SMALL EMBEDDINGS OF INTEGRAL DOMAINS

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Abstract. Let A be a geometrically integral algebra over a field k. We prove that for any affine k-domain R, if there exists an extension field K of k such that $R \subseteq K \otimes_k A$ and $R \nsubseteq K$, then there exists an extension field L of k such that $R \subseteq L \otimes_k A$ and $\operatorname{trdeg}_k(L) < \operatorname{trdeg}_k(R)$. This generalizes a result of Freudenburg, namely, the fact that this is true for $A = k^{[1]}$.

1. Introduction

Recall that by a *domain*, one means an integral domain. If k is a field, then a *k-domain* is a domain that is also a k-algebra, and an *affine k-domain* is a domain that is a finitely generated k-algebra.

Let k be a field and consider k-domains R and A. An *A-embedding* of R is a pair (K, f) where K is an extension field of k and $f : R \to K \otimes_k A$ is an injective k-homomorphism. An A-embedding (K, f) of R is *trivial* if f factors through the canonical homomorphism $K \to K \otimes_k A$; (K, f) is *small* if $\operatorname{trdeg}_k(K) < \operatorname{trdeg}_k(R)$. Observe that all small A-embeddings of R are nontrivial.

It is clear that given any field k and any k-domains R and A, there exists a trivial A-embedding of R: let K be the field of fractions of R, then $R \to K \to K \otimes_k A$ is a trivial A-embedding of R. However, there may or may not exist a nontrivial A-embedding of R. If there exists a nontrivial (resp. small) A-embedding of R, we say that R *admits* a nontrivial (resp. small) A-embedding.

1.1. **Definition.** Let k be a field. We say that a k-domain A *has the small embedding property* if every affine k-domain that admits a nontrivial A-embedding also admits a small A-embedding.

We abbreviate "A has the small embedding property" to "A has property (SE)". If we want to emphasize k, we say "A has property (SE) over k".

Remark. Let us rephrase the definition of property (SE) in more concrete terms. Let k be a field and A a k-domain. Then A has property (SE) over k if and only if for every field extension K/k and every affine k-domain R such that $R \subseteq K \otimes_k A$ and $R \nsubseteq \mu_K(K)$ (where $\mu_K : K \to K \otimes_k A$ is the canonical map), there exists a pair (L, f) where L/k is a field extension satisfying $\operatorname{trdeg}_k(L) < \operatorname{trdeg}_k(R)$ and $f : R \to L \otimes_k A$ is an injective k-homomorphism.

Keeping in mind the above definition, consider the following result of Freudenburg (the main result of [Fre15]):

> *Let R be a finitely generated algebra over a field k. Suppose that there exists a field extension K/k and an element x transcendental over K such that $R \subseteq K[x]$ and $R \nsubseteq K$. Then there exists a field extension L/k and an element y transcendental over L such that $R \subseteq L[y]$ and $L[y]$ is algebraic over R.*

Clearly, Freudenburg's result is equivalent to the statement: *for any field k, the polynomial ring $k[X]$ has property (SE) over k.* In the same paper, Freudenburg asks whether the analogue of

2010 *Mathematics Subject Classification.* Primary: 14R10, 13B25, 13G05.
Key words and phrases: Integral domains, tensor products, ring extensions.
Research of both authors supported by grant RGPIN/104976-2010 from NSERC Canada.

his result holds for Laurent polynomial rings. This is equivalent to asking: *does* $k[X, X^{-1}]$ *have property (SE) over* k*?*

In this paper we partially solve the problem of describing the class of domains having property (SE). Let us recall:

1.2. **Definition.** Let k be a field. A k-algebra A is *geometrically integral over* k if, for every field extension K/k, the ring $K \otimes_k A$ is a domain.

Then our main result is:

1.3. **Theorem.** *Let* k *be a field and* A *a* k*-domain that is geometrically integral over* k*. Then* A *has property* (SE) *over* k*.*

The proof is given in Section 2. It follows in particular that if k is a field and $n \geq 1$ then the polynomial ring $k[x_1, \dots, x_n]$ and the Laurent polynomial ring $k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ have property (SE). Setting $n = 1$, we recover Freudenburg's result and we give an affirmative answer to his question.

Note that if k is algebraically closed then every k-domain is geometrically integral. Thus:

1.4. **Corollary.** *If* k *is an algebraically closed field then every* k*-domain has property (SE).*

1.5. **Remark.** If k is not algebraically closed then there exist k-domains that do not have property (SE). For instance, if A/k is an algebraic field extension such that $A \neq k$ then A does not have property (SE) over k. (Proof: Let x be an indeterminate over k, and note that $k(x) \otimes_k A$ can be identified with $A(x)$. Let $a \in A \setminus k$ and let $R = k[x + a]$. Then $R \subseteq A(x) = k(x) \otimes_k A$ is a nontrivial A-embedding of R but R does not admit a small A-embedding.)

1.6. **Remark.** Let k be a field and consider k-domains R and A. We say that R is A*-refractory* if all A-embeddings of R are trivial. Theorem 1.3 provides us with a useful criterion for deciding whether a given k-domain R is A-refractory. The criterion is particularly manageable when R has transcendence degree 1:

> *Let* R *be a 1-dimensional affine* k*-domain and* A *a geometrically integral* k*-domain. Then* R *is* A*-refractory if and only if there does not exist an injective* k*-homomorphism* $R \to \bar{k} \otimes_k A$*, where* $\bar{k}$ *is the algebraic closure of* k*.*

This follows from the fact that (by Thm 1.3) A has property (SE) over k.

