [0, 0, 0],\ [A] \text{ or } [0, 0, A] \text{ (for some } A \in \mathcal{N}^*)$$

(4) *Let $X \in \mathbb{Z}^*$ be such that $\mathcal{L} = [X]$; then the equivalence class of X is either a prime class or the successor of a prime class.*

This fact is interesting in connection with the paragraph before 7.6. We don't know a reference for 7.8, but at least part of it is known. Compare with 3.2.4 of [6]. The proof of 7.8 requires the following fact:

7.9. *Let $\mathcal{G}$ be a geometric weighted graph.*

(1) *$\|\mathcal{G}\| \leq 1$ or $\det(\mathcal{G}) = 0$.*
(2) *If $\mathcal{G}' \sim \mathcal{G}$ then $\mathcal{G}'$ is geometric.*
(3) *Every induced subgraph of $\mathcal{G}$ is geometric.*
(4) *Let $\mathcal{G}'$ be a weighted graph with the same underlying graph as $\mathcal{G}$ and such that $w(v, \mathcal{G}') \leq w(v, \mathcal{G})$ holds for every vertex v. Then $\mathcal{G}'$ is geometric.*

Note that a subgraph G' of a graph G is "induced" if every edge of G which has its two endpoints in G' is an edge of G'.

Result 7.9 is well known (the first assertion is a consequence of the Hodge Index Theorem, see for instance [6]; (2) and (3) are trivial and (4) follows from (2) and (3)).

Proof of 7.8. It is clear that (3) is equivalent to (4); we prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Suppose that $\mathcal{L}$ is geometric and that $\det(\mathcal{L}) = 0$. By 4.23.2, $\mathcal{L} \sim [0^{2n+1}]$ for some $n \in \mathbb{N}$; by parts (2) and (3) of 7.9, it follows that $[0^{2n+1}]$ is geometric and then that $[0^{2n}]$ is geometric. We have $\det[0^{2n}] = (-1)^n$ and $\|[0^{2n}]\| = n$ by 4.12 and 4.13, so $n \leq 1$ by part (1) of 7.9. Thus:

$$\textit{If } \mathcal{L} \textit{ is geometric and } \det \mathcal{L} = 0 \textit{ then } \mathcal{L} \textit{ is equivalent to } [0] \textit{ or } [0, 0, 0].$$

The fact that (1) implies (2) follows from this and part (1) of 7.9.

Consider a canonical linear chain $[0^r, A]$ equivalent to $\mathcal{L}$ (with $r \in \mathbb{N}$, $A \in \mathcal{N}^*$ and r is even if $A \neq \varnothing$). If (2) holds then $r < 4$, so $[0^r, A]$ is one of the chains displayed in assertion (3). So (2) implies (3).

To show that (3) implies (1), we have to check that each of $[0]$, $[0,0,0]$, $[A]$, $[0,0,A]$ (where $A \in \mathcal{N}^*$) is geometric; by part (3) of 7.9, it suffices to prove that $[0,0,0]$ and $[0,0,A]$ are geometric, where we may assume that $A \neq \varnothing$. Considering a pair of lines in $\mathbb{P}^2$ shows that $[1,1]$ is geometric; so $[0,0,0] \sim [1,1]$ is geometric. Let $n \geq 1$ be such that $A = (a_1, \dots, a_n)$. If $n = 1$ then $[0,0,A]$ is geometric by applying part (4) of 7.9 to $[0,0,A]$ and $[0,0,0]$; if $n > 1$ then $[0,0,-1,-2^{n-2},-1] \sim [0,0,0]$ is geometric and, by part (4) of 7.9 applied to $[0,0,A]$ and $[0,0,-1,-2^{n-2},-1]$, $[0,0,A]$ is geometric. $\square$

8. Pseudo-minimal forests with a given skeleton

Throughout this section we fix a skeleton S (see 2.1).

8.1. **Notation.** Let $\mathrm{FO}(S)$ denote the set of pseudo-minimal forests $\mathcal{G}$ satisfying:

There exists at least one skeletal map $\sigma : S \dashrightarrow \mathcal{G}$.

Problem 3. *Classify the elements of* $\mathrm{FO}(S)$ *up to equivalence of weighted graphs.*

This is the fundamental problem that has to be solved since, by 3.9 and 3.10, a solution to Problem 3 for every S includes a classification of all minimal weighted forests (and hence of all weighted forests).

8.2. **Definition.** Let $\mathrm{FO}^+(S)$ be the set of ordered pairs $(\sigma, \mathcal{G})$ where $\mathcal{G}$ is a pseudo-minimal forest and $\sigma : S \dashrightarrow \mathcal{G}$ is a skeletal map. Two elements $(\sigma, \mathcal{G}), (\sigma', \mathcal{G}')$ of $\mathrm{FO}^+(S)$ are *isomorphic* if there exists an isomorphism $f : \mathcal{G} \to \mathcal{G}'$ of weighted graphs such that $f \circ \sigma = \sigma'$. Let $p_2 : \mathrm{FO}^+(S) \to \mathrm{FO}(S)$ be the surjection given by $p_2(\sigma, \mathcal{G}) = \mathcal{G}$.

8.3. **Definition.** Let $(\sigma, \mathcal{G}), (\sigma', \mathcal{G}') \in \mathrm{FO}^+(S)$. If $\mathcal{G}'$ is a strict blowing-down of $\mathcal{G}$, and if the blowing-down map $\pi : \mathcal{G} \dashrightarrow \mathcal{G}'$ (defined in 3.4) satisfies $\pi \circ \sigma = \sigma'$, then we say that $(\sigma', \mathcal{G}')$ is a *blowing-down* of $(\sigma, \mathcal{G})$ and that $(\sigma, \mathcal{G})$ is a *blowing-up* of $(\sigma', \mathcal{G}')$. Two elements of $\mathrm{FO}^+(S)$ are *equivalent* (notation: "$\sim$") if one can be obtained from the other via a finite sequence of blowings-up and blowings-down.

Problem 4. *Classify the elements of* $\mathrm{FO}^+(S)$ *up to equivalence* $(\sim)$.

We shall first solve Problem 4 and then derive a solution to Problem 3. The precise relation between the two problems will be described after the solution to Problem 4; we will see that Problem 3 is essentially Problem 4 modulo automorphisms of S. For instance, if S is $\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$ then Problem 3 asks for the classification of all linear chains not equivalent to $\boldsymbol{\varnothing}$ (solved in section 5) and Problem 4 can be seen to be equivalent to the classification of sequences not equivalent to $\varnothing$ (solved in section 4).

Paragraphs 8.4–8.32 solve Problem 4. The machinery developed for solving the problem is, we think, as meaningful as the final answer, stated in 8.32.

8.4. **Definition.** An *edge map* for S is a set map $W : P(S) \to \mathbb{Z}^*$ satisfying the two conditions:

(1) $\forall_{\gamma \in P(S)} \quad W(\gamma^-) = W(\gamma)^-$

(2) $\forall_{\gamma \in P(S)} \quad W(\gamma) \sim \varnothing \implies \gamma$ is of type $(+,+)$.

Two edge maps W and W' for S are *equivalent* if $\forall_{\gamma \in P(S)} \ W(\gamma) \sim W'(\gamma)$ (equivalences in $\mathbb{Z}^*$). The symbol $\Omega(S)$ denotes the set of equivalence classes of edge maps for S. If $\omega \in \Omega(S)$ and $\gamma \in P(S)$, we define

$$\omega(\gamma) = \big\{ \, W(\gamma) \mid W \in \omega \, \big\},$$

so $\omega(\gamma) \subset \mathbb{Z}^*$ is an equivalence class of sequences of integers and it makes sense to speak of the determinant of $\omega(\gamma)$ (see 4.5).

Clearly, if $(\sigma, \mathcal{G}) \in \mathrm{FO}^+(S)$ then the composite $P(S) \xrightarrow{\vec{\sigma}} P(\mathcal{G}) \xrightarrow{W_{\mathcal{G}}} \mathbb{Z}^*$ is an edge map for S (see 2.4 for $\vec{\sigma}$ and 3.7 for $W_{\mathcal{G}}$).

8.5. **Definition.** Given $\omega \in \Omega(S)$, let $\mathrm{FO}^+(S,\omega)$ be the set of pairs $(\sigma, \mathcal{G}) \in \mathrm{FO}^+(S)$ satisfying:

The composite $P(S) \xrightarrow{\vec{\sigma}} P(\mathcal{G}) \xrightarrow{W_{\mathcal{G}}} \mathbb{Z}^*$ *is an element of* ω.

Note that $\big\{ \, \mathrm{FO}^+(S,\omega) \mid \omega \in \Omega(S) \, \big\}$ is a partition of $\mathrm{FO}^+(S)$.

8.6. **Lemma.** *Let* $(\sigma, \mathcal{G}), (\sigma', \mathcal{G}') \in \mathrm{FO}^+(S)$ *and* $\omega \in \Omega(S)$. *If* $(\sigma, \mathcal{G}) \sim (\sigma', \mathcal{G}')$ *and* $(\sigma, \mathcal{G}) \in \mathrm{FO}^+(S,\omega)$, *then* $(\sigma', \mathcal{G}') \in \mathrm{FO}^+(S,\omega)$.

Proof. If $(\sigma', \mathcal{G}')$ is a blowing-down of $(\sigma, \mathcal{G})$, and if $\pi : \mathcal{G} \dashrightarrow \mathcal{G}'$ is the blowing-down map, then for each $\gamma \in P(\mathcal{G})$ the sequences $W_{\mathcal{G}}(\gamma)$ and $W_{\mathcal{G}'}\big(\vec{\pi}(\gamma)\big)$ are equivalent; thus the composites $P(S) \xrightarrow{\vec{\sigma}} P(\mathcal{G}) \xrightarrow{W_{\mathcal{G}}} \mathbb{Z}^*$ and $P(S) \xrightarrow{\vec{\sigma'}} P(\mathcal{G}') \xrightarrow{W_{\mathcal{G}'}} \mathbb{Z}^*$ are equivalent edge maps for S and, consequently, $(\sigma, \mathcal{G}) \in \mathrm{FO}^+(S,\omega) \Leftrightarrow (\sigma', \mathcal{G}') \in \mathrm{FO}^+(S,\omega)$. The desired result follows. $\square$

By 8.6, Problem 4 reduces to classifying the elements of $\mathrm{FO}^+(S,\omega)$ for each $\omega \in \Omega(S)$.

Set-up. Recall that S was fixed at the beginning of the section. From here to 8.28, we fix $\omega \in \Omega(S)$ and classify elements of $\mathrm{FO}^+(S,\omega)$.

8.7. **Definition.** If $(\sigma, \mathcal{G}) \in \mathrm{FO}^+(S,\omega)$, define $T(\sigma, \mathcal{G}) = (W, w)$ where $W : P(S) \to \mathbb{Z}^*$ is the composite $P(S) \xrightarrow{\vec{\sigma}} P(\mathcal{G}) \xrightarrow{W_{\mathcal{G}}} \mathbb{Z}^*$ and $w : \mathrm{Vtx}_{>2}(S) \to \mathbb{Z}$ is the map given by $w(v) = w(\sigma(v), \mathcal{G})$. It is clear that

$$T : \mathrm{FO}^+(S,\omega) \to \omega \times \mathbb{Z}^{\mathrm{Vtx}_{>2}(S)}$$

is surjective and that the inverse image of any element of $\omega \times \mathbb{Z}^{\mathrm{Vtx}_{>2}(S)}$ is an isomorphism class of pairs $(\sigma, \mathcal{G})$ (isomorphism is defined in 8.2).

The transplant operation

8.8. Given $\gamma = (v_0, v_1) \in P(S)$ of type τ, define a map $\mathcal{X}_\gamma : \omega \times \mathbb{Z}^{\mathrm{Vtx}_{>2}(S)} \to \mathbb{Z}^*_\tau$ by

$$\mathcal{X}_\gamma(W, w) = \begin{cases} W(\gamma) & \text{if } \tau = (-,-) \\ \big(w(v_0), W(\gamma)\big) & \text{if } \tau = (+,-) \\ \big(W(\gamma), w(v_1)\big) & \text{if } \tau = (-,+) \\ \big(w(v_0), W(\gamma), w(v_1)\big) & \text{if } \tau = (+,+). \end{cases}$$

8.9. **Definition.** Let $(W, w) \in \omega \times \mathbb{Z}^{\mathrm{Vtx}_{>2}(S)}$, let $\gamma = (v_0, v_1) \in P(S)$ be of type τ and let $\mathcal{Y} \in \mathbb{Z}^*_\tau$ be such that $\mathcal{X}_\gamma(W, w) \overset{\tau}{\sim} \mathcal{Y}$. Then a unique pair $(W', w') \in \omega \times \mathbb{Z}^{\mathrm{Vtx}_{>2}(S)}$ is determined by

- $\mathcal{X}_\gamma(W', w') = \mathcal{Y}$
- W' agrees with W on $P(S) \setminus \{\gamma, \gamma^-\}$
- w' agrees with w on $\mathrm{Vtx}_{>2}(S) \setminus \{v_0, v_1\}$.

We say that (W', w') is obtained by *transplanting* $(\gamma, \mathcal{Y})$ *into* (W, w) and write

$$(W', w') = \mathrm{Transp}(\gamma, \mathcal{Y}; W, w).$$

If this is the case, and if $(\sigma, \mathcal{G}), (\sigma', \mathcal{G}') \in \mathrm{FO}^+(S, \omega)$ satisfy $T(\sigma, \mathcal{G}) = (W, w)$ and $T(\sigma', \mathcal{G}') = (W', w')$, we also say that $(\sigma', \mathcal{G}')$ is obtained by *transplanting* $(\gamma, \mathcal{Y})$ *into* $(\sigma, \mathcal{G})$.

8.9.1. **Lemma.** *Let $(\sigma, \mathcal{G}), (\sigma', \mathcal{G}') \in \mathrm{FO}^+(S, \omega)$ and suppose that $(\sigma', \mathcal{G}')$ is obtained by transplanting some pair $(\gamma, \mathcal{Y})$ into $(\sigma, \mathcal{G})$. Then $(\sigma, \mathcal{G}) \sim (\sigma', \mathcal{G}')$.*

Proof. The weighted graph $\mathcal{G}'$ is obtained from $\mathcal{G}$ by performing the operation described in 6.3; more precisely, the operation is performed on $(u_1, \dots, u_m) = \vec{\sigma}(\gamma) \in P(\mathcal{G})$. So 6.3 gives $\mathcal{G} \sim \mathcal{G}'$ and it is easy to see that $(\sigma, \mathcal{G}) \sim (\sigma', \mathcal{G}')$. ☐

Congruence

See 8.4 for the definition of $\omega(\gamma)$.

8.10. **Definition.** (1) Let $(S, \omega)^\sharp$ denote the forest (not weighted) whose vertex set is $\mathrm{Vtx}_{>2}(S)$ and whose edges are the pairs $\{u, v\}$ of vertices satisfying:

$\gamma = (u, v)$ belongs to $P(S)$ and $\omega(\gamma)$ has determinant zero.

(2) A vertex u of $(S, \omega)^\sharp$ is *special* if it satisfies:

there exists $v \in \mathrm{Vtx}(S)$ such that $\gamma = (u, v)$ belongs to $P(S)$, γ is of type $(+,-)$ and $\omega(\gamma)$ has determinant zero.

(3) Let $Z(S, \omega)$ be the set of all maps $z : \mathrm{Vtx}_{>2}(S) \to \mathbb{Z}$ satisfying: *For each connected component C of $(S, \omega)^\sharp$ which contains no special vertex,*

$$\sum_{v \in \mathrm{Vtx}(C)} z(v) = 0.$$

We give generators for the submodule $Z(S, \omega)$ of the $\mathbb{Z}$-module $\mathbb{Z}^{\mathrm{Vtx}_{>2}(S)}$. For each $\varepsilon = (u, u') \in P(S)$ of type $(+,+)$ and such that $\omega(\varepsilon)$ has determinant zero, define $z_\varepsilon \in Z(S,\omega)$ by $z_\varepsilon(u) = 1$, $z_\varepsilon(u') = -1$ and $z_\varepsilon(v) = 0$ for all $v \in \text{Vtx}_{>2}(S) \setminus \{u, u'\}$; let $Z_e(S,\omega)$ be the set of these z_ε. For each special vertex u of $(S,\omega)^\sharp$, define $z_u \in Z(S,\omega)$ by $z_u(u) = 1$ and $z_u(v) = 0$ for all $v \in \text{Vtx}_{>2}(S) \setminus \{u\}$; let $Z_s(S,\omega)$ be the set of these z_u. Then the reader may verify that

$$Z(S,\omega) \textit{ is generated as a } \mathbb{Z}\textit{-module by } Z_e(S,\omega) \cup Z_s(S,\omega).$$

(4) Define the $\mathbb{Z}$-module $N(S,\omega) = \mathbb{Z}^{\text{Vtx}_{>2}(S)}/Z(S,\omega)$.

(5) Let $\bar{T} : \text{FO}^+(S,\omega) \to \omega \times N(S,\omega)$ be the composite:

$$\begin{array}{ccc} \text{FO}^+(S,\omega) \xrightarrow{T} \omega \times \mathbb{Z}^{\text{Vtx}_{>2}(S)} & \longrightarrow & \omega \times N(S,\omega) \\ \big(W, w\big) & \longmapsto & \big(W, \pi(w)\big) \end{array}$$

where $\pi : \mathbb{Z}^{\text{Vtx}_{>2}(S)} \to N(S,\omega)$ is the canonical epimorphism and T is defined in 8.7. Note that $\bar{T}$ is surjective.

(6) Let $(\sigma, \mathcal{G}), (\sigma', \mathcal{G}') \in \text{FO}^+(S,\omega)$. If $\bar{T}(\sigma,\mathcal{G}) = \bar{T}(\sigma',\mathcal{G}')$ then we write $(\sigma,\mathcal{G}) \equiv (\sigma',\mathcal{G}')$ and say that $(\sigma,\mathcal{G}), (\sigma',\mathcal{G}')$ are *congruent.*

8.11. **Lemma.** *If $(\sigma,\mathcal{G}), (\sigma',\mathcal{G}') \in \text{FO}^+(S,\omega)$ and $(\sigma,\mathcal{G}) \equiv (\sigma',\mathcal{G}')$, then $(\sigma,\mathcal{G}) \sim (\sigma',\mathcal{G}')$.*

Proof. Write $(W,w) = T(\sigma,\mathcal{G})$ and $(W,w') = T(\sigma',\mathcal{G}')$. Since $(\sigma,\mathcal{G}) \equiv (\sigma',\mathcal{G}')$, we have $w' - w \in Z(S,\omega)$ and consequently there exists a sequence $(w_0, \dots, w_n)$ in $\mathbb{Z}^{\text{Vtx}_{>2}(S)}$ satisfying $w_0 = w$, $w_n = w'$ and, for all $i > 0$, $\pm(w_{i+1} - w_i) \in Z_e(S,\omega) \cup Z_s(S,\omega)$ (notation as in part (3) of 8.10). Define $(\sigma_0, \mathcal{G}_0) = (\sigma,\mathcal{G})$, $(\sigma_n,\mathcal{G}_n) = (\sigma',\mathcal{G}')$ and, for each i such that $0 < i < n$, choose $(\sigma_i,\mathcal{G}_i) \in \text{FO}^+(S,\omega)$ such that $T(\sigma_i,\mathcal{G}_i) = (W, w_i)$. Note that $(\sigma_i,\mathcal{G}_i) \equiv (\sigma_{i+1},\mathcal{G}_{i+1})$ for every i; so it suffices to prove that $(\sigma,\mathcal{G}) \sim (\sigma',\mathcal{G}')$ under the assumption that

$$w' - w \in Z_e(S,\omega) \cup Z_s(S,\omega).$$

Assume that $w' - w = z_\varepsilon \in Z_e(S,\omega)$ and write $\varepsilon = (v_0, v_1)$; recall that $\varepsilon \in P(S)$ is of type $(+,+)$ and that $\det W(\varepsilon) = 0$. From $w' = w + z_\varepsilon$ and part (2a) of 6.8 we get

$$\big(w'(v_0), W(\varepsilon), w'(v_1)\big) = \big(w(v_0) + 1, W(\varepsilon), w(v_1) - 1\big) \overset{(+,+)}{\sim} \big(w(v_0), W(\varepsilon), w(v_1)\big),$$

which we rewrite as $\mathcal{X}_\varepsilon(W,w') \overset{(+,+)}{\sim} \mathcal{X}_\varepsilon(W,w)$. So it makes sense to transplant $(\varepsilon, \mathcal{Y})$ into $(\sigma,\mathcal{G})$, where $\mathcal{Y} = \mathcal{X}_\varepsilon(W,w')$, and clearly $\text{Transp}(\varepsilon,\mathcal{Y}; W, w) = (W, w')$. So $(\sigma',\mathcal{G}')$ can be obtained by transplanting $(\varepsilon,\mathcal{Y})$ into $(\sigma,\mathcal{G})$ and, by 8.9.1, $(\sigma,\mathcal{G}) \sim (\sigma',\mathcal{G}')$.

The case where $w' - w \in Z_s(S,\omega)$ is proved in a similar way, except that part (2b) of 6.8 is used in place of (2a). $\square$

TRANSPLANT AS AN ACTION

8.12. For each type τ (6.1), define a restriction map $\mathbb{Z}^*_\tau \to \mathbb{Z}^*$, $X \mapsto X|_\tau$, by declaring that if $X = (x_1, \dots, x_n) \in \mathbb{Z}^*_\tau$ then

$$X|_\tau = \begin{cases} X, & \text{if } \tau = (-,-) \\ (x_2, \dots, x_n), & \text{if } \tau = (+,-) \\ (x_1, \dots, x_{n-1}), & \text{if } \tau = (-,+) \\ (x_2, \dots, x_{n-1}), & \text{if } \tau = (+,+). \end{cases}$$

8.13. **Definition.** Let $\mathcal{R}(S, \omega)$ be the set of ordered pairs $\binom{Y}{\gamma}$ (written vertically) such that $\gamma \in P(S)$ and $Y \in \omega(\gamma)$.

8.14. **Proposition.** *Let $y = \binom{Y}{\gamma} \in \mathcal{R}(S, \omega)$. There is a unique set map*

$$\Theta_y \ : \ \omega \times N(S, \omega) \longrightarrow \omega \times N(S, \omega)$$

which satisfies the following condition:

Let $(W, w) \in \omega \times \mathbb{Z}^{\mathrm{Vtx}_{>2}(S)}$ and $\mathcal{Y} \in \mathbb{Z}^*_\tau$ (where τ is the type of γ) be such that $\mathcal{Y}|_\tau = Y$ and $\mathcal{Y} \overset{\tau}{\sim} \mathcal{X}_\gamma(W, w)$; let $(W', w') = \mathrm{Transp}(\gamma, \mathcal{Y}; W, w)$; then $\Theta_y(W, \pi(w)) = (W', \pi(w'))$, where $\pi : \mathbb{Z}^{\mathrm{Vtx}_{>2}(S)} \to N(S, \omega)$ is the canonical epimorphism.

Proof. Given $(W, w) \in \omega \times \mathbb{Z}^{\mathrm{Vtx}_{>2}(S)}$, choose $\mathcal{Y} \in \mathbb{Z}^*_\tau$ satisfying

$$\mathcal{Y}|_\tau = Y \quad \text{and} \quad \mathcal{Y} \overset{\tau}{\sim} \mathcal{X}_\gamma(W, w) \tag{23}$$

(by 6.5, there exists at least one such $\mathcal{Y}$); then let $(W', w') = \mathrm{Transp}(\gamma, \mathcal{Y}; W, w)$ and define $\theta_y(W, w) = (W', \pi(w'))$. The proof of 8.14 consists in showing that

$$\theta_y : \omega \times \mathbb{Z}^{\mathrm{Vtx}_{>2}(S)} \to \omega \times N(S, \omega)$$

is a well-defined map and satisfies

$$\theta_y(W, w) = \theta_y(W, w + z), \quad \text{for all } W \in \omega,\ w \in \mathbb{Z}^{\mathrm{Vtx}_{>2}(S)} \text{ and } z \in Z(S, \omega). \tag{24}$$

Let $(W, w) \in \omega \times \mathbb{Z}^{\mathrm{Vtx}_{>2}(S)}$. For $i = 1, 2$, consider $\mathcal{Y}_i \in \mathbb{Z}^*_\tau$ such that $\mathcal{Y}_i|_\tau = Y$ and $\mathcal{Y}_i \overset{\tau}{\sim} \mathcal{X}_\gamma(W, w)$; let $(W'_i, w'_i) = \mathrm{Transp}(\gamma, \mathcal{Y}_i; W, w)$. From $\mathcal{Y}_1|_\tau = \mathcal{Y}_2|_\tau$, we obtain $W'_1 = W'_2$. To prove that $\pi(w'_1) = \pi(w'_2)$, we may assume that $\mathcal{Y}_1 \neq \mathcal{Y}_2$; then 6.8 implies that $\det(Y) = 0$ and also describes how $\mathcal{Y}_1$ differs from $\mathcal{Y}_2$; that description implies that $w'_1 - w'_2$ is a multiple of an element of $Z_e(S, \omega) \cup Z_s(S, \omega)$, so in particular $w'_1 - w'_2 \in Z(S, \omega)$ and $\pi(w'_1) = \pi(w'_2)$. Thus $(W'_1, \pi(w'_1)) = (W'_2, \pi(w'_2))$, which shows that θ_y is a well-defined map.

Let $W \in \omega$, $w \in \mathbb{Z}^{\mathrm{Vtx}_{>2}(S)}$ and $z \in Z(S, \omega)$. Choose $\mathcal{Y} \in \mathbb{Z}^*_\tau$ such that $\mathcal{Y}|_\tau = Y$ and $\mathcal{Y} \overset{\tau}{\sim} \mathcal{X}_\gamma(W, w)$; let $(W', w') = \mathrm{Transp}(\gamma, \mathcal{Y}; W, w)$. By 6.6, there exits $\mathcal{Y}^* \in \mathbb{Z}^*_\tau$ such that

$$\mathcal{Y}^*|_\tau = Y, \quad \mathcal{Y}^* \overset{\tau}{\sim} \mathcal{X}_\gamma(W, w + z) \quad \text{and} \quad \mathrm{Transp}(\gamma, \mathcal{Y}^*; W, w + z) = (W', w' + z)$$

(for instance, if $\gamma = (v_0, v_1)$ is of type $(+,-)$ then $\mathcal{X}_\gamma(W, w + z)$ is obtained from $\mathcal{X}_\gamma(W, w)$ by adding $z(v_0)$ to the leftmost term of the sequence; then let $\mathcal{Y}^*$ be the sequence obtained by adding $z(v_0)$ to the leftmost term of $\mathcal{Y}$). By definition of θ_y, we have $\theta_y(W, w+z) = \big(W', \pi(w'+z)\big) = \big(W', \pi(w')\big) = \theta_y(W, w)$, which proves that θ_y satisfies (24). $\square$

8.15. **Definition.** (1) The symbol $\mathcal{R}^*(S,\omega)$ denotes the free monoid on the set $\mathcal{R}(S,\omega)$. That is, the elements of $\mathcal{R}^*(S,\omega)$ are the words $y_1 \cdots y_n$ where $n \in \mathbb{N}$ and $y_1, \dots, y_n \in \mathcal{R}(S,\omega)$, and the operation is concatenation.

(2) If $y \in \mathcal{R}(S,\omega)$ and $(W,\eta) \in \omega \times N(S,\omega)$, define $y(W,\eta) = \Theta_y(W,\eta)$. This extends uniquely to a left-action of the monoid $\mathcal{R}^*(S,\omega)$ on the set $\omega \times N(S,\omega)$.

Because of 8.28 below, this left-action of $\mathcal{R}^*(S,\omega)$ on $\omega \times N(S,\omega)$ plays a central role in the classification. We now investigate the properties of that action. The first fact is easily verified:

8.16. *Let $(W,\eta) \in \omega \times N(S,\omega)$.*

(1) *If $\gamma \in P(S)$ then the element $y = \binom{W(\gamma)}{\gamma}$ of $\mathcal{R}(S,\omega)$ satisfies $y(W,\eta) = (W,\eta)$.*

(2) *Let $y = \binom{Y}{\gamma} \in \mathcal{R}(S,\omega)$ and let $y^- = \binom{Y^-}{\gamma^-}$. Then y^- belongs to $\mathcal{R}(S,\omega)$ and satisfies $y^-(W,\eta) = y(W,\eta)$.*

(3) *If $y = \binom{Y}{\gamma} \in \mathcal{R}(S,\omega)$ and $(W',\eta') = y(W,\eta)$, then W' is determined by:*

$$W'(\gamma) = Y, \quad W'(\gamma^-) = Y^- \quad \textit{and} \quad W'(\gamma') = W(\gamma') \textit{ for all } \gamma' \in P(S) \setminus \{\gamma, \gamma^-\}.$$

8.17. **Definition.** (1) Two elements $\binom{Y_1}{\gamma_1}$ and $\binom{Y_2}{\gamma_2}$ of $\mathcal{R}(S,\omega)$ are said to be *disjoint* if $\{\gamma_1, \gamma_1^-\} \cap \{\gamma_2, \gamma_2^-\} = \varnothing$.

(2) A word $y \in \mathcal{R}^*(S,\omega)$ is *self-disjoint* if the unique $y_1, \cdots, y_n \in \mathcal{R}(S,\omega)$ satisfying $y = y_1 \cdots y_n$ are pairwise disjoint (i.e., y_i, y_j are disjoint whenever $i \neq j$). In particular, the empty word and all elements of $\mathcal{R}(S,\omega)$ are self-disjoint.

8.18. **Lemma.** *Let $\xi \in \omega \times N(S,\omega)$.*

(1) *If $y_1, y_2 \in \mathcal{R}(S,\omega)$ are not disjoint, then $y_2 y_1 \xi = y_2 \xi$.*

(2) *If $y_1, y_2 \in \mathcal{R}(S,\omega)$ are disjoint, then $y_2 y_1 \xi = y_1 y_2 \xi$.*

(3) *Given any $y \in \mathcal{R}^*(S,\omega)$, there exists a self-disjoint word $y' \in \mathcal{R}^*(S,\omega)$ such that $y\xi = y'\xi$.*

Proof. The last assertion easily follows from the first two, so we prove only (1) and (2). Let $\pi : \mathbb{Z}^{\mathrm{Vtx}_{>2}(S)} \to N(S,\omega)$ be the canonical epimorphism and consider an arbitrary element $(W, \pi(w))$ of $\omega \times N(S,\omega)$ (where $w \in \mathbb{Z}^{\mathrm{Vtx}_{>2}(S)}$). We have to prove:

$$(\Theta_{y_2} \circ \Theta_{y_1})(W, \pi(w)) = \begin{cases} \Theta_{y_2}(W, \pi(w)) & \text{if } y_1, y_2 \text{ are not disjoint,} \\ (\Theta_{y_1} \circ \Theta_{y_2})(W, \pi(w)) & \text{if } y_1, y_2 \text{ are disjoint.} \end{cases} \tag{25}$$

For each $i \in \{1,2\}$, write $y_i = \binom{Y_i}{\gamma_i}$ and $\gamma_i = (v_0^i, v_1^i)$, and let τ_i be the type of γ_i. Choose $\mathcal{Y}_i \in \mathbb{Z}^*_{\tau_i}$ satisfying $\mathcal{Y}_i|_{\tau_i} = Y_i$ and $\mathcal{Y}_i \overset{\tau_i}{\sim} \mathcal{X}_{\gamma_i}(W, w)$; let

$$(W_i, w_i) = \mathrm{Transp}(\gamma_i, \mathcal{Y}_i; W, w); \tag{26}$$

then $\Theta_{y_i}(W, \pi(w)) = (W_i, \pi(w_i))$. We record the following consequences of (26) (cf. 8.9):

$$\mathcal{X}_{\gamma_i}(W_i, w_i) = \mathcal{Y}_i \tag{27}$$

$$W_i(\gamma') = W(\gamma') \text{ for all } \gamma' \in P(S) \setminus \{\gamma_i, \gamma_i^-\} \tag{28}$$

$$w_i(v) = w(v) \text{ for all } v \in \mathrm{Vtx}_{>2}(S) \setminus \{v_0^i, v_1^i\}. \tag{29}$$

For each $(i,j) \in \{(1,2),(2,1)\}$, choose $\mathcal{Y}_{ij} \in \mathbb{Z}^*_{\tau_j}$ satisfying $\mathcal{Y}_{ij}|_{\tau_j} = Y_j$ and $\mathcal{Y}_{ij} \overset{\tau_j}{\sim} \mathcal{X}_{\gamma_j}(W_i, w_i)$; let $(W_{ij}, w_{ij}) = \mathrm{Transp}(\gamma_j, \mathcal{Y}_{ij}; W_i, w_i)$; then $\Theta_{y_j}(W_i, \pi(w_i)) = (W_{ij}, \pi(w_{ij}))$ or equivalently

$$(\Theta_{y_j} \circ \Theta_{y_i})(W, \pi(w)) = (W_{ij}, \pi(w_{ij})).$$

Suppose that y_1, y_2 are not disjoint. Then $\gamma_2 \in \{\gamma_1, \gamma_1^-\}$ and, by the second part of 8.16, we may assume that $\gamma_2 = \gamma_1$; in fact we write $\gamma_1 = \gamma_2 = \gamma = (v_0, v_1)$ and $\tau_1 = \tau_2 = \tau$. Note that

$$\mathcal{Y}_2|_\tau = Y_2 \quad \text{and} \quad \mathcal{Y}_2 \overset{\tau}{\sim} \mathcal{X}_\gamma(W, w) \overset{\tau}{\sim} \mathcal{Y}_1 \overset{(27)}{=} \mathcal{X}_\gamma(W_1, w_1)$$

so, when we choose $\mathcal{Y}_{12}$, we may set $\mathcal{Y}_{12} = \mathcal{Y}_2$. If we do this, then

$$(W_{12}, w_{12}) = \mathrm{Transp}(\gamma, \mathcal{Y}_2; W_1, w_1) = \mathrm{Transp}(\gamma, \mathcal{Y}_2; W, w) = (W_2, w_2) \tag{30}$$

where the middle equality is a consequence of (28) and (29). Now (30) implies that

$$(\Theta_{y_2} \circ \Theta_{y_1})(W, \pi(w)) = (W_{12}, \pi(w_{12})) = (W_2, \pi(w_2)) = \Theta_{y_2}(W, \pi(w)),$$

which proves the first part of (25).

Suppose that y_1, y_2 are disjoint. Then $\{v_0^1, v_1^1\} \neq \{v_0^2, v_1^2\}$, but these sets may or may not be disjoint. We prove the second part of (25) in the case where these sets are not disjoint, as this is the most delicate of the two cases. In view of part (2) of 8.16, we may arrange:

$$\gamma_1 = (v_0^1, v_*) \text{ and } \gamma_2 = (v_0^2, v_*), \text{ where } v_0^1, v_0^2, v_* \text{ are distinct.}$$

Note that $\deg(v_*, S) > 1$, so $\deg(v_*, S) > 2$ and $\tau_1, \tau_2 \in \{(-,+),(+,+)\}$. We adopt the following notation: If $\tau \in \{(-,+),(+,+)\}$, $\mathcal{Y} \in \mathbb{Z}^*_\tau$ and $n \in \mathbb{Z}$, then $\mathcal{Y}^n \in \mathbb{Z}^*_\tau$ is the sequence obtained by adding n to the rightmost term of $\mathcal{Y}$.

Let $(i,j) \in \{(1,2),(2,1)\}$. With $n_i = w_i(v_*) - w(v_*)$, we have

$$\mathcal{Y}_j^{n_i}|_{\tau_j} = Y_j \quad \text{and} \quad \mathcal{Y}_j^{n_i} \overset{\tau_j}{\sim} \mathcal{X}_{\gamma_j}(W, w)^{n_i} = \mathcal{X}_{\gamma_j}(W_i, w_i) \tag{31}$$

where the τ_j-equivalence follows from 6.6 and where the last equality is verified directly: If $\tau_j = (-,+)$ (resp. $(+,+)$) then

$$\begin{aligned} \mathcal{X}_{\gamma_j}(W, w)^{n_i} &= (W(\gamma_j), w(v_*) + n_i) \qquad && (\text{resp. } (w(v_0^j), W(\gamma_j), w(v_*) + n_i)) \\ \mathcal{X}_{\gamma_j}(W_i, w_i) &= (W_i(\gamma_j), w_i(v_*)) \qquad && (\text{resp. } (w_i(v_0^j), W_i(\gamma_j), w_i(v_*))) \end{aligned}$$

and these two sequences are equal by (28) and (29). Statement (31) allows us to set $\mathcal{Y}_{ij} = \mathcal{Y}_j^{n_i}$. Then $(W_{ij}, w_{ij}) = \text{Transp}(\gamma_j, \mathcal{Y}_j^{n_i}; W_i, w_i)$ and consequently

$$\mathcal{X}_{\gamma_j}(W_{ij}, w_{ij}) = \mathcal{Y}_j^{n_i} \tag{32}$$

$$W_{ij}(\gamma) = W_i(\gamma) \text{ for all } \gamma \in P(S) \setminus \{\gamma_j, \gamma_j^-\} \tag{33}$$

$$w_{ij}(v) = w_i(v) \text{ for all } v \in \text{Vtx}_{>2}(S) \setminus \{v_0^j, v_*\}. \tag{34}$$

We claim that $w_{12} = w_{21}$. If $v \in \text{Vtx}_{>2}(S) \setminus \{v_0^1, v_0^2, v_*\}$ then, by (34) and (29), $w_{ij}(v) = w_i(v) = w(v)$ and hence $w_{12}(v) = w_{21}(v)$. By (32), $w_{ij}(v_*) = w_j(v_*) + n_i = w_j(v_*) + w_i(v_*) - w(v_*)$, so $w_{12}(v_*) = w_{21}(v_*)$. Finally, let $k \in \{1, 2\}$ and assume that $v_0^k \in \text{Vtx}_{>2}(S)$. If $k = j$ (resp. $k = i$) then by (32) (resp. by (34)) we obtain $w_{ij}(v_0^k) = w_k(v_0^k)$; thus $w_{12}(v_0^k) = w_{21}(v_0^k)$ and we proved that $w_{12} = w_{21}$.

Again, let $k \in \{1, 2\}$. If $k = j$ (resp. $k = i$) then (32) (resp. (33)) implies that $W_{ij}(\gamma_k) = Y_k$; it follows that $W_{12}(\gamma_k) = W_{21}(\gamma_k)$ and that $W_{12}(\gamma_k^-) = W_{21}(\gamma_k^-)$. If $\gamma \in P(S) \setminus \{\gamma_1, \gamma_1^-, \gamma_2, \gamma_2^-\}$ then, by (33) and (28), $W_{ij}(\gamma) = W_i(\gamma) = W(\gamma)$, so $W_{12}(\gamma) = W_{21}(\gamma)$. Hence, $W_{12} = W_{21}$ and we conclude that

$$(\Theta_{y_2} \circ \Theta_{y_1})(W, \pi(w)) = (W_{12}, \pi(w_{12})) = (W_{21}, \pi(w_{21})) = (\Theta_{y_1} \circ \Theta_{y_2})(W, \pi(w)),$$

which proves the second part of (25). □

Since $\mathcal{R}^*(S, \omega)$ is not a group, the following remark is relevant:

8.19. **Corollary.** *If $\xi \in \omega \times N(S, \omega)$, define $\mathcal{O}_\xi = \{\, y\xi \mid y \in \mathcal{R}^*(S, \omega) \,\}$.*

(1) *$\{\, \mathcal{O}_\xi \mid \xi \in \omega \times N(S, \omega) \,\}$ is a partition of the set $\omega \times N(S, \omega)$. The elements of this partition are called the "orbits" of the left-action* 8.15.

(2) *If ξ, ξ' belong to the same orbit, then there exists $y \in \mathcal{R}^*(S, \omega)$ such that $y\xi = \xi'$.*

Proof. This reduces to showing that, if $y \in \mathcal{R}^*(S, \omega)$ and $\xi \in \omega \times N(S, \omega)$, then there exists $y' \in \mathcal{R}^*(S, \omega)$ such that $y'y\xi = \xi$.

It suffices to prove this in the case where $y = \binom{Y}{\gamma} \in \mathcal{R}(S, \omega)$. In this case we define $y' = \binom{W(\gamma)}{\gamma}$, where W is defined by $\xi = (W, \eta)$. Then $y, y' \in \mathcal{R}(S, \omega)$ are not disjoint and, by 8.18, $y'y\xi = y'\xi$. We have $y'\xi = \xi$ by the first part of 8.16, so we are done. □

Note the following consequence of part (3) of 8.16:

8.20. *Consider $y = y_1 \cdots y_n \in \mathcal{R}^*(S, \omega)$, where for each i we have $y_i = \binom{Y_i}{\gamma_i} \in \mathcal{R}(S, \omega)$. Let $(W', \eta') = y\xi$, where ξ is any element of $\omega \times N(S, \omega)$. If y is self-disjoint, then $W'(\gamma_i) = Y_i$ for all $i \in \{1, \dots, n\}$.*

8.21. **Corollary.** *Let $\mathcal{O}$ be an orbit of the left-action defined in* 8.15. *Then the map*

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & \omega \\ (W, \eta) & \longmapsto & W \end{array}$$

is bijective.

Proof. Let the map $(W, \eta) \mapsto W$ be denoted $p : \mathcal{O} \to \omega$. Suppose that $\xi = (W, \eta)$ and $\xi' = (W', \eta')$ belong to $\mathcal{O}$. By 8.19, $\xi' = y\xi$ for some $y \in \mathcal{R}^*(S, \omega)$; by 8.18, we may choose y to be a self-disjoint word. Write $y = y_1 \cdots y_n$ with $y_i = \binom{Y_i}{\gamma_i} \in \mathcal{R}(S, \omega)$. Since

y is self-disjoint, 8.20 gives $W'(\gamma_i) = Y_i$ for all $i \in \{1, \dots, n\}$. If $p(\xi) = p(\xi')$, then $W' = W$ and consequently

$$y = \binom{W(\gamma_1)}{\gamma_1} \cdots \binom{W(\gamma_n)}{\gamma_n}; \tag{35}$$

then $y\xi = \xi$ by part (1) of 8.16, showing that p is injective.

To prove surjectivity, select a subset $\Sigma = \{\gamma_1, \dots, \gamma_n\}$ of $P(S)$ satisfying

$$\forall_{\gamma \in P(S)} \quad \{\gamma, \gamma^-\} \cap \Sigma \text{ is a singleton.} \tag{36}$$

Given $W \in \omega$, define a word $y \in \mathcal{R}^*(S, \omega)$ by (35) and note that y is self-disjoint by (36). This and 8.20 imply that if ξ is an arbitrary element of $\mathcal{O}$ then $y\xi = (W, \eta)$ for some η, i.e., $p(y\xi) = W$. So p is surjective. $\square$

8.22. The following remarks are useful for computing elements of $N(S, \omega)$.

8.22.1. Let $C_1, \dots, C_p$ be the distinct connected components of $(S, \omega)^\sharp$ which contain no special vertex. Then $N(S, \omega)$ may be identified with the free module $\mathbb{Z}^{\{C_1, \dots, C_p\}} = \mathbb{Z}^p$. Indeed, given any map $w : \mathrm{Vtx}_{>2}(S) \to \mathbb{Z}$ define $\bar{w} : \{C_1, \dots, C_p\} \to \mathbb{Z}$ by $\bar{w}(C_i) = \sum_{v \in C_i} w(v)$. Then $\mathbb{Z}^{\mathrm{Vtx}_{>2}(S)} \to \mathbb{Z}^{\{C_1, \dots, C_p\}}$, $w \mapsto \bar{w}$, is surjective and has kernel $Z(S, \omega)$.

8.22.2. **Definition.** Given a connected component C of $(S, \omega)^\sharp$ which contains no special vertex, define

$$P_C^1(S) = \big\{ (u, v) \in P(S) \mid u \in C \text{ and } v \notin C \big\}, \quad P_C^0(S) = \big\{ (u, v) \in P(S) \mid u, v \in C \big\}.$$

See 6.7 for $\delta(X, Y)$; given $(W, W') \in \omega \times \omega$, define

$$\delta_C(W, W') = \sum_{\gamma \in P_C^1(S)} \delta(W(\gamma), W'(\gamma)) \; + \; \frac{1}{2} \sum_{\gamma \in P_C^0(S)} \delta(W(\gamma), W'(\gamma)).$$

Note that $\delta_C(W, W')$ is an integer. Indeed, if $\gamma \in P_C^0(S)$ then $\gamma^- \in P_C^0(S)$ and $\det W(\gamma) = 0$; so $\delta(W(\gamma), W'(\gamma)) = \delta(W(\gamma^-), W'(\gamma^-))$ and $\sum_{\gamma \in P_C^0(S)} \delta(W(\gamma), W'(\gamma))$ is even. Also, if $W'' \in \omega$ then

$$\delta_C(W, W') + \delta_C(W', W'') = \delta_C(W, W''). \tag{37}$$

8.22.3. **Lemma.** *Let $\mathcal{O} \subseteq \omega \times N(S, \omega)$ be an orbit of the left-action* 8.15. *If $(W, \eta), (W', \eta') \in \mathcal{O}$, then*

$$\eta'(C_i) = \eta(C_i) + \delta_{C_i}(W, W') \qquad (1 \le i \le p) \tag{38}$$

where we view η and η' as maps $\{C_1, \dots, C_p\} \to \mathbb{Z}$, as in 8.22.1.

Proof. Since the map $\mathcal{O} \to \omega$ of 8.21 is injective, it is a priori clear that η' is uniquely determined by η, W and W'. We have $(W', \eta') = y(W, \eta)$ for some $y = y_n \cdots y_1 \in \mathcal{R}^*(S, \omega)$ with $y_j \in \mathcal{R}(S, \omega)$. Formula (38) holds trivially if $n = 0$; the case $n = 1$ follows from part (3) of 8.16 and 6.8. If $n > 1$ write $(W_0, \eta_0) = (W, \eta)$ and, for $0 < j \le n$, $(W_j, \eta_j) = y_j(W_{j-1}, \eta_{j-1})$; then $\eta_j(C_i) - \eta_{j-1}(C_i) = \delta_{C_i}(W_{j-1}, W_j)$ holds for every j, so $\eta'(C_i) - \eta(C_i) = \sum_{j=1}^n (\eta_j(C_i) - \eta_{j-1}(C_i)) = \sum_{j=1}^n \delta_{C_i}(W_{j-1}, W_j)$, and this is equal to $\delta_{C_i}(W, W')$ by (37). $\square$

8.23. **Definition.** An edge map W for S is *canonical* if

$$\forall_{\gamma \in P(S)} \text{ at least one of } W(\gamma),\ W(\gamma^-) \text{ is a canonical sequence}$$

(see 4.22 for the notion of canonical sequence).

8.24. **Lemma.** *Let W, W' be canonical elements of ω and let C be a connected component of $(S,\omega)^\sharp$ which contains no special vertex. Then $\delta_C(W, W') = 0$.*

Proof. It suffices to show that if $X, Y \in \mathbb{Z}^*$ are equivalent sequences satisfying:

$$\text{at least one of } X,\ X^- \text{ and one of } Y,\ Y^- \text{ is a canonical sequence,}$$

then $\delta(X, Y) = 0$. Indeed, if this is true then $\delta(W(\gamma), W'(\gamma)) = 0$ for all $\gamma \in P(S)$, and consequently $\delta_C(W, W') = 0$. We may assume that $X \neq Y$, otherwise $\delta(X, Y) = 0$ holds trivially. This implies that $\det X \neq 0$ and $\{X, Y\} = \big\{(0^{2i}, A), (A, 0^{2i})\big\}$, where $i > 0$ and $\varnothing \neq A \in \mathcal{N}^*$. Since $\delta(Y, X) = -\delta(X, Y)$, we may assume that $X = (0^{2i}, A)$ and $Y = (A, 0^{2i})$. Then one can obtain $\delta(X, Y) = 0$ by a direct calculation. Alternatively, observe that the equivalence $(0, 0^{2i}, A, 0) \sim (0, A, 0^{2i}, 0)$ given by 4.15 is in fact a $(+,+)$-equivalence; thus $(0, X, 0) \overset{(+,+)}{\sim} (0, Y, 0)$ and we obtain $\delta(X, Y) = 0$ by part (1a) of 6.8. $\square$

Recall from 4.23 that each equivalence class of sequences in $\mathbb{Z}^*$ contains exactly one canonical sequence; so $\big\{\, W \in \omega \mid W \text{ is canonical} \,\big\}$ is nonempty and finite. So 8.21 implies:

8.25. **Corollary.** *Let $\mathcal{O}$ be an orbit of the left-action defined in 8.15. Then the set $\big\{\, (W, \eta) \in \mathcal{O} \mid W$ is canonical$\,\big\}$ is finite and nonempty.*

8.26. **Lemma.** *Let $(W, \eta), (W', \eta') \in \omega \times N(S, \omega)$ where W and W' are canonical edge maps. Then the following are equivalent:*

(1) *(W, η) and (W', η') belong to the same orbit $\mathcal{O} \subseteq \omega \times N(S, \omega)$*
(2) *$\eta = \eta'$.*

Proof. The fact that (1) implies (2) is an immediate consequence of 8.22.3 and 8.24. To prove the converse, assume that $(W, \eta), (W', \eta) \in \omega \times N(S, \omega)$ are such that W and W' are canonical. By 8.21, the orbit $\mathcal{O}$ of (W, η) contains an element (W', η^*) for some $\eta^* \in N(S, \omega)$. By applying the fact that (1) implies (2) to the pairs (W, η) and (W', η^*), we obtain $\eta = \eta^*$; so $(W', \eta) \in \mathcal{O}$, as desired. $\square$

8.27. **Corollary.** *There exists a unique map*

$$\pi_2 \ :\ \big\{\, \mathcal{O}_\xi \mid \xi \in \omega \times N(S, \omega) \,\big\} \longrightarrow N(S, \omega)$$

satisfying the following condition:

$(*)$ *If $\xi = (W, \eta) \in \omega \times N(S, \omega)$ and W is a canonical edge map, $\pi_2(\mathcal{O}_\xi) = \eta$.*

Moreover, π_2 is bijective.

Proof. That $(*)$ determines a unique set map is clear from 8.25 and 8.26. In fact, 8.26 also implies that π_2 is injective. If η is any element of $N(S,\omega)$ then pick a canonical edge map $W \in \omega$ and set $\xi = (W,\eta)$; then $\pi_2(\mathcal{O}_\xi) = \eta$, so π_2 is also surjective. $\square$

8.28**. Proposition.** *Consider the map $\bar{T} : \mathrm{FO}^+(S,\omega) \to \omega \times N(S,\omega)$ defined in* 8.10*. If $\mathcal{C} \subseteq \mathrm{FO}^+(S,\omega)$ is an equivalence class then $\bar{T}$ maps $\mathcal{C}$ onto an orbit $\mathcal{O} \subseteq \omega \times N(S,\omega)$ of the action* 8.15*. Moreover, $\bar{T}^{-1}(\mathcal{O}) = \mathcal{C}$.*

Proof. Let $(\sigma,\mathcal{G}),(\sigma',\mathcal{G}') \in \mathrm{FO}^+(S,\omega)$; it suffices to show that the following are equivalent:

$$(\sigma,\mathcal{G}) \sim (\sigma',\mathcal{G}') \tag{39}$$

$$y\bar{T}(\sigma,\mathcal{G}) = \bar{T}(\sigma',\mathcal{G}'), \text{ for some } y \in \mathcal{R}^*(S,\omega). \tag{40}$$

Let $(W,w) = T(\sigma,\mathcal{G})$ and $(W',w') = T(\sigma',\mathcal{G}')$. Let $\pi : \mathbb{Z}^{\mathrm{Vtx}_{>2}(S)} \to N(S,\omega)$ be the canonical epimorphism.

If $(\sigma',\mathcal{G}')$ is either a blowing-up or a blowing-down of $(\sigma,\mathcal{G})$ then it is quite clear that $(\sigma',\mathcal{G}')$ can be obtained by transplanting a suitable $(\gamma,\mathcal{Y})$ into $(\sigma,\mathcal{G})$, so $(W',w') = \mathrm{Transp}(\gamma,\mathcal{Y};W,w)$; let $Y = \mathcal{Y}|_\tau$ (where τ is the type of γ) and $y = \binom{Y}{\gamma} \in \mathcal{R}(S,\omega)$, then $\Theta_y(W,\pi(w)) = (W',\pi(w'))$ and hence (40) holds. It follows that (39) implies (40).

Conversely, suppose that (40) holds.

If y is the empty word then $(\sigma,\mathcal{G}) \equiv (\sigma',\mathcal{G}')$, so (39) holds by 8.11.

If $y = \binom{Y}{\gamma} \in \mathcal{R}(S,\omega)$ then, since (40) holds, we have $\Theta_y(W,\pi(w)) = (W',\pi(w'))$. Choose $\mathcal{Y} \in \mathbb{Z}^*_\tau$ (where τ is the type of γ) such that $\mathcal{Y}|_\tau = Y$ and $\mathcal{Y} \overset{\tau}{\sim} \mathcal{X}_\gamma(W,w)$; let

$$(W^*,w^*) = \mathrm{Transp}(\gamma,\mathcal{Y};W,w). \tag{41}$$

Then by definition of Θ_y we have $\Theta_y(W,\pi(w)) = (W^*,\pi(w^*))$, so

$$(W',\pi(w')) = (W^*,\pi(w^*)). \tag{42}$$

Let $(\sigma^*,\mathcal{G}^*) \in \mathrm{FO}^+(S,\omega)$ be such that $T(\sigma^*,\mathcal{G}^*) = (W^*,w^*)$. Then (41) means that $(\sigma^*,\mathcal{G}^*)$ is obtained by transplanting $(\gamma,\mathcal{Y})$ into $(\sigma,\mathcal{G})$, so $(\sigma,\mathcal{G}) \sim (\sigma^*,\mathcal{G}^*)$ by 8.9.1; and (42) means that $(\sigma',\mathcal{G}') \equiv (\sigma^*,\mathcal{G}^*)$, so $(\sigma',\mathcal{G}') \sim (\sigma^*,\mathcal{G}^*)$ by 8.11. So (39) holds.

By induction on the length of the word y, it follows that (40) implies (39). $\square$

Solution to Problem 4

Recall that S is fixed (but, from here on, ω is no longer fixed).

8.29**. Definition.** A weighted forest $\mathcal{G}$ is *canonical* if

$$\forall_{\gamma \in P(\mathcal{G})} \text{ one of } W_\mathcal{G}(\gamma),\ W_\mathcal{G}(\gamma^-) \text{ is a canonical sequence}$$

(note that this condition implies that $\mathcal{G}$ is minimal).

8.30**. Lemma.** *If $(\sigma,\mathcal{G}) \in \mathrm{FO}^+(S)$ then there exists $(\sigma',\mathcal{G}') \in \mathrm{FO}^+(S)$ such that $(\sigma,\mathcal{G}) \sim (\sigma',\mathcal{G}')$ and $\mathcal{G}'$ is a canonical forest.*

Proof. Let $\omega \in \Omega(S)$ be such that $(\sigma, \mathcal{G}) \in \mathrm{FO}^+(S, \omega)$; by 8.28, $\bar{T} : \mathrm{FO}^+(S, \omega) \to \omega \times N(S, \omega)$ maps the equivalence class $\mathcal{C} \subseteq \mathrm{FO}^+(S, \omega)$ of $(\sigma, \mathcal{G})$ onto an orbit $\mathcal{O} \subseteq \omega \times N(S, \omega)$. So the composite $\mathcal{C} \xrightarrow{\bar{T}} \mathcal{O} \xrightarrow{p} \omega$ (where p is the bijection of 8.21) is surjective and some $(\sigma', \mathcal{G}') \in \mathcal{C}$ gets mapped to a canonical element of ω. Then $\mathcal{G}'$ is canonical. $\square$

8.31. **Definition.** Let $\mathfrak{X}(S)$ be the set of pairs (ω, η) such that $\omega \in \Omega(S)$ and $\eta \in N(S, \omega)$.

Here is the solution to Problem 4:

8.32. **Theorem.** *There exists a unique surjective map*

$$Q : \mathrm{FO}^+(S) \longrightarrow \mathfrak{X}(S)$$

which satisfies the following conditions:

(1) *For* $(\sigma, \mathcal{G}), (\sigma', \mathcal{G}') \in \mathrm{FO}^+(S)$, $(\sigma, \mathcal{G}) \sim (\sigma', \mathcal{G}') \iff Q(\sigma, \mathcal{G}) = Q(\sigma', \mathcal{G}')$.

(2) *For any* $(\sigma, \mathcal{G}) \in \mathrm{FO}^+(S)$ *such that* $\mathcal{G}$ *is a canonical forest, let* ω *be the element of* $\Omega(S)$ *such that* $(\sigma, \mathcal{G}) \in \mathrm{FO}^+(S, \omega)$ *and let* $\eta \in N(S, \omega)$ *be such that* $\bar{T}(\sigma, \mathcal{G}) = (W, \eta)$ *for some* W*; then* $Q(\sigma, \mathcal{G}) = (\omega, \eta)$.

Proof. In view of 8.30, it is clear that Q is completely determined by conditions (1) and (2). So it suffices to prove the existence of Q.

For each $\omega \in \Omega(S)$, let $Q_\omega : \mathrm{FO}^+(S, \omega) \to \mathfrak{X}(S)$ be the composite

$$\begin{array}{ccccccc} \mathrm{FO}^+(S, \omega) & \xrightarrow{\alpha} & \{\, \mathcal{O}_\xi \mid \xi \in \omega \times N(S, \omega) \,\} & \xrightarrow{\beta} & \{\omega\} \times N(S, \omega) & \hookrightarrow & \mathfrak{X}(S) \\ (\sigma, \mathcal{G}) & \longmapsto & \text{orbit of } \bar{T}(\sigma, \mathcal{G}) & & & & \\ & & \mathcal{O} & \longmapsto & \big(\omega, \pi_2(\mathcal{O})\big) & & \end{array}$$

(see 8.27 for π_2). Since $\{\, \mathrm{FO}^+(S, \omega) \mid \omega \in \Omega(S) \,\}$ is a partition of $\mathrm{FO}^+(S)$, we obtain a map $Q : \mathrm{FO}^+(S) \to \mathfrak{X}(S)$ by taking the union of the Q_ω. Note that α is surjective, because $\bar{T} : \mathrm{FO}^+(S, \omega) \to \omega \times N(S, \omega)$ is surjective; since β is bijective by 8.27, it follows that the image of Q_ω is $\{\omega\} \times N(S, \omega)$. Consequently, Q is surjective.

If $(\omega, \eta) \in \{\omega\} \times N(S, \omega)$ then, since β is bijective, $Q_\omega^{-1}(\omega, \eta)$ is equal to the inverse image by $\bar{T}$ of some orbit; by 8.28, this is an equivalence class in $\mathrm{FO}^+(S, \omega)$ (hence also in $\mathrm{FO}^+(S)$), so Q satisfies condition (1). Because β is defined in terms of π_2, Q satisfies condition (2) (see condition $(*)$ in 8.27). $\square$

Solution to Problem 3

We consider the group $A = \mathrm{Aut}(S)$ of graph automorphisms of S.

8.33. **Definition.** We define a right-action of the group A on the set $\mathfrak{X}(S)$.

(1) If W is an edge map for S and $\alpha \in A$, let $W^\alpha = W \circ \vec{\alpha} : P(S) \to \mathbb{Z}^*$ and note that W^α is an edge map for S; this defines a right-action of A on the set of edge maps for S.

(2) If $\omega \in \Omega(S)$ and $\alpha \in A$, let $\omega^\alpha = \{\, W^\alpha \mid W \in \omega \,\} \in \Omega(S)$. This is a right-action of A on the set $\Omega(S)$.

(3) Again, let $\omega \in \Omega(S)$ and $\alpha \in A$. Then the restriction $\alpha^\sharp : \mathrm{Vtx}_{>2}(S) \to \mathrm{Vtx}_{>2}(S)$ of α is an isomorphism of graphs, $\alpha^\sharp : (S, \omega^\alpha)^\sharp \to (S, \omega)^\sharp$, and maps the special vertices of $(S, \omega^\alpha)^\sharp$ to those of $(S, \omega)^\sharp$ (refer to 8.10 for all this). Consequently, the automorphism of $\mathbb{Z}$-modules

$$\mathbb{Z}^{\mathrm{Vtx}_{>2}(S)} \longrightarrow \mathbb{Z}^{\mathrm{Vtx}_{>2}(S)}, \quad w \longmapsto w \circ \alpha^\sharp$$

maps $Z(S, \omega)$ onto $Z(S, \omega^\alpha)$ and hence determines an isomorphism of $\mathbb{Z}$-modules

$$N(S, \omega) \to N(S, \omega^\alpha), \quad \eta \longmapsto \eta^\alpha.$$

(4) If $(\omega, \eta) \in \mathfrak{X}(S)$ and $\alpha \in A$, let $(\omega, \eta)^\alpha = (\omega^\alpha, \eta^\alpha) \in \mathfrak{X}(S)$ where ω^α and η^α are defined in parts (2) and (3) of this definition. This defines a right-action of the group A on the set $\mathfrak{X}(S)$. The set of orbits is denoted $\mathfrak{X}(S)/A$.

See 8.1 and 8.2 for the definitions of $\mathrm{FO}(S)$, $\mathrm{FO}^+(S)$ and $p_2 : \mathrm{FO}^+(S) \to \mathrm{FO}(S)$. The following result solves Problem 3:

8.34. **Theorem.** *There exists a unique map* $\bar{Q} : \mathrm{FO}(S) \longrightarrow \mathfrak{X}(S)/A$ *such that*

$$\begin{array}{ccc} \mathrm{FO}^+(S) & \xrightarrow{Q} & \mathfrak{X}(S) \\ \Big\downarrow{\scriptstyle p_2} & & \Big\downarrow \\ \mathrm{FO}(S) & \xrightarrow{\bar{Q}} & \mathfrak{X}(S)/A \end{array} \tag{43}$$

commutes, where Q is defined in 8.32 *and where $\mathfrak{X}(S) \to \mathfrak{X}(S)/A$ is the canonical quotient map. Moreover, $\bar{Q}$ is surjective and given any $\mathcal{G}, \mathcal{G}' \in \mathrm{FO}(S)$ we have*

$$\mathcal{G} \sim \mathcal{G}' \iff \bar{Q}(\mathcal{G}) = \bar{Q}(\mathcal{G}').$$

Some facts are needed for the proof of 8.34.

8.35. **Definition.** Given $(\sigma, \mathcal{G}) \in \mathrm{FO}^+(S)$ and $\alpha \in A$, let $(\sigma, \mathcal{G})^\alpha = (\sigma \circ \alpha, \mathcal{G}) \in \mathrm{FO}^+(S)$. This defines a right-action of the group A on the set $\mathrm{FO}^+(S)$. The set of orbits is denoted $\mathrm{FO}^+(S)/A$.

8.36. **Lemma.** *If $\mathcal{G} \in \mathrm{FO}(S)$, then $p_2^{-1}(\mathcal{G})$ is an orbit of the right-action* 8.35. *Moreover, $\mathcal{G} \mapsto p_2^{-1}(\mathcal{G})$ defines a bijection* $\mathrm{FO}(S) \to \mathrm{FO}^+(S)/A$.

Proof. By 2.7, the skeletal map $S \dashrightarrow \mathcal{G}$ is unique up to automorphism of S. □

8.37. **Lemma.** *Let $(\sigma, \mathcal{G}) \in \mathrm{FO}^+(S)$ and $\alpha \in A$. If $\bar{T}(\sigma, \mathcal{G}) = (W, \eta) \in \omega \times N(S, \omega)$, then $\bar{T}\big((\sigma, \mathcal{G})^\alpha\big) = (W^\alpha, \eta^\alpha) \in \omega^\alpha \times N(S, \omega^\alpha)$.*

Proof. Let $(W, w) = T(\sigma, \mathcal{G})$, then $T\big((\sigma, \mathcal{G})^\alpha\big) = T(\sigma \circ \alpha, \mathcal{G}) = (W^\alpha, w \circ \alpha^\sharp)$ is clear and the result follows. □

8.38. **Lemma.** *Given $(\sigma, \mathcal{G}), (\sigma', \mathcal{G}') \in \mathrm{FO}^+(S)$ and $\alpha \in A$,*

$$(\sigma, \mathcal{G}) \sim (\sigma', \mathcal{G}') \iff (\sigma, \mathcal{G})^\alpha \sim (\sigma', \mathcal{G}')^\alpha.$$

Proof. It suffices to prove that if $(\sigma', \mathcal{G}')$ is a blowing-down of $(\sigma, \mathcal{G})$ then $(\sigma', \mathcal{G}')^\alpha$ is a blowing-down of $(\sigma, \mathcal{G})^\alpha$. Let $\pi : \mathcal{G} \dashrightarrow \mathcal{G}'$ be the blowing-down map; then $\sigma' = \pi \circ \sigma$ and consequently $\sigma' \circ \alpha = \pi \circ (\sigma \circ \alpha)$, so $(\sigma' \circ \alpha, \mathcal{G}') = (\sigma', \mathcal{G}')^\alpha$ is a blowing-down of $(\sigma \circ \alpha, \mathcal{G}) = (\sigma, \mathcal{G})^\alpha$. □

8.39. **Lemma.** *$Q\big((\sigma, \mathcal{G})^\alpha\big) = Q(\sigma, \mathcal{G})^\alpha$, for all $(\sigma, \mathcal{G}) \in \mathrm{FO}^+(S)$ and $\alpha \in A$.*

Proof. By 8.30, we may choose $(\sigma', \mathcal{G}') \in \mathrm{FO}^+(S)$ such that $(\sigma', \mathcal{G}') \sim (\sigma, \mathcal{G})$ and $\mathcal{G}'$ is a canonical forest. Then $(\sigma, \mathcal{G}) \sim (\sigma', \mathcal{G}')$ implies $Q(\sigma, \mathcal{G}) = Q(\sigma', \mathcal{G}')$ and hence $Q(\sigma, \mathcal{G})^\alpha = Q(\sigma', \mathcal{G}')^\alpha$. By 8.38 we have $(\sigma, \mathcal{G})^\alpha \sim (\sigma', \mathcal{G}')^\alpha$, so $Q\big((\sigma, \mathcal{G})^\alpha\big) = Q\big((\sigma', \mathcal{G}')^\alpha\big)$. So it suffices to prove that $Q\big((\sigma', \mathcal{G}')^\alpha\big) = Q(\sigma', \mathcal{G}')^\alpha$, i.e., we may assume that $\mathcal{G}$ is canonical.

Let ω be the element of $\Omega(S)$ such that $(\sigma, \mathcal{G}) \in \mathrm{FO}^+(S, \omega)$ and let $(W, \eta) = \bar{T}(\sigma, \mathcal{G}) \in \omega \times N(S, \omega)$; then 8.37 gives $\bar{T}\big((\sigma, \mathcal{G})^\alpha\big) = (W^\alpha, \eta^\alpha) \in \omega^\alpha \times N(S, \omega^\alpha)$. Since $\mathcal{G}$ is canonical, $Q(\sigma, \mathcal{G})$ and $Q\big((\sigma, \mathcal{G})^\alpha\big) = Q(\sigma \circ \alpha, \mathcal{G})$ may be computed by applying part (2) of 8.32; this gives $Q(\sigma, \mathcal{G}) = (\omega, \eta)$ and $Q\big((\sigma, \mathcal{G})^\alpha\big) = (\omega^\alpha, \eta^\alpha)$, which is the desired statement. □

8.40. **Proposition.** *Let $(\sigma, \mathcal{G}) \in \mathrm{FO}^+(S)$. For a pseudo-minimal forest $\mathcal{G}'$, the following are equivalent:*

(1) $\mathcal{G} \sim \mathcal{G}'$
(2) *There exists $\sigma' : S \dashrightarrow \mathcal{G}'$ such that $(\sigma, \mathcal{G}) \sim (\sigma', \mathcal{G}')$ in $\mathrm{FO}^+(S)$.*

Proof. Only (1) $\Rightarrow$ (2) requires a proof. Recall from 3.10 that if (1) holds then $\mathcal{G}$ is strictly equivalent to $\mathcal{G}'$. So we may assume that $\mathcal{G}'$ is a strict blowing-down or a strict blowing-up of $\mathcal{G}$.

Suppose that $\mathcal{G}'$ is a strict blowing-down of $\mathcal{G}$. Let $\pi : \mathcal{G} \dashrightarrow \mathcal{G}'$ be the blowing-down map defined in 3.4 and let $\sigma' = \pi \circ \sigma$; then it is immediate that $(\sigma', \mathcal{G}')$ belongs to $\mathrm{FO}^+(S)$ and is a blowing-down of $(\sigma, \mathcal{G})$.

If $\mathcal{G}'$ is a strict blowing-up of $\mathcal{G}$ then, by 2.6, σ factors through the blowing-down map $\pi : \mathcal{G}' \dashrightarrow \mathcal{G}$ and consequently there exists a skeletal map $\sigma' : S \dashrightarrow \mathcal{G}'$ such that $(\sigma', \mathcal{G}')$ is a blowing-up of $(\sigma, \mathcal{G})$. □

Proof of 8.34. By 8.36 and 8.39, if two elements of $\mathrm{FO}^+(S)$ have the same image via p_2 then they have the same image via the composite $\mathrm{FO}^+(S) \xrightarrow{Q} \mathfrak{X}(S) \to \mathfrak{X}(S)/A$; thus there exists a unique map $\bar{Q}$ such that (43) is commutative. Since $\mathfrak{X}(S) \to \mathfrak{X}(S)/A$ and Q are surjective, so is $\bar{Q}$. Let $\mathcal{G}, \mathcal{G}' \in \mathrm{FO}(S)$. Pick any skeletal maps $\sigma : S \dashrightarrow \mathcal{G}$

and $\sigma' : S \dashrightarrow \mathcal{G}'$ and consider $(\sigma, \mathcal{G}), (\sigma', \mathcal{G}') \in \mathrm{FO}^+(S)$. Then

$$\begin{aligned}
\bar{Q}(\mathcal{G}) = \bar{Q}(\mathcal{G}') &\overset{(43)}{\iff} Q(\sigma, \mathcal{G}) \text{ and } Q(\sigma', \mathcal{G}') \text{ belong to the same } A\text{-orbit} \\
&\iff \exists_{\alpha \in A}\ Q(\sigma, \mathcal{G}) = Q(\sigma', \mathcal{G}')^\alpha \overset{8.39}{=} Q\big((\sigma', \mathcal{G}')^\alpha\big) \\
&\overset{8.32}{\iff} \exists_{\alpha \in A}\ (\sigma, \mathcal{G}) \sim (\sigma', \mathcal{G}')^\alpha \\
&\overset{8.36}{\iff} \exists_{\sigma^* : S \dashrightarrow \mathcal{G}'}\ (\sigma, \mathcal{G}) \sim (\sigma^*, \mathcal{G}') \overset{8.40}{\iff} \mathcal{G} \sim \mathcal{G}',
\end{aligned}$$

which completes the proof of 8.34. $\square$

We conclude this section with:

Problem 5. *Given weighted forests $\mathcal{G}$ and $\mathcal{G}'$, decide whether they are equivalent.*

Solution:

(1) *Blow-down the two graphs until they are minimal; check that the two minimal weighted forests have isomorphic skeletons.*

From now-on, assume that the skeletons are isomorphic (by 3.6, this is a necessary condition for equivalence).

(2) *Find canonical weighted forests $\mathcal{G}_*$ and $\mathcal{G}'_*$ such that $\mathcal{G} \sim \mathcal{G}_*$ and $\mathcal{G}' \sim \mathcal{G}'_*$.*

One can design an algorithm which accomplishes step (2); we leave this to the reader.

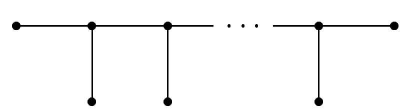
(3) *Let S be the skeleton of $\mathcal{G}_*$ (and of $\mathcal{G}'_*$), fix a skeletal map $\sigma : S \dashrightarrow \mathcal{G}_*$ and compute $(W, \eta) = \bar{T}(\sigma, \mathcal{G}_*)$. For each $\sigma' : S \dashrightarrow \mathcal{G}'_*$, compute $(W', \eta') = \bar{T}(\sigma', \mathcal{G}'_*)$. The condition $\mathcal{G} \sim \mathcal{G}'$ is equivalent to:*

For some σ' such that $W' \sim W$, we have $\eta' = \eta$.

The claim contained in step (3) is a consequence of 8.34. Note that if σ' satisfies $W' \sim W$ then η and η' belong to the same module $N(S, \omega)$ (where ω is the equivalence class of W) and hence can be compared. One can use 8.22.1 to compare η and η'.

By 8.36, the number of skeletal maps $\sigma' : S \dashrightarrow \mathcal{G}'_*$ is equal to the order of the group $A = \mathrm{Aut}(S)$, which is often a small number (8.41). Moreover, we are only interested in those σ' which satisfy $W' \sim W$; if σ'_0 is such a map, then the set of σ' satisfying $W' \sim W$ is $\{\, \sigma'_0 \circ \alpha \mid \alpha \in A \text{ and } \omega^\alpha = \omega \,\}$ (see 8.33 for the right-action of A on $\Omega(S)$).

8.41**. Example.** In the study of algebraic surfaces, skeletons of the form

$S \ =$ 

are not uncommon. Such an S satisfies $|\,\mathrm{Aut}(S)| = 8$ if at least two vertices have degree 3.

9. Minimal weighted forests

This section reduces Problem 1 to Problem 2. Recall that Section 7 partially solves Problem 2.

9.1. Definition. Let S be a skeleton and $\omega \in \Omega(S)$. By a *minimal element of* ω, we mean an element $W \in \omega$ such that

$$\forall_{\gamma \in P(S)}\ W(\gamma) \text{ is a minimal element of } \omega(\gamma)$$

(see 8.4 for the element $\omega(\gamma)$ of $\mathbb{Z}^*/{\sim}$). The symbol $\min(\omega)$ denotes the set of minimal elements of ω.

9.1.1. Pick a subset $\Sigma = \{\gamma_1, \dots, \gamma_n\}$ of $P(S)$ such that $\Sigma \cap \{\gamma, \gamma^-\}$ is a singleton for every $\gamma \in P(S)$. Write $\mathcal{C}_i = \omega(\gamma_i)$ for each i. Then $W \mapsto (W(\gamma_1), \dots, W(\gamma_n))$ is a bijection $\omega \to \prod_{i=1}^n \mathcal{C}_i$ and maps $\min(\omega)$ onto $\prod_{i=1}^n \min(\mathcal{C}_i)$. Consequently, a description of $\min(\omega)$ follows immediately from a solution to Problem 2.

9.2. Suppose that we want to describe the set $\mathcal{M}$ of minimal elements of an equivalence class $\mathcal{C}$ of weighted forests.

Pick $\mathcal{G} \in \mathcal{M}$. We may choose a skeleton S and a skeletal map $\sigma : S \dashrightarrow \mathcal{G}$; then $(\sigma, \mathcal{G}) \in \mathrm{FO}^+(S)$ and we may consider the equivalence class $\mathcal{C}^+ \subseteq \mathrm{FO}^+(S)$ of $(\sigma, \mathcal{G})$. By 8.40, $p_2(\mathcal{C}^+)$ is the set of pseudo-minimal forests belonging to $\mathcal{C}$; so

$$\mathcal{M} \subseteq p_2(\mathcal{C}^+) \subseteq \mathcal{C}. \tag{44}$$

By 8.6, we may choose $\omega \in \Omega(S)$ such that $\mathcal{C}^+ \subseteq \mathrm{FO}^+(S, \omega)$. Then, by 8.28, the map $\bar{T} : \mathrm{FO}^+(S, \omega) \to \omega \times N(S, \omega)$ maps $\mathcal{C}^+$ onto an orbit $\mathcal{O} \subseteq \omega \times N(S, \omega)$; also consider the bijection $p : \mathcal{O} \to \omega$ of 8.21, so we have:

$$\mathcal{C} \xleftarrow{p_2} \mathcal{C}^+ \xrightarrow{\bar{T}} \mathcal{O} \xrightarrow{p} \omega. \tag{45}$$

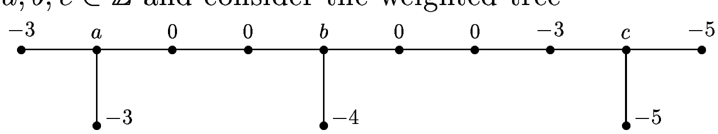
9.2.1. **Proposition.** $\mathcal{M} = p_2(\mathcal{M}^+)$, *where* $\mathcal{M}^+ \subseteq \mathcal{C}^+$ *denotes the inverse image of* $\min(\omega)$ *via* $p \circ \bar{T} : \mathcal{C}^+ \to \omega$.

Proof. It is clear that, for an element $(\sigma', \mathcal{G}')$ of $\mathcal{C}^+$, $\mathcal{G}'$ is a minimal weighted graph if and only if $p(\bar{T}(\sigma', \mathcal{G}'))$ is a minimal element of ω; in other words, $\mathcal{M}^+ = p_2^{-1}(\mathcal{M})$. We get $p_2(\mathcal{M}^+) = \mathcal{M}$ by (44). $\square$

Remark. In applications of 9.2.1, one needs to evaluate $p^{-1} : \omega \to \mathcal{O}$ at some elements of ω; 8.22.3 can be used for this.

9.3. **Example.** Fix $a, b, c \in \mathbb{Z}$ and consider the weighted tree

$\mathcal{G}$ =

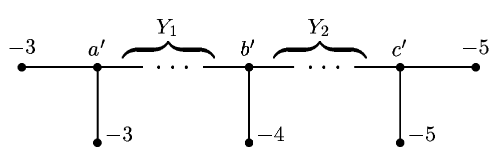


Write $X_1 = (0,0)$ and $X_2 = (0,0,-3)$ and, for each $i \in \{1,2\}$, let $\mathcal{C}_i \in \mathbb{Z}^*/{\sim}$ be the equivalence class of X_i. Given $(Y_1, Y_2) \in \min(\mathcal{C}_1) \times \min(\mathcal{C}_2)$, set

$$a' = a + \delta(X_1, Y_1), \quad b' = b + \delta(X_1^-, Y_1^-) + \delta(X_2, Y_2), \quad c' = c + \delta(X_2^-, Y_2^-) \tag{46}$$

and define

$\mathcal{G}'(Y_1, Y_2)$ =



By 9.2.1 (and noting that (46) is obtained from 8.22.3), it follows that

$$\left\{ \mathcal{G}'(Y_1, Y_2) \;\middle|\; (Y_1, Y_2) \in \min(\mathcal{C}_1) \times \min(\mathcal{C}_2) \right\}$$

is the set of minimal weighted graphs equivalent to $\mathcal{G}$. Since the sets $\min(\mathcal{C}_1)$ and $\min(\mathcal{C}_2)$ are explicitely described in section 7 (because each $\mathcal{C}_i$ is the successor of a prime class), and since (46) is an explicit formula for a', b', c', we know all minimal weighted graphs equivalent to $\mathcal{G}$.

We conclude that 9.2.1 reduces Problem 1 to the problem of describing $\min(\omega)$. Hence, in view of 9.1.1, Problem 1 reduces to Problem 2.

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