HILBERT'S FOURTEENTH PROBLEM AND LOCALLY NILPOTENT DERIVATIONS

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The first section of this text is a survey of Hilbert's Fourteenth Problem. However the reader will quickly realize that the survey is somewhat biased, as more attention is given to the questions pertaining to derivations than to those having to do with algebraic group actions. In fact the aim is to arrive quickly at destination, namely, the following special case of Hilbert's question:

($\star$) *If D is a locally nilpotent derivation of a polynomial ring $\mathbf{k}[X_1, \dots, X_n]$, where $\mathbf{k}$ is a field of characteristic zero, is $\ker(D)$ finitely generated as a $\mathbf{k}$-algebra?*

Also, we will not pay much attention to the historical aspect but will make a serious effort to explain the logical organization of the various subcases of the problem.

The question addressed by the second section can be phrased as follows:

Which subalgebras of $\mathbf{k}[X_1, \dots, X_n]$ are kernels of locally nilpotent derivations?

In this perspective, the question whether these algebras are finitely generated (that is, question ($\star$)) is only one aspect of the problem. We shall discuss this question for small values of n, and in particular for $n = 3$, in which case these algebras are known to be finitely generated.

Throughout, $\mathbf{k}$ is a field and $R = \mathbf{k}^{[n]}$ (which means that R is a polynomial ring in n variables over $\mathbf{k}$). The field of fractions of a domain A is denoted $\operatorname{Frac} A$, and $\operatorname{Frac}(\mathbf{k}^{[n]})$ is denoted $\mathbf{k}^{(n)}$. If A is a ring, A^* denotes its group of units.

1. A biased survey of Hilbert's Fourteenth Problem

Let $\mathbf{k}$ be a field and $R = \mathbf{k}^{[n]}$. In Hilbert's famous paper [21] proposing 23 mathematical problems, the 14th item of the list is the following:

H14. *If K is a field such that $\mathbf{k} \subseteq K \subseteq \operatorname{Frac}(R)$, is $K \cap R$ finitely generated as a $\mathbf{k}$-algebra?*

We begin by stating five special cases of this problem. The first three are concerned with group actions and invariants, and the other two with derivations and rings of constants.

Group Actions

If A is an affine $\mathbf{k}$-domain and H is a subgroup of $\operatorname{Aut}_{\mathbf{k}}(A)$, define a $\mathbf{k}$-subalgebra A^H of A (called the *ring of invariants*) by:

$$A^H = \{ a \in A \mid \forall_{h \in H}\ h(a) = a \}. \tag{1}$$

It is immediate that $\mathrm{Frac}(A^H) \cap A = A^H$ (the intersection being taken in $\mathrm{Frac}\, A$), so if $A = \mathbf{k}^{[n]}$ then the question whether A^H is finitely generated is a special case of H14.

Suppose now that G is an algebraic group acting on an affine algebraic variety $V = \mathrm{Spec}\, A$. Since each element of G determines an automorphism of V and hence an automorphism of A, the action determines a subgroup $\overline{G}$ of $\mathrm{Aut}_{\mathbf{k}}(A)$ in a natural way and one defines the ring of invariants A^G of the action by:

$$A^G = A^{\overline{G}}$$

where $A^{\overline{G}}$ is defined in (1). This leads to the following special case of H14 (where "GA" refers to "group actions"):

H14-GA. *If G is an algebraic group acting algebraically on $\mathbb{A}^n = \mathrm{Spec}\, R$, is R^G finitely generated as a $\mathbf{k}$-algebra?*

We shall mention two special cases of H14-GA but first we need some definitions. Recalll that $R = \mathbf{k}^{[n]}$. By a *coordinate system* of R we mean an ordered n-tuple $\gamma = (X_1, \dots, X_n)$ of elements of R such that $R = \mathbf{k}[X_1, \dots, X_n]$. Choosing a coordinate system γ determines an embedding $\mathrm{GL}_n(\mathbf{k}) \hookrightarrow \mathrm{Aut}_{\mathbf{k}}(R)$ whose image we shall denote $GL_n^\gamma(\mathbf{k})$.