1.7. **Remark.** We thank Gene Freudenburg for reminding us that Lemma 14 in Makar-Limanov's lecture notes [ML] is essentially the same as the following statement, which is related to the subject matter of this article: *Let* R *be a* $\mathbb{C}$*-domain with* $\operatorname{trdeg}_{\mathbb{C}}(R) = 1$*. If there exists a field extension* $K/\mathbb{C}$ *and an element* x *transcendental over* K *such that* $R \subseteq K[x]$ *and* $R \nsubseteq K$*, then* R *is* $\mathbb{C}$*-affine and there exists an injective* $\mathbb{C}$*-homomorphism* $R \to \mathbb{C}[x]$*.*

1.8. **Remark** (Small embeddings of fields)**.** As indicated by the title of [Fre15], the result of Freudenburg stated above is an affine version of a result of Nagata about fields. That result of Nagata is Thm 2 of [Nag67], which we state here in a slightly different formulation:

(∗) *Let* E *and* K *be two field extensions of a field* k *and let* x *be a transcendental element over* K*. Assume that* K/k *is finitely generated, that* $E \subseteq K(x)$ *and that* $E \nsubseteq K$*. Then there exists an extension* L/k *and a transcendental element* y *over* L *such that* $E \subseteq L(y)$ *and* $L(y)$ *is algebraic over* E*.*

One can define a "small embedding property for fields" (SEF) as follows. First, if $K \otimes_k A$ is a domain then we write $K \circledast_k A$ for the field of fractions of $K \otimes_k A$. Then let us say that a geometrically integral field extension A/k has property (SEF) if for every field extension K/k and every finitely generated field extension E/k such that $E \subseteq K \circledast_k A$ and $E \nsubseteq K$, there exists a field extension L/k such that $E \subseteq L \circledast_k A$ and $\operatorname{trdeg}_k(L) < \operatorname{trdeg}_k(E)$. With this definition, Nagata's result $(*)$ is equivalent to the statement that, for any field k, if x is transcendental over k then $k(x)/k$ has property (SEF) (to see this equivalence, one has to see that the statement obtained from $(*)$ by replacing the assumption "K/k is finitely generated" by "E/k is finitely generated" is actually equivalent to $(*)$).

It would be interesting to ask for a description of the class of field extensions that have property (SEF). In this regard, we note that it is not hard to obtain, as a corollary to Nagata's result, that every unirational field extension has property (SEF).

Conventions. All rings are commutative and have a unity 1. If B is an algebra over a ring A, the notation $B = A^{[n]}$ means that B is isomorphic (as an A-algebra) to a polynomial ring in n variables over A. If L/K is a field extension, $L = K^{(n)}$ means that L is a purely transcendental extension of K of transcendence degree n. If A is a ring then A^* is the set of units of A. If A is a domain then $\operatorname{Frac}(A)$ is its field of fractions. We write "trdeg" for transcendence degree, "$\setminus$" for set difference, and we adopt the convention that $0 \in \mathbb{N}$.

2. Proof of Theorem 1.3

Our proof is inspired by that of [Fre15, Thm 2.1], and we also borrow some of the notations from that source. However, the ideas of [Fre15] have to be significantly elaborated in order to be applied to the present setting.

2.1. **Definitions.** Let A be a ring and $(G, +, \leq)$ a totally ordered abelian group. Given a formal sum $f = \sum_{i \in G} a_i t^i$ where $a_i \in A$ for all $i \in G$, define $\operatorname{supp}(f) = \{ i \in G \mid a_i \neq 0 \}$. Then let $A\langle\langle G \rangle\rangle$ be the set of all formal sums $f = \sum_{i \in G} a_i t^i$ such that $a_i \in A$ for all $i \in G$ and such that $\operatorname{supp}(f)$ is a well-ordered subset of G. Then one can add and multiply elements of $A\langle\langle G \rangle\rangle$ as if they were power series, and it is straightforward to check that $A\langle\langle G \rangle\rangle$ is a ring. It is sometimes called a ring of "long power series".

Given $a \in A$ and $j \in G$, define the element $at^j \in A\langle\langle G \rangle\rangle$ by $at^j = \sum_{i \in G} \alpha_i t^i$ with $\alpha_j = a$ and $\alpha_i = 0$ for all $i \in G \setminus \{j\}$. Then $1t^0$ is the element 1 of the ring $A\langle\langle G \rangle\rangle$ and the map $a \mapsto at^0$ is an injective ring homomorphism $A \to A\langle\langle G \rangle\rangle$. Given $f = \sum_{i \in G} a_i t^i \in A\langle\langle G \rangle\rangle$, we define

$$\operatorname{ord}(f) = \begin{cases} \min \operatorname{supp}(f) & \text{if } f \neq 0, \\ \infty & \text{if } f = 0, \end{cases} \qquad \bar{f} = \begin{cases} a_r t^r \text{ where } r = \operatorname{ord}(f) & \text{if } f \neq 0, \\ 0 & \text{if } f = 0. \end{cases}$$

This defines two maps,

$$\begin{array}{rcl} A\langle\langle G \rangle\rangle & \to & G \cup \{\infty\} \\ f & \mapsto & \operatorname{ord}(f) \end{array} \qquad \begin{array}{rcl} A\langle\langle G \rangle\rangle & \to & A\langle\langle G \rangle\rangle \\ f & \mapsto & \bar{f} \end{array}.$$

We shall say that an element f of $A\langle\langle G \rangle\rangle$ is *homogeneous* if $f = \bar{f}$.

Remark. For any ring A we have $A\langle\langle \mathbb{Z} \rangle\rangle = A((t))$, the ring of formal Laurent series over A. If k is a field and G a totally ordered abelian group, then $k\langle\langle G \rangle\rangle$ is a field and $\operatorname{ord} : k\langle\langle G \rangle\rangle \to G \cup \{\infty\}$ is a valuation.

2.2. **Lemma.** *Let G be a totally ordered abelian group, K/k a field extension, R a k-subalgebra of $K\langle\langle G \rangle\rangle$ and $\bar{R} = k[\{ \bar{r} \mid r \in R \}]$. If $\operatorname{trdeg}_k R < \infty$, then $\operatorname{trdeg}_k \bar{R} \leq \operatorname{trdeg}_k R$.*

Proof. Let $S = \{\, \bar{r} \mid r \in R \setminus \{0\} \,\}$. Since $\bar{R} = k[S]$, there exists a subset of S which is a transcendence basis of $\operatorname{Frac}(\bar{R})$ over k. So, to prove the claim, it's enough to show that given any integer $n > \operatorname{trdeg}_k(R)$, every family $(s_1, \dots, s_n) \in S^n$ is algebraically dependent over k.

Let $n > \operatorname{trdeg}_k(R)$ and $(s_1, \dots, s_n) \in S^n$. Choose $(r_1, \dots, r_n) \in R^n$ such that $\bar{r}_i = s_i$ for all $i = 1, \dots, n$. Then there exists a polynomial $F(x_1, \dots, x_n) \in k[x_1, \dots, x_n] \setminus \{0\}$ such that $F(r_1, \dots, r_n) = 0$. Define a G-grading, $k[x_1, \dots, x_n] = \bigoplus_{d \in G} B_d$, by declaring that $k \subseteq B_0$ and that $x_i \in B_{\operatorname{ord}(r_i)}$ for all $i = 1, \dots, n$. Let $H(x_1, \dots, x_n) \neq 0$ be the homogeneous component of $F(x_1, \dots, x_n)$ of smallest degree.[1] Then $H(s_1, \dots, s_n) = 0$, showing that $(s_1, \dots, s_n)$ is algebraically dependent over k. Thus $\operatorname{trdeg}_k \bar{R} \le \operatorname{trdeg}_k R$. □

2.3. **Lemma.** *Let K/k be a field extension and let R be a k-subalgebra of $K\langle\langle \mathbb{Z}^n \rangle\rangle$, where $n \ge 1$ and where $\mathbb{Z}^n$ is endowed with some total order. Assume that $R \nsubseteq K$ and that $R = k[S]$ for some set S of homogeneous elements of $K\langle\langle \mathbb{Z}^n \rangle\rangle$, and let $L = \operatorname{Frac}(R) \cap K$. Then there exist $m \in \{1, \dots, n\}$ and elements $c_1, \dots, c_m \in K^*$ and $d_1, \dots, d_m \in \mathbb{Z}^n$ such that $d_1, \dots, d_m$ are linearly independent over $\mathbb{Z}$ and*

$$R \subseteq L[(c_1 t^{d_1})^{\pm 1}, \dots, (c_m t^{d_m})^{\pm 1}] \subseteq \operatorname{Frac}(R).$$

Moreover, we have $\operatorname{Frac}(R) = L^{(m)}$ *and consequently* $\operatorname{trdeg}_L(\operatorname{Frac} R) = m \ge 1$.

Proof. Let H be the subgroup of $\mathbb{Z}^n$ generated by the set $\{\, \operatorname{ord}(s) \mid s \in S \setminus \{0\} \,\}$. Since $k[S] = R \nsubseteq K$, H is not the trivial group, so $H \cong \mathbb{Z}^m$ for some $m \in \{1, \dots, n\}$. Let $\{d_1, \dots, d_m\}$ be a basis of H. For each $i \in \{1, \dots, m\}$, there exist $s_{i1}, \dots, s_{in_i} \in S$ and $e_{i1}, \dots, e_{in_i} \in \mathbb{Z}$ such that $\operatorname{ord}(s_{i1}^{e_{i1}} \cdots s_{in_i}^{e_{in_i}}) = d_i$. Then $s_{i1}^{e_{i1}} \cdots s_{in_i}^{e_{in_i}} = c_i t^{d_i}$ for some $c_i \in K^*$, and clearly $c_i t^{d_i} \in \operatorname{Frac}(R)$. As $L \subseteq \operatorname{Frac}(R)$, this shows that

$$L[(c_1 t^{d_1})^{\pm 1}, \dots, (c_m t^{d_m})^{\pm 1}] \subseteq \operatorname{Frac}(R).$$

Let $s \in S \setminus \{0\}$. Then $\operatorname{ord}(s) = a_1 d_1 + \cdots + a_m d_m$ for some $a_1, \dots, a_m \in \mathbb{Z}$, so $\operatorname{ord}(s) = \operatorname{ord}\big((c_1 t^{d_1})^{a_1} \cdots (c_m t^{d_m})^{a_m}\big)$, so

$$\frac{s}{(c_1 t^{d_1})^{a_1} \cdots (c_m t^{d_m})^{a_m}} \in \operatorname{Frac}(R) \cap K^* = L^*.$$

It then follows that $s \in L[(c_1 t^{d_1})^{\pm 1}, \dots, (c_m t^{d_m})^{\pm 1}]$, so $R \subseteq L[(c_1 t^{d_1})^{\pm 1}, \dots, (c_m t^{d_m})^{\pm 1}]$.

Since $d_1, \dots, d_m$ are linearly independent over $\mathbb{Z}$, the family $(c_1 t^{d_1}, \dots, c_m t^{d_m})$ is algebraically independent over K, hence algebraically independent over L because $L \subseteq K$. It follows that $\operatorname{Frac}(R) = L(c_1 t^{d_1}, \dots, c_m t^{d_m}) = L^{(m)}$. □

2.4. **Lemma.** *Let k be a field, B a k-algebra and G a totally ordered abelian group. Then there is an injective B-homomorphism $B \otimes_k k\langle\langle G \rangle\rangle \to B\langle\langle G \rangle\rangle$ given by $b \otimes \sum_{i \in G} a_i t^i \mapsto \sum_{i \in G} (a_i b) t^i$, where $b \in B$ and $\sum_{i \in G} a_i t^i \in k\langle\langle G \rangle\rangle$.*

Proof. Applying the universal property of $B \otimes_k k\langle\langle G \rangle\rangle$ (i.e., the pushout property) to the natural homomorphisms $B \to B\langle\langle G \rangle\rangle \leftarrow k\langle\langle G \rangle\rangle$ shows that there exists a ring homomorphism

$$\theta : B \otimes_k k\langle\langle G \rangle\rangle \to B\langle\langle G \rangle\rangle$$

satisfying $\theta(b \otimes \sum_i a_i t^i) = \sum_i (a_i b) t^i$ for all $b \in B$ and $\sum_i a_i t^i \in k\langle\langle G \rangle\rangle$. It is clear that θ is a B-homomorphism, so our task is to show that θ is injective. Let $x \in \ker \theta$. Let $(e_j)_{j \in J}$ be a basis

[1]Consider the unique decomposition $F(x_1, \dots, x_n) = F_{d_1}(x_1, \dots, x_n) + \cdots + F_{d_r}(x_1, \dots, x_n)$ where $d_1 < \cdots < d_r$ are elements of G and $F_d(x_1, \dots, x_n) \in B_d \setminus \{0\}$ for all d. Then the polynomials $F_{d_i}(x_1, \dots, x_n)$ are called the *homogeneous components* of $F(x_1, \dots, x_n)$, and $H(x_1, \dots, x_n) = F_{d_1}(x_1, \dots, x_n)$ is the one of smallest degree.

of B over k and note that $(e_j \otimes 1)_{j\in J}$ is a basis of $B \otimes_k k\langle\langle G\rangle\rangle$ over $k\langle\langle G\rangle\rangle$. So $x = \sum_{j\in J_0} e_j \otimes f_j$ where J_0 is a finite subset of J and $f_j = \sum_{i\in G} a_{ij}t^i \in k\langle\langle G\rangle\rangle$ for all $j \in J_0$ ($a_{ij} \in k$). Then

$$0 = \theta(x) = \sum_{j\in J_0} \theta(e_j \otimes \sum_{i\in G} a_{ij}t^i) = \sum_{j\in J_0}\sum_{i\in G} a_{ij}e_jt^i = \sum_{i\in G}(\sum_{j\in J_0} a_{ij}e_j)t^i$$

so $\sum_{j\in J_0} a_{ij}e_j = 0$ for each $i \in G$, so $a_{ij} = 0$ for all i, j and hence $x = 0$. □

2.5. **Notation.** Let k be a field, let B and A be k-algebras, and let $\varphi : A \to k\langle\langle G\rangle\rangle$ be an injective k-homomorphism where G is a totally ordered abelian group. We define

$$\Theta^B_\varphi : B \otimes_k A \to B\langle\langle G\rangle\rangle$$

to be the composition $B \otimes_k A \xrightarrow{B\otimes\varphi} B \otimes_k k\langle\langle G\rangle\rangle \to B\langle\langle G\rangle\rangle$, where $B \otimes_k k\langle\langle G\rangle\rangle \to B\langle\langle G\rangle\rangle$ is the map of Lemma 2.4. Note that Θ^B_φ is an injective B-homomorphism. It is given explicitly by the following rule: if $b \in B$, $x \in A$ and $\varphi(x) = \sum_{i\in G} a_it^i$, then $\Theta^B_\varphi(b\otimes x) = \sum_{i\in G}(a_ib)t^i$. We may also write this rule as $\Theta^B_\varphi(b \otimes x) = b\varphi(x)$, because $\varphi(x) \in k\langle\langle G\rangle\rangle \subseteq B\langle\langle G\rangle\rangle$ and $B\langle\langle G\rangle\rangle$ is a B-algebra.

2.6. **Lemma.** *Let k be a field, A a k-algebra, and $\varphi : A \to k\langle\langle G\rangle\rangle$ an injective k-homomorphism where G is a totally ordered abelian group. Let B be a k-algebra and K a field that contains B. Consider the commutative diagram*

$$\begin{tikzcd} K \otimes_k A \arrow[r, "\Theta^K_\varphi"] & K\langle\langle G\rangle\rangle \\ B \otimes_k A \arrow[u] \arrow[r, "\Theta^B_\varphi"'] & B\langle\langle G\rangle\rangle \arrow[u] \end{tikzcd} .$$

Then $(\Theta^K_\varphi)^{-1}\big(B\langle\langle G\rangle\rangle\big) = B \otimes_k A$.

Proof. Let $\xi \in (K \otimes_k A) \setminus \{0\}$. Write $\xi = u_1 \otimes \alpha_1 + \cdots + u_n \otimes \alpha_n$ where $u_i \in K$, $\alpha_i \in A$, and where $\alpha_1, \dots, \alpha_n$ are linearly independent over k. Then the elements $\varphi(\alpha_1), \dots, \varphi(\alpha_n)$ of $k\langle\langle G\rangle\rangle$ are linearly independent over k. For each $j = 1, \dots, n$, write $\varphi(\alpha_j) = \sum_{i\in G} c_{ij}t^i$ ($c_{ij} \in k$). By linear independence, we may choose a finite subset $\{i_1, \dots, i_m\}$ of G such that the rank of the matrix $C = \begin{pmatrix} c_{i_1 1} & \cdots & c_{i_1 n} \\ \vdots & & \vdots \\ c_{i_m 1} & \cdots & c_{i_m n}\end{pmatrix}$ is equal to n. Then the K-linear map

$$\Phi : K^n \to K^m, \quad \begin{pmatrix} x_1 \\ \vdots \\ x_n\end{pmatrix} \mapsto C\begin{pmatrix} x_1 \\ \vdots \\ x_n\end{pmatrix}$$

is injective and we claim that

$$\Phi^{-1}(B^m) = B^n. \tag{1}$$

To see this, note that B is a subspace of the k-linear space K, so $K = B \oplus V$ for some subspace V of K. Now $K^n = B^n \oplus V^n$ as k-vector spaces, and the fact that C is a matrix over k implies that $\Phi(B^n) \subseteq B^m$ and $\Phi(V^n) \subseteq V^m$. Suppose that $b + v \in \Phi^{-1}(B^m)$ where $b \in B^n$ and $v \in V^n$. Then $\Phi(b) + \Phi(v) = \Phi(b + v) \in B^m$ implies that $\Phi(v) = 0$, so $v = 0$, so $b + v \in B^n$, proving (1).

Consider $\Theta^K_\varphi(\xi) \in K\langle\langle G\rangle\rangle$. If we write $\Theta^K_\varphi(\xi) = \sum_{i\in G} y_it^i$ ($y_i \in K$) then

$$\Phi(\begin{pmatrix} u_1 \\ \vdots \\ u_n\end{pmatrix}) = \begin{pmatrix} y_{i_1} \\ \vdots \\ y_{i_m}\end{pmatrix}.$$

If we now assume that $\Theta_\varphi^K(\xi) \in B\langle\!\langle G \rangle\!\rangle$, we have $\Phi\Big(\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}\Big) \in B^m$, so $\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in B^n$ by (1), so $\xi \in B \otimes_k A$. This shows that $(\Theta_\varphi^K)^{-1}\big(B\langle\!\langle G \rangle\!\rangle\big) = B \otimes_k A$, so the Lemma is proved. $\square$

2.7. **Definition.** Let k be a field and G a totally ordered abelian group. We shall say that a k-algebra A *has enough embeddings in* $k\langle\!\langle G \rangle\!\rangle$ if, for every field extension K/k and every element $\xi \in (K \otimes_k A) \setminus K$, there exists an injective k-homomorphism $\varphi : A \to k\langle\!\langle G \rangle\!\rangle$ such that $\Theta_\varphi^K : K \otimes_k A \to K\langle\!\langle G \rangle\!\rangle$ maps ξ to an element of nonzero order.

2.8. **Theorem.** *Let $n \geq 0$, and let G denote $\mathbb{Z}^n$ endowed with some total order. If k is a field and A is a k-algebra that has enough embeddings in $k\langle\!\langle G \rangle\!\rangle$, then A has property (SE) over k.*

Proof. If $n = 0$ then the fact that A has enough embeddings in $k\langle\!\langle G \rangle\!\rangle = k$ implies that $A = k$, in which case it is clear that A has property (SE). From now-on, we assume that $n \geq 1$.

Let A be a k-algebra that has enough embeddings in $k\langle\!\langle G \rangle\!\rangle = k\langle\!\langle \mathbb{Z}^n \rangle\!\rangle$. Let K be a field extension of k and let R be an affine k-domain satisfying $R \subseteq K \otimes_k A$ and $R \nsubseteq K$. It has to be shown that there exists a field extension L of k such that $R \subseteq L \otimes_k A$ and $\operatorname{trdeg}_k(L) < \operatorname{trdeg}_k(R)$.

We may choose an element $\xi \in R \setminus K \subseteq (K \otimes_k A) \setminus K$; since A has enough embeddings in $k\langle\!\langle \mathbb{Z}^n \rangle\!\rangle$, there exists an injective k-homomorphism $\varphi : A \to k\langle\!\langle \mathbb{Z}^n \rangle\!\rangle$ such that $\Theta_\varphi^K : K \otimes_k A \to K\langle\!\langle \mathbb{Z}^n \rangle\!\rangle$ maps ξ to an element of nonzero order. Consider the k-subalgebra $\Theta_\varphi^K(R)$ of $K\langle\!\langle \mathbb{Z}^n \rangle\!\rangle$ and define $\bar{R} = k[\{\, \bar{\rho} \mid \rho \in \Theta_\varphi^K(R) \,\}]$, which is a k-subalgebra of $K\langle\!\langle \mathbb{Z}^n \rangle\!\rangle$; then

$$\operatorname{trdeg}_k(\bar{R}) \leq \operatorname{trdeg}_k\big(\Theta_\varphi^K(R)\big) = \operatorname{trdeg}_k(R) \tag{2}$$

by Lemma 2.2. Since $\Theta_\varphi^K(\xi)$ has nonzero order, we have $\bar{R} \nsubseteq K$. Let $L' = \operatorname{Frac}(\bar{R}) \cap K$. Then Lemma 2.3 implies that there exist $m \in \{1, \dots, n\}$ and elements $c_1, \dots, c_m \in K^*$ and $d_1, \dots, d_m \in \mathbb{Z}^n \setminus \{0\}$ such that $d_1, \dots, d_m$ are linearly independent over $\mathbb{Z}$ and

$$\bar{R} \subseteq L'[(c_1 t^{d_1})^{\pm 1}, \dots, (c_m t^{d_m})^{\pm 1}] \subseteq \operatorname{Frac}(\bar{R}).$$

Moreover, also by Lemma 2.3, $\operatorname{Frac}(\bar{R})$ is a transcendental extension of L'. So

$$\operatorname{trdeg}_k(L') < \operatorname{trdeg}_k(\bar{R}) \leq \operatorname{trdeg}_k(R), \tag{3}$$

where the second inequality follows from (2).

Consider $w_1, \dots, w_N \in K \otimes_k A$ such that $R = k[w_1, \dots, w_N]$. For each $i = 1, \dots, N$, write $w_i = u_{i1} \otimes \beta_{i1} + \cdots + u_{in_i} \otimes \beta_{in_i}$ with $u_{ij} \in K$ and $\beta_{ij} \in A$. Let S be the finite subset of K whose elements are $c_1, \dots, c_m$, $\frac{1}{c_1}, \dots, \frac{1}{c_m}$ and all u_{ij}, and consider the subring $B = L'[S]$ of K. Note that B is an affine L'-domain. For each i, j we have $\Theta_\varphi^K(u_{ij} \otimes \beta_{ij}) = u_{ij}\varphi(\beta_{ij}) \in B\langle\!\langle \mathbb{Z}^n \rangle\!\rangle$; so $\Theta_\varphi^K(R) \subseteq B\langle\!\langle \mathbb{Z}^n \rangle\!\rangle$ and hence $R \subseteq (\Theta_\varphi^K)^{-1}(B\langle\!\langle \mathbb{Z}^n \rangle\!\rangle) = B \otimes_k A$ by Lemma 2.6. In other words, there exists an injective k-homomorphism $f : R \to B \otimes_k A$. As $B \neq 0$, we may choose a maximal ideal $\mathfrak{m}$ of B and consider the commutative diagram

$$\begin{tikzcd}
 & K \otimes_k A \arrow[r] \arrow[rr, bend left, "\Theta_\varphi^K"] & K \otimes_k k\langle\!\langle \mathbb{Z}^n \rangle\!\rangle \arrow[r] & K\langle\!\langle \mathbb{Z}^n \rangle\!\rangle \\
R \arrow[r, "f"'] & B \otimes_k A \arrow[r] \arrow[u, hook] \arrow[d, "g"'] & B \otimes_k k\langle\!\langle \mathbb{Z}^n \rangle\!\rangle \arrow[r] \arrow[u, hook] \arrow[d] & B\langle\!\langle \mathbb{Z}^n \rangle\!\rangle \arrow[u, hook] \arrow[d] \\
 & (B/\mathfrak{m}) \otimes_k A \arrow[r] & (B/\mathfrak{m}) \otimes_k k\langle\!\langle \mathbb{Z}^n \rangle\!\rangle \arrow[r] & (B/\mathfrak{m})\langle\!\langle \mathbb{Z}^n \rangle\!\rangle
\end{tikzcd} \tag{4}$$

Note that all horizontal arrows in this diagram are injective k-homomorphisms. Let $r \in R\setminus\{0\}$ and let $\rho = \sum_{i\in G} b_i t^i$ (where $G = \mathbb{Z}^n$ and $b_i \in B$ for all i) be the image of r in $B\langle\langle \mathbb{Z}^n\rangle\rangle$ via the maps of diagram (4). As ρ is a nonzero element of $\Theta^K_\varphi(R)$, we may consider $\bar\rho \in \bar R\setminus\{0\}$. Write $\theta = \operatorname{ord}(\rho)$, then $\bar\rho = b_\theta t^\theta$ where $b_\theta \in B\setminus\{0\}$. Note that $0 \neq b_\theta t^\theta \in \bar R \subseteq L'[(c_1t^{d_1})^{\pm 1}, \dots, (c_m t^{d_m})^{\pm 1}]$. Consequently, $b_\theta t^\theta$ is a finite sum of terms of the form

$$\lambda (c_1t^{d_1})^{a_1}\cdots(c_mt^{d_m})^{a_m} \quad \text{with } \lambda \in (L')^* \text{ and } a_1,\dots,a_m \in \mathbb{Z}.$$

By linear independence of $d_1,\dots,d_m$ over $\mathbb{Z}$, it follows that

$$b_\theta t^\theta = \lambda (c_1t^{d_1})^{a_1}\cdots(c_mt^{d_m})^{a_m} \quad \text{for some } \lambda \in (L')^* \text{ and } a_1,\dots,a_m \in \mathbb{Z}.$$

Then $b_\theta = \lambda c_1^{a_1}\cdots c_m^{a_m} \in B^*$, so the image $\sum_{i\in G}(b_i+\mathfrak{m})t^i$ of r in $(B/\mathfrak{m})\langle\langle\mathbb{Z}^n\rangle\rangle$ is nonzero. It follows that $(g\circ f)(r) \neq 0$, and this proves that $g\circ f : R \to (B/\mathfrak{m})\otimes_k A$ is an injective k-homomorphism.

Write $L = B/\mathfrak{m}$, then $R \subseteq L\otimes_k A$. Since B is an affine L'-domain, L is a finite extension of L'. So $\operatorname{trdeg}_k(L) = \operatorname{trdeg}_k(L') < \operatorname{trdeg}_k(R)$ by (3). $\square$

2.9. **Lemma.** *Let G be a totally ordered abelian group. Let A be an algebra over a field k and suppose that A has enough embeddings in $k\langle\langle G\rangle\rangle$. Then every k-subalgebra of A has enough embeddings in $k\langle\langle G\rangle\rangle$.*

Proof. Let A_0 be a k-subalgebra of A, let $i : A_0 \to A$ be the inclusion map, let K/k be a field extension and let $\mu_0 : K \to K\otimes_k A_0$ and $\mu : K \to K\otimes_k A$ be the canonical maps. Note the commutative diagrams (ignore φ and Θ^K_φ for now):

$$\begin{tikzcd}
K\otimes_k A_0 \arrow[r, "K\otimes i"] & K\otimes_k A \\
K \arrow[u, "\mu_0"] \arrow[ur, "\mu"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
K\otimes_k A_0 \arrow[r, "K\otimes i"'] \arrow[rr, bend left, "\Theta^K_\varphi\circ(K\otimes i)=\Theta^K_{\varphi\circ i}"] & K\otimes_k A \arrow[r, "\Theta^K_\varphi"'] & K\langle\langle G\rangle\rangle \\
A_0 \arrow[u] \arrow[r, "i"'] & A \arrow[u] \arrow[r, "\varphi"'] & k\langle\langle G\rangle\rangle \arrow[u]
\end{tikzcd}$$

Let $\xi \in (K\otimes_k A_0)\setminus\mu_0(K)$. If $(K\otimes i)(\xi) \in \mu(K)$ then there exists $a \in K$ such that $(K\otimes i)(\xi) = \mu(a) = (K\otimes i)(\mu_0(a))$, so $\xi = \mu_0(a)$ since $K\otimes i$ is injective, and this contradicts the hypothesis $\xi \notin \mu_0(K)$. So $(K\otimes i)(\xi) \notin \mu(K)$. Since A has enough embeddings in $k\langle\langle G\rangle\rangle$, there exists $\varphi : A \to k\langle\langle G\rangle\rangle$ such that $\Theta^K_\varphi\big((K\otimes i)(\xi)\big)$ has nonzero order. We have $\Theta^K_\varphi\circ(K\otimes i) = \Theta^K_{\varphi\circ i}$, so $\Theta^K_{\varphi\circ i}(\xi)$ has nonzero order. $\square$

2.10. **Proposition.** *Let A be an algebra over a field k and let G be a totally ordered abelian group. Suppose that for each $x \in A\setminus k$, there exists an injective k-homomorphism $\varphi : A \to k\langle\langle G\rangle\rangle$ such that $\varphi(x)$ has negative order. Then A has enough embeddings in $k\langle\langle G\rangle\rangle$.*

Proof. Let K/k be a field extension and let $\xi \in (K\otimes_k A)\setminus K$. Choose a basis $(u_i)_{i\in I}$ of K over k; then $(u_i\otimes 1)_{i\in I}$ is a basis of $K\otimes_k A$ over A, so there exists a finite subset $I(\xi)$ of I and a family $(x_i)_{i\in I(\xi)}$ of elements of A such that $\xi = \sum_{i\in I(\xi)} u_i\otimes x_i$. Since $\xi \notin K$, we have $x_{i_0} \notin k$ for some $i_0 \in I(\xi)$. Choose an injective k-homomorphism $\varphi : A \to k\langle\langle G\rangle\rangle$ such that $\varphi(x_{i_0})$ has negative order.

$$\begin{tikzcd}
K \arrow[r] & K\otimes_k A \arrow[r] \arrow[rr, bend left, "\Theta^K_\varphi"] & K\otimes_k k\langle\langle G\rangle\rangle \arrow[r] & K\langle\langle G\rangle\rangle \\
k \arrow[u] \arrow[r] & A \arrow[u] \arrow[r, "\varphi"'] & k\langle\langle G\rangle\rangle \arrow[u] &
\end{tikzcd}$$

For each $i \in I(\xi)$, we may write $\varphi(x_i) = \sum_{j \in G} a_{ij} t^j$ ($a_{ij} \in k$ for all i, j). Then

$$\Theta_\varphi^K(\xi) = \sum_{i \in I(\xi)} \Theta_\varphi^K(u_i \otimes x_i) = \sum_{i \in I(\xi)} \sum_{j \in G} a_{ij} u_i t^j = \sum_{j \in G} \Big(\sum_{i \in I(\xi)} a_{ij} u_i \Big) t^j. \tag{5}$$

As $\varphi(x_{i_0}) = \sum_{j \in G} a_{i_0 j} t^j$ has negative order, we have $a_{i_0 j_0} \neq 0$ for some $j_0 < 0$. By linear independence of $(u_i)_{i \in I}$ over k, we have $\sum_{i \in I(\xi)} a_{i j_0} u_i \neq 0$, i.e., the coefficient of t^{j_0} in $\Theta_\varphi^K(\xi)$ is nonzero (see (5)). So the order of $\Theta_\varphi^K(\xi)$ in $K\langle\langle G \rangle\rangle$ is negative and we have shown that A has enough embeddings in $k\langle\langle G \rangle\rangle$. □

By a *function field of dimension* n, we mean a finitely generated field extension of transcendence degree n.

2.11. **Proposition.** *Let L/k be a function field of dimension $n \geq 1$, where k is an algebraically closed field. Then L has enough embeddings in $k\langle\langle \mathbb{Z}^n \rangle\rangle$, where $\mathbb{Z}^n$ is endowed with lexicographic order.*

Proof. Let $x \in L \setminus k$. Choose a nonsingular projective variety X over k whose function field is L. Then there exists a closed point $P \in X$ such that $1/x$ belongs to the maximal ideal of $\mathcal{O}_{X,P}$. Let $\varphi_1 : \mathcal{O}_{X,P} \to k[\![x_1, \dots, x_n]\!]$ be the canonical homomorphism in the completion of $\mathcal{O}_{X,P}$. Let G denote $\mathbb{Z}^n$ with lexicographical order and define $\varphi_2 : k[\![x_1, \dots, x_n]\!] \to k\langle\langle G \rangle\rangle$ by

$$\sum_{(e_1,\dots,e_n) \in \mathbb{N}^n} a_{e_1,\dots,e_n} x_1^{e_1} \cdots x_n^{e_n} \quad \longmapsto \quad \sum_{(e_1,\dots,e_n) \in \mathbb{N}^n} a_{e_1,\dots,e_n} t^{(e_1,\dots,e_n)}$$

(note that $\mathbb{N}^n$ is a well-ordered subset of G). Then $\varphi_2 \circ \varphi_1 : \mathcal{O}_{X,P} \to k\langle\langle G \rangle\rangle$ is an injective k-homomorphism that maps $1/x$ to an element of positive order. The extension

$$L = \operatorname{Frac}(\mathcal{O}_{X,P}) \xrightarrow{\varphi} k\langle\langle G \rangle\rangle$$

of $\varphi_2 \circ \varphi_1$ maps x to an element of negative order. It follows from Prop. 2.10 that L has enough embeddings in $k\langle\langle \mathbb{Z}^n \rangle\rangle$. □

2.12. **Corollary.** *Let k be an algebraically closed field. Then every k-domain has property (SE).*

Proof. Let A be a k-domain. To show that A has property (SE), we consider a field extension K/k and an affine k-domain R satisfying $R \subseteq K \otimes_k A$ and $R \nsubseteq K$, and we have to show that there exists a field extension L of k such that $R \subseteq L \otimes_k A$ and $\operatorname{trdeg}_k(L) < \operatorname{trdeg}_k(R)$.

Write $R = k[r_1, \dots, r_m]$ and, for each $i \in \{1, \dots, m\}$, $r_i = \sum_{j=1}^{n_i} u_{ij} \otimes t_{ij}$ ($u_{ij} \in K$, $t_{ij} \in A$). Let A_0 be the k-subalgebra of A generated by all the t_{ij}. Then A_0 is an affine k-domain, so, by Proposition 2.11, $\operatorname{Frac}(A_0)$ has enough embeddings in $k\langle\langle \mathbb{Z}^n \rangle\rangle$ for some n. By Lemma 2.9, it follows that A_0 has enough embeddings in $k\langle\langle \mathbb{Z}^n \rangle\rangle$. So A_0 has property (SE) by Theorem 2.8. Since $R \subseteq K \otimes_k A_0$ and $R \nsubseteq K$, there exists a field extension L of k such that $R \subseteq L \otimes_k A_0$ and $\operatorname{trdeg}_k(L) < \operatorname{trdeg}_k(R)$. As $L \otimes_k A_0 \subseteq L \otimes_k A$, we have $R \subseteq L \otimes_k A$, so we are done. □

2.13. **Remarks.** See Definition 1.2 for the notion of a geometrically integral algebra. The following facts are well known.

(1) If k is an algebraically closed field then every k-domain is geometrically integral over k.
(2) If k is a field and A is a k-algebra geometrically integral over k, then A is a domain and k is algebraically closed in $\operatorname{Frac}(A)$.

(3) If k is a field and A is a k-domain, then A is geometrically integral over k if and only if $\text{Frac}(A)$ is geometrically integral over k. Moreover, if A is geometrically integral over k then so is every subalgebra of A.

(4) If k is a field and A a k-algebra that is geometrically integral over k then, for every field extension K/k, $K \otimes_k A$ is geometrically integral over K and K is algebraically closed in $\text{Frac}(K \otimes_k A)$.

(5) If k is a field and G a totally ordered abelian group, then $k\langle\langle G\rangle\rangle$ is geometrically integral over k (this follows from Lemma 2.4). Consequently, every k-subalgebra of $k\langle\langle G\rangle\rangle$ is geometrically integral over k.

2.14. **Lemma.** *Let k be a field and A a geometrically integral k-algebra. Suppose that there exists an algebraic extension κ/k such that $\kappa \otimes_k A$ has property (SE) over κ. Then A has property (SE) over k.*

Proof. Define $\bar{A} = \kappa \otimes_k A$. Let K be a field extension of k and let R be an affine k-domain satisfying $R \subseteq K \otimes_k A$ and $R \nsubseteq K$. It has to be shown that there exists a field extension L of k such that $R \subseteq L \otimes_k A$ and $\text{trdeg}_k(L) < \text{trdeg}_k(R)$.

Choose a maximal ideal $\mathfrak{m}$ of $K \otimes_k \kappa$, define $\bar{K} = (K \otimes_k \kappa)/\mathfrak{m}$, and let $\pi : K \otimes_k \kappa \to \bar{K}$ be the canonical homomorphism. Define $\varphi : K \otimes_k A \to \bar{K} \otimes_\kappa \bar{A}$ to be the composite $g \circ f$, where $f : K \otimes_k A \to K \otimes_k A \otimes_k \kappa$ is the canonical map $x \mapsto x \otimes 1$, and where $g : K \otimes_k A \otimes_k \kappa \to \bar{K} \otimes_\kappa \bar{A}$ is the following composition:

$$\begin{array}{ccccccccc} K \otimes_k A \otimes_k \kappa & \cong & K \otimes_k \kappa \otimes_k A & \cong & K \otimes_k \kappa \otimes_\kappa \kappa \otimes_k A = & K \otimes_k \kappa \otimes_\kappa \bar{A} & \xrightarrow{\pi \otimes \text{id}} & \bar{K} \otimes_\kappa \bar{A} \\ \alpha \otimes t \otimes y & \mapsto & \alpha \otimes y \otimes t & \mapsto & \alpha \otimes y \otimes 1 \otimes t & = \alpha \otimes y \otimes (1 \otimes t) & \mapsto & \pi(\alpha \otimes y) \otimes (1 \otimes t) \end{array}$$

where $\alpha \in K$, $t \in A$ and $y \in \kappa$. Note that given $\alpha \in K$ and $t \in A$,

(6) $\quad \varphi$ sends the element $\alpha \otimes t \in K \otimes_k A$ to $\pi(\alpha \otimes 1) \otimes (1 \otimes t) \in \bar{K} \otimes_\kappa (\kappa \otimes A) = \bar{K} \otimes_\kappa \bar{A}$.

Let $\psi : K \to \bar{K}$ be the composition $K \to K \otimes_k \kappa \xrightarrow{\pi} \bar{K}$. Then (6) implies that the diagram

(7)
$$\begin{tikzcd} \bar{K} \arrow[r, "\bar{u}"] & \bar{K} \otimes_\kappa \bar{A} \arrow[r, equal] & \bar{K} \otimes_\kappa \kappa \otimes_k A \\ K \arrow[u, "\psi"] \arrow[r, "u"] & K \otimes_k A \arrow[u, "\varphi"] \arrow[r, "\psi \otimes \text{id}"'] & \bar{K} \otimes_k A \arrow[u, "\theta", "\cong"'] \end{tikzcd}$$

commutes, where u (resp. $\bar{u}$) is the map $x \mapsto x \otimes 1$ and $\theta(x \otimes t) = x \otimes 1 \otimes t$ (for $x \in \bar{K}$, $t \in A$). Since ψ is injective, so is $\psi \otimes \text{id} : K \otimes_k A \to \bar{K} \otimes_k A$; so, by commutativity of diagram (7),

(8) $$\varphi \text{ is injective.}$$

We claim:

(9) $$\text{if } x \in K \otimes_k A \text{ satisfies } \varphi(x) \in \bar{u}(\bar{K}), \text{ then } x \in u(K).$$

To see this, consider $a \in \bar{K}$ such that $\bar{u}(a) = \varphi(x)$. Since ψ is the composition $K \to K \otimes_k \kappa \xrightarrow{\pi} \bar{K}$ where the first map is an integral homomorphism and the second is surjective, $\bar{K}$ is an algebraic extension of its subfield $\psi(K)$. So there exists a monic polynomial $P \in K[X] \setminus \{0\}$ such that $P^{(\psi)}(a) = 0$ (where for $P(X) = \sum_i \alpha_i X^i$, $\alpha_i \in K$, we write $P^{(\psi)}(X) = \sum_i \psi(\alpha_i) X^i \in \bar{K}[X]$). Then

$$\varphi(P^{(u)}(x)) = P^{(\varphi \circ u)}(\varphi(x)) = P^{(\varphi \circ u)}(\bar{u}(a)) = P^{(\bar{u} \circ \psi)}(\bar{u}(a)) = \bar{u}(P^{(\psi)}(a)) = \bar{u}(0) = 0;$$

since φ is injective, we get $P^{(u)}(x) = 0$ and hence x is integral over $u(K)$; since $u(K)$ is integrally closed in $K \otimes_k A$ (because A is geometrically integral over k, see Rem. 2.13(4)), $x \in u(K)$, which proves (9).

Note that $R \otimes_k \kappa$ is a finitely generated κ-algebra and that $\bar{K} \otimes_\kappa \bar{A}$ is a domain (because $\bar{A}$ is geometrically integral over κ, by Rem. 2.13(4)). Define $\bar{R} \subseteq \bar{K} \otimes_\kappa \bar{A}$ to be the image of the κ-homomorphism $R \otimes_k \kappa \to K \otimes_k A \otimes_k \kappa \xrightarrow{g} \bar{K} \otimes_\kappa \bar{A}$. Then $\bar{R}$ is an affine κ-domain and we have the commutative diagram

$$\text{(10)} \quad \begin{array}{ccccc} K \otimes_k A & \xrightarrow{f} & K \otimes_k A \otimes_k \kappa & \xrightarrow{g} & \bar{K} \otimes_\kappa \bar{A} \\ \uparrow & & \uparrow & & \uparrow \\ R & \longrightarrow & R \otimes_k \kappa & \longrightarrow & \bar{R} \end{array} \quad \text{which we simplify to} \quad \begin{array}{ccc} K \otimes_k A & \xrightarrow{\varphi} & \bar{K} \otimes_\kappa \bar{A} \\ \uparrow & & \uparrow \\ R & \longrightarrow & \bar{R} \end{array}$$

(recall that $\varphi = g \circ f$). If $\bar{R} \subseteq \bar{u}(\bar{K})$ then $\varphi(R) \subseteq \bar{u}(\bar{K})$, so (9) implies that $R \subseteq u(K)$, which is not the case. So $\bar{R} \nsubseteq \bar{u}(\bar{K})$. Since $\bar{A}$ has property (SE) over κ, it follows that there exists an extension L/κ such that $\operatorname{trdeg}_\kappa(L) < \operatorname{trdeg}_\kappa(\bar{R})$ and $\bar{R} \subseteq L \otimes_\kappa \bar{A}$. Observe that in $R \to R \otimes_k \kappa \to \bar{R}$, the first homomorphism is integral and the second is surjective; so $R \to \bar{R}$ is integral. By (8) and (10), $R \to \bar{R}$ is also injective. So $\operatorname{Frac}(\bar{R})$ is an algebraic extension of $\operatorname{Frac}(R)$ and consequently $\operatorname{trdeg}_\kappa(\bar{R}) = \operatorname{trdeg}_k(R)$. Then L is an extension of k and $\operatorname{trdeg}_k(L) = \operatorname{trdeg}_\kappa(L) < \operatorname{trdeg}_\kappa(\bar{R}) = \operatorname{trdeg}_k(R)$. Moreover,

$$R \subseteq \bar{R} \subseteq L \otimes_\kappa \bar{A} = L \otimes_\kappa \kappa \otimes_k A = L \otimes_k A,$$

so the proof is complete. ☐

We may now complete the proof of our main result:

Proof of Theorem 1.3. Let k be a field and A a k-domain that is geometrically integral over k. Let $\bar{k}$ be the algebraic closure of k and let $\bar{A} = \bar{k} \otimes_k A$. Then $\bar{A}$ is a $\bar{k}$-domain so, by Corollary 2.12, $\bar{A}$ has property (SE) over $\bar{k}$. By Lemma 2.14, A has property (SE) over k. ☐

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