Suppose that G is an algebraic group acting on $\mathbb{A}^n = \mathrm{Spec}\, R$ and consider the corresponding subgroup $\overline{G}$ of $\mathrm{Aut}_{\mathbf{k}}(R)$ as before. If there exists a coordinate system γ of R such that $GL_n^\gamma(\mathbf{k}) \supseteq \overline{G}$, we say that G acts "by linear automorphisms."[1] So we may consider the following special case of H14-GA:

H14-LinGA. *If G is an algebraic group acting algebraically on $\mathbb{A}^n = \mathrm{Spec}\, R$* **by linear automorphisms**, *is R^G finitely generated as a $\mathbf{k}$-algebra?*

This problem was Hilbert's motivation for H14 and, for that reason, is sometimes referred to as the *Original 14th Problem*. However we stress that the problem that Hilbert proposed in paper [21] was H14, not H14-LinGA (so "original" refers to the origin of the problem, not to its authenticity as a member of Hilbert's list).

The second special case of H14-GA which we consider is:

H14-ConnGA. *If G is a* **connected** *algebraic group acting algebraically on $\mathbb{A}^n = \mathrm{Spec}\, R$, is R^G finitely generated as a $\mathbf{k}$-algebra?*

However we note that H14-ConnGA and H14-GA are equivalent, by virtue of the following:

1.1. **Lemma.** *Let G be an algebraic group acting algebraically on $\mathbb{A}^n = \mathrm{Spec}\, R$, and let G_0 be the connected component of G containing the identity. Then*

$$R^G \text{ is finitely generated over } \mathbf{k} \iff R^{G_0} \text{ is finitely generated over } \mathbf{k}.$$

Proof. Consider the inclusions $\mathbf{k} \subseteq R^G \subseteq R^{G_0}$. As G_0 has finite index in G it follows that R^{G_0} is integral over R^G. Moreover, $R^G = \mathrm{Frac}(R^G) \cap R$ is an intersection of normal domains, and so is normal; similarly, R^{G_0} is normal. The desired conclusion follows from these observations together with finiteness of integral closure. □

[1] In this situation one also says that the group action is *linearizable*.

It also follows from 1.1 that H14-GA has an affirmative answer whenever G is a finite group (G_0 is the trivial group, so R^{G_0} is finitely generated, so R^G is finitely generated by 1.1). There are other cases where H14-GA is known to have an affirmative answer, for instance when G is reductive, or in the special case $G = G_a$ of H14-LinGA, but we shall not discuss this (some details can be found in [19]).

Derivations

Let $R = \mathbf{k}^{[n]}$ where $\mathbf{k}$ is a field of characteristic zero.

If A is a $\mathbf{k}$-domain and $D : A \to A$ a $\mathbf{k}$-derivation then $\ker D = \left\{ x \in A \mid Dx = 0 \right\}$ is a $\mathbf{k}$-subalgebra of A and $\operatorname{Frac}(\ker D) \cap A = \ker D$. So the following is a special case of H14:

H14-Der. *If $D : R \to R$ is a $\mathbf{k}$-derivation, is $\ker D$ finitely generated as a $\mathbf{k}$-algebra?*

and of course the following is a special case of H14-Der:

H14-LND. *If $D : R \to R$ is a locally nilpotent derivation, is $\ker D$ finitely generated as a $\mathbf{k}$-algebra?*

Recall that a derivation $D : R \to R$ is *locally nilpotent* if for each $f \in R$ there exists $N > 0$ (depending on f) such that $D^N(f) = 0$. Consider the algebraic group G_a (which can be identified with $(\mathbf{k}, +)$ when $\mathbf{k}$ is algebraically closed). It is well-known that each locally nilpotent derivation $D : R \to R$ determines an algebraic action of G_a on $\mathbb{A}^n$ satisfying $R^{G_a} = \ker D$, and that all G_a-actions on $\mathbb{A}^n$ are obtained in this way. Thus H14-LND can also be viewed as the special case $G = G_a$ of H14-ConnGA.

Moreover, Nowicki [28] showed that H14-ConnGA is a special case of H14-Der. We explain this. Let $A \subseteq B$ be integral domains. An element $b \in B$ is *algebraic* over A if there exists a nonzero polynomial $f(T) \in A[T]$ such that $f(b) = 0$ (where $f(T)$ is not required to be monic). If the elements of A are the only elements of B which are algebraic over A, we say that A is *algebraically closed in B*. The following are Theorems 5.4 and 6.4 of [28]. In both statements, $\mathbf{k}$ is an arbitrary field of characteristic zero; in the second result, $R = \mathbf{k}^{[n]}$.

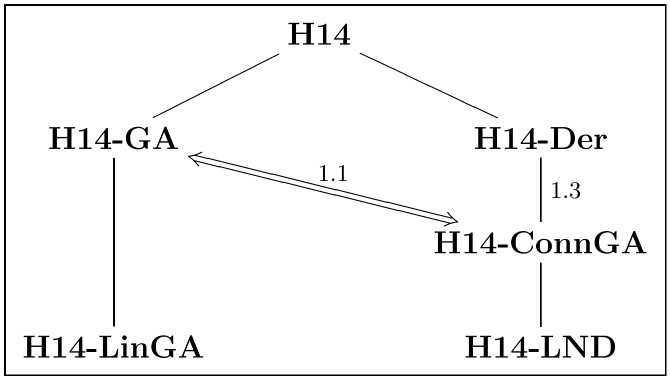
1.2. **Theorem** (Nowicki, 1994)**.** *For a subalgebra A of an affine $\mathbf{k}$-domain B, the following are equivalent:*

(a) *A is the kernel of some $\mathbf{k}$-derivation $D : B \to B$*
(b) *A is algebraically closed in B.*

1.3. **Theorem** (Nowicki, 1994)**.** *If G is a connected algebraic group acting on $\mathbb{A}^n = \operatorname{Spec} R$, then R^G is algebraically closed in R (and hence $R^G = \ker D$ for some $\mathbf{k}$-derivation $D : R \to R$).*

Remark. As far as this author knows, the following is an open question: *If A is the kernel of some $\mathbf{k}$-derivation of $R = \mathbf{k}^{[n]}$, does there exist an algebraic group action on $\mathbb{A}^n$ such that $A = R^G$?*

So we have the following hierarchy of special cases of H14:



Status of H14

The last open cases of H14 and H14-Der have been settled by Kuroda in 2005, but the other cases (H14-GA, H14-LinGA and H14-LND) are still open. The present subsection briefly describes the situation concerning H14, and the next one discusses H14-LND and H14-Der.

According to a comment by Nagata in [27], no contribution to H14 was made until 1953 when Zariski obtained the following:

Zariski's Theorem (cf. [34])**.** *Let A be a normal affine domain over a field $\mathbf{k}$ and let K be a field such that $\mathbf{k} \subseteq K \subseteq \operatorname{Frac} A$. If $\operatorname{trdeg}(K/\mathbf{k}) \leq 2$, then $K \cap A$ is finitely generated as a $\mathbf{k}$-algebra.*

So H14 has an affirmative answer whenever $\operatorname{trdeg}(K/\mathbf{k}) \leq 2$. Having obtained this result, Zariski proposed a generalised version of H14 in which R would be any normal affine $\mathbf{k}$-domain. In 1957 Rees [30] gave a counterexample to that generalized problem, but this was not a counterexample to H14. However Rees' idea to use symbolic blow-up of prime ideals would eventually be revisited by others and give rise to important developments (notably 1.5, below). The first counterexamples to H14 were found by Nagata, and were in fact counterexamples to H14-LinGA (which he calls the original 14-th problem). In particular, one example given in [27] implies:

1.4 (Nagata, 1959)**.** *Let $\mathbf{k}$ be a field (of any characteristic) whose transcendence degree over the prime field is at least 48. Then there exists an algebraic group G acting algebraically on $R = \mathbf{k}^{[32]}$ by linear automorphisms such that R^G is not finitely generated over $\mathbf{k}$. Moreover, $K = \operatorname{Frac}(R^G)$ has transcendence degree 4 over $\mathbf{k}$.*

The next counterexamples to H14 were found 30 years later by Roberts, and imply the following statement:

1.5 (Roberts 1990, [32])**.** *Let $\mathbf{k}$ be a field of characteristic zero and $R = \mathbf{k}^{[7]}$. Then there is a field K such that $\mathbf{k} \subset K \subset \operatorname{Frac} R$, $\operatorname{trdeg}(K/\mathbf{k}) = 6$ and $K \cap R$ is not finitely generated.*

This reignited the interest in the subject and, in the following years, several authors contributed counterexamples. However in all cases K had transcendence degree at least 4, and in fact the case $\operatorname{trdeg}(K/\mathbf{k}) = 3$ of H14 remained completely open until Kuroda found several counterexamples in 2005–2006. In particular:

1.6 (Kuroda, [25]). *Let $\mathbf{k}$ be a field of characteristic zero and $e \geq 3$ an integer. Then there exists a field K such that $\mathbf{k} \subset K \subset \mathbf{k}(X,Y,Z)$, $[\mathbf{k}(X,Y,Z) : K] = e$ and $K \cap \mathbf{k}[X,Y,Z]$ is not finitely generated.*

An easy consequence of 1.6 is:

1.7. **Corollary.** *Let $\mathbf{k}$ be a field of characteristic zero, $3 \leq d \leq n$ integers, $R = \mathbf{k}^{[n]}$. Then there exists a field K such that $\mathbf{k} \subset K \subset \operatorname{Frac}(R)$, $\operatorname{trdeg}(K/\mathbf{k}) = d$ and $K \cap R$ is not finitely generated as a $\mathbf{k}$-algebra.*

Zariski's Theorem and 1.7 settle H14, in the narrow sense that for every pair of integers $n \geq d \geq 0$ we know whether or not there exists a counterexample (to H14) with $R = \mathbf{k}^{[n]}$ and $\operatorname{trdeg}(K/\mathbf{k}) = d$. However, we understand very little of the problem!

Status of H14-Der and of H14-LND

Throughout this subsection, $\mathbf{k}$ is any field of characteristic zero. From Zariski's Theorem and the elementary fact that the kernel of a derivation of $R = \mathbf{k}^{[n]}$ is algebraically closed in R, one immediately obtains:

1.8. **Corollary.** *If $R = \mathbf{k}^{[n]}$ with $n \leq 3$, then the kernel of any $\mathbf{k}$-derivation $D : R \to R$ is a finitely generated $\mathbf{k}$-algebra.*

Kuroda [24] constructed some examples in 2005 showing that there exists a $\mathbf{k}$-derivation of $\mathbf{k}^{[4]}$ whose kernel is not finitely generated as a $\mathbf{k}$-algebra (note that Kuroda's derivation is not locally nilpotent). From this and Nowicki's result 1.2, it is easy to derive the following statement:

1.9. **Corollary.** *Given integers $n > d \geq 3$, there exists a $\mathbf{k}$-derivation of $\mathbf{k}^{[n]}$ whose kernel has transcendence degree d over $\mathbf{k}$ and is not finitely generated.*

By 1.8 and 1.9, H14-Der is settled (in the narrow sense that we have already explained). Before discussing the status of H14-LND, we make two remarks.

So far we have been parametrizing Hilbert's Problem by the pair (n, d), where $R = \mathbf{k}^{[n]}$ and $d = \operatorname{trdeg}(K/\mathbf{k})$, but here the reader should note that if $D \neq 0$ is a locally nilpotent derivation of $\mathbf{k}^{[n]}$ then $\ker D$ has transcendence degree $n-1$ over $\mathbf{k}$. In other words, H14-LND depends on n alone.

Let $R = \mathbf{k}[X_1, \dots, X_n] = \mathbf{k}^{[n]}$ and recall that a $\mathbf{k}$-derivation $D : R \to R$ is *triangular* if $DX_i \in \mathbf{k}[X_1, \dots, X_{i-1}]$ holds for all i. Our second remark is that triangular derivations have the following (well-known and elementary) property:

If $D : R \to R$ is triangular then it is locally nilpotent and moreover $\ker D$ contains a variable of R, where by a *variable* of R we mean an element $f \in R$ for which there exist $f_2, \dots, f_n$ satisfying $R = \mathbf{k}[f, f_2, \dots, f_n]$.

The first counterexamples to H14-LND were obtained by A'Campo-Neuen [1] and Deveney and Finston [15], by showing that Robert's counterexamples 1.5 to H14 were in fact kernels of locally nilpotent derivations of $\mathbf{k}^{[7]}$. So the case $n = 7$ of H14-LND has a negative answer, and so does the case $n = 6$ by an example of Freudenburg [18]. Then Freudenburg and this author gave the following example in [9]:

1.10. *Let $R = \mathbf{k}[a, b, x, y, z] = \mathbf{k}^{[5]}$ and define a $\mathbf{k}$-derivation $D : R \to R$ by*

$$D = a^2 \frac{\partial}{\partial x} + (ax + b)\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}.$$

Then $\ker D$ *is not finitely generated as a* $\mathbf{k}$*-algebra.*

Note that D (in 1.10) is triangular, hence locally nilpotent. The current status of H14-LND can be summarized by:

- H14-LND has an affirmative answer when $n \leq 3$ (by 1.8),
- it follows from 1.10 that, for each $n \geq 5$, there exists a locally nilpotent derivation of $\mathbf{k}^{[n]}$ whose kernel is not finitely generated.

The case $n = 4$ of H14-LND is still open, and is quite interesting. Freudenburg and this author gave the following results in [11] and [10]:

1.11. *The kernel of any triangular derivation of $\mathbf{k}^{[4]}$ is finitely generated.*

1.12. *Given $m \in \mathbb{N}$, there exists a triangular derivation of $\mathbf{k}^{[4]}$ whose kernel cannot be generated by fewer than m elements.*

Result 1.11 is particularly interesting if one remembers that D in 1.10 is a triangular derivation of $\mathbf{k}^{[5]}$. We point out that 1.11 was obtained in [11] as a corollary of the case $A = \mathbf{k}^{[1]}$ of the following result (also proved in [11]):

1.13. **Theorem.** *If A is a $\mathbf{k}$-affine Dedekind domain then the kernel of any triangular A-derivation of $A^{[3]}$ is finitely generated as a $\mathbf{k}$-algebra.*

In unpublished work, Bhatwadekar showed that 1.13 remains valid (with almost the same proof, modulo some clever observations) if the word "triangular" is replaced by "locally nilpotent". This gives the following improved versions (due to Bhatwadekar) of 1.13 and 1.11:

1.13.1. **Theorem.** *If A is a $\mathbf{k}$-affine Dedekind domain then the kernel of any locally nilpotent A-derivation of $A^{[3]}$ is finitely generated as a $\mathbf{k}$-algebra.*

1.11.1. **Corollary.** *Let $D : R \to R$ be a locally nilpotent derivation, where $R = \mathbf{k}^{[4]}$. If* $\ker D$ *contains a variable of R, then* $\ker D$ *is finitely generated as a* $\mathbf{k}$*-algebra.*

Regarding 1.13 and 1.13.1, one should note that the hypothesis that $\dim A = 1$ is important: if $A = \mathbf{k}^{[2]}$ then (by 1.10) there exists an A-triangular derivation of $A^{[3]}$ whose kernel is not finitely generated.

Not all locally nilpotent derivations D of $R = \mathbf{k}^{[4]}$ have the property that $\ker D$ contains a variable of R (cf. [17]); so it is still not known whether or not there exists a locally nilpotent derivation of $\mathbf{k}^{[4]}$ whose kernel is not finitely generated.

Status of H14-GA and of H14-LinGA

Much could be said here, but we limit ourselves to the following utterly incomplete remarks.

Freudenburg [20] gave a counterexample to H14-LinGA with $n = 11$; this appears to be the lowest dimension for which a counterexample is known, and the question remains open for $5 \le n \le 10$.

By Zariski's Theorem, 1.1 and 1.3, H14-GA has an affirmative answer whenever $n \le 3$; by 1.10, it has a negative answer when $n \ge 5$. As far as this author knows, the case $n = 4$ is open (see also the remark after 1.3).

2. Kernels of locally nilpotent derivations

Throughout this section, $R = \mathbf{k}^{[n]}$ where $\mathbf{k}$ is an arbitrary field of characteristic zero. We shall discuss the following question:

(∗) *Which subalgebras of R are kernels of locally nilpotent derivations $D : R \to R$?*

It is convenient to introduce a notation: if B is a ring, we define

$$\text{KLND}(B) = \{ \ker D \mid D : B \to B \text{ is a locally nilpotent derivation other than the zero derivation} \}.$$

Then the question (∗) under consideration is:

2.1. **Problem.** *Describe the set* KLND(R)*, where* $R = \mathbf{k}^{[n]}$.

Hilbert's Problem H14-LND asks whether each element of KLND(R) is finitely generated as a $\mathbf{k}$-algebra, which is one aspect of 2.1.

There is good motivation for Problem 2.1. Solving it would immediately lead to a description of all locally nilpotent derivations of $\mathbf{k}^{[n]}$, which is equivalent to describing all G_a-actions on $\mathbb{A}^n$ (note that Problem 2.1 asks for a classification of the algebraic quotient maps $\mathbb{A}^n \to \mathbb{A}^n /\!/ G_a$). Also, the subgroup E of $\text{Aut}_{\mathbf{k}}(R)$ generated by the set

$$\{ \exp(D) \mid D : R \to R \text{ is a locally nilpotent derivation} \}$$

is in fact normal, $E \lhd \text{Aut}_{\mathbf{k}}(R)$, and is thought to be essentially all of $\text{Aut}_{\mathbf{k}}(R)$ (it is conjectured that $\text{Aut}_{\mathbf{k}}(R)$ is generated by E and the linear automorphisms); so one hopes that progress in 2.1 would lead to progress in the understanding of automorphisms of $\mathbf{k}^{[n]}$. As another motivation, observe that any subalgebra A of R satisfying $R = A^{[1]}$ is an element of KLND(R); so this is closely related to the Cancellation Problem, which asks: *if A is a subalgebra of $R = \mathbf{k}^{[n]}$ such that $R = A^{[1]}$, does it follow that $A = \mathbf{k}^{[n-1]}$?*

In what follows we sometimes refer to the fact that the group $\text{Aut}_{\mathbf{k}}(R)$ acts on the set KLND(R); the action is the obvious one: if $\theta \in \text{Aut}_{\mathbf{k}}(R)$ and $A \in$ KLND(R) then $\theta A := \theta(A)$. Also, it is good to keep in mind the following fact, valid for all $n \ge 1$:

Let $R = \mathbf{k}^{[n]}$ and let $A \in$ KLND(R). Then

$$\text{Frac}\, A \otimes_A R = (\text{Frac}\, A)^{[1]}.$$

Consequently, $\text{Frac}\, A$ *is stably rational:* $(\text{Frac}\, A)^{(1)} = \mathbf{k}^{(n)}$.

We shall now make a few very basic comments on Problem 2.1 for small values of n. The case $n = 3$ will be elaborated afterwards.

2.1.1. The case $n = 1$ of 2.1 is trivial: $\text{KLND}(\mathbf{k}^{[1]}) = \{\mathbf{k}\}$.

2.1.2. The case $n = 2$ is not trivial, but is solved: Rentschler proved in [31] that if $A \in \text{KLND}(R)$ where $R = \mathbf{k}^{[2]}$, then there exist X, Y such that $R = \mathbf{k}[X, Y]$ and $A = \mathbf{k}[X]$. Thus, if $R = \mathbf{k}^{[n]}$ with $n \in \{1, 2\}$, each element $A \in \text{KLND}(R)$ satisfies

$$A = \mathbf{k}^{[n-1]} \text{ and } R = A^{[1]}. \tag{2}$$

It follows that $\text{Aut}_{\mathbf{k}}(R)$ acts transitively on $\text{KLND}(R)$ when $n \leq 2$.

2.1.3. Case $n = 3$. By a theorem of Miyanishi [26], each $A \in \text{KLND}(\mathbf{k}^{[3]})$ satisfies $A = \mathbf{k}^{[2]}$ (which is a great improvement over 1.8); in other words, the first part of (2) is still valid. However the second part fails, as one can prove:

If $n \geq 3$ then the action of $\text{Aut}_{\mathbf{k}}(R)$ on $\text{KLND}(R)$ is not transitive, and the set of orbits has cardinality $|\mathbf{k}|$.

2.1.4. Case $n = 4$. Here, it is no longer the case that any two elements of $\text{KLND}(R)$ are isomorphic to each other; indeed, one can show:

If $n \geq 4$ then $\{ [A] \mid A \in \text{KLND}(R) \}$ is an infinite set, where $[A]$ denotes the isomorphism class of the the $\mathbf{k}$-algebra A.

However, Deveney and Finston [14] showed that (if $\mathbf{k} = \mathbb{C}$) the field of fractions of any element of $\text{KLND}(\mathbf{k}^{[4]})$ is $\mathbf{k}^{(3)}$. It is not known whether every $A \in \text{KLND}(\mathbf{k}^{[4]})$ is finitely generated over $\mathbf{k}$, as we saw in the first section. Moreover, one knows examples of kernels $A \in \text{KLND}(R)$ (where $R = \mathbf{k}^{[4]}$) such that R is not a flat A-module (in contrast with the case $n \leq 3$, where R is always a faithfully flat A-module; cf. 2.2).

2.1.5. If $n \geq 5$ then some elements of $\text{KLND}(R)$ are not finitely generated. The question whether $A \in \text{KLND}(\mathbf{k}^{[n]})$ implies $\text{Frac}\, A = \mathbf{k}^{(n-1)}$ is open, as far as this author knows (however $(\text{Frac}\, A)^{(1)} = \mathbf{k}^{(n)}$ is always true).

We summarize the above discussion with the following table:

For all $A \in \text{KLND}(R)$	$n = 1$	2	3	4	5
$A = \mathbf{k}^{[n-1]}$ and $R = A^{[1]}$	yes	yes[a]	no	no	no
$A = \mathbf{k}^{[n-1]}$	yes	yes	yes[b]	no	no
$\text{Frac}\, A = \mathbf{k}^{(n-1)}$	yes	yes	yes	yes[c]	?
A is finitely generated	yes	yes	yes	?	no

[a]Rentschler 1968, [31].
[b]Miyanishi 1985, [26].
[c]Deveney and Finston 1994, [14].

THE CASE $n = 3$

We now restrict ourselves to the problem of describing $\text{KLND}(R)$ when $R = \mathbf{k}^{[3]}$ (and where $\mathbf{k}$ is any field of characteristic zero). Although this is still far from being understood, many partial results are known and there is steady progress, for instance:

2.2. **Theorem** (Bonnet 2002, [2]). *If $A \in \text{KLND}(R)$ then the morphism* $\operatorname{Spec} R \to \operatorname{Spec} A$, *determined by the inclusion $A \hookrightarrow R$, is surjective. Furthermore, R is faithfully flat as an A-module.*

2.3. **Theorem** (Kaliman 2002, [22]). *A fixed point free G_a-action on $\mathbb{A}^3$ is a translation.*

2.4. There has been significant progress also in the homogeneous case of the problem. Given a positive grading $\mathfrak{g}$ of R (that is, an $\mathbb{N}$-grading $R = \oplus_{i\in\mathbb{N}} R_i$ satisfying $R_0 = \mathbf{k}$), one considers the set $\text{KLND}(R, \mathfrak{g})$ of kernels of $\mathfrak{g}$-homogeneous locally nilpotent derivations of R; the problem then is to describe $\text{KLND}(R, \mathfrak{g})$ for each positive grading $\mathfrak{g}$. Early work on this question was done in [4], [5], and a geometric version of the problem was solved in [12], [13]. Although we cannot claim to have a complete solution, work in progress [8] describes $\text{KLND}(R, \mathfrak{g})$ to a large extent. For instance we will mention below that certain conjectures have been proved in the homogeneous case; the proofs are in [8]. We note that [8] makes crucial use of [12], [13].

Local slice construction. In the homogeneous case as well as in the general case, the main hope is to obtain a description of $\text{KLND}(R)$ in terms of the "local slice construction" (LSC). The LSC is a method, introduced by Freudenburg in [16], for modifying a kernel. That is, given a kernel $A \in \text{KLND}(R)$ and a parameter σ (which has to be chosen in a certain set), the process constructs a new element $\text{LSC}(A, \sigma)$ of $\text{KLND}(R)$ distinct from A. One says that $A' = \text{LSC}(A, \sigma)$ is obtained from A by local slice construction. We recall the definition of $\text{LSC}(A, \sigma)$ in 2.7, below, but first we need a notation.

2.5. If $A \in \text{KLND}(R)$ then, up to multiplication by an element of $\mathbf{k}^*$, there exists a unique locally nilpotent derivation $\Delta_A : R \to R$ which satisfies (i) $\ker(\Delta_A) = A$; and (ii) Δ_A is "irreducible", i.e., the only principal ideal of R which contains $\Delta_A(R)$ is R itself. Concretely, Δ_A can be obtained as follows: choose X, Y, Z such that $R = \mathbf{k}[X, Y, Z]$ and (cf. 2.1.3) choose f, g such that $A = \mathbf{k}[f, g]$; then for any $h \in R$ define $\Delta_A(h) = \left|\frac{\partial(f,g,h)}{\partial(X,Y,Z)}\right|$. (The fact that this jacobian derivation coincides with Δ_A is proved in [3].)

2.6. **Definition.** Given $A \in \text{KLND}(R)$, let Σ_A be the set of ordered triples (f, g, s) of elements of R satisfying $A = \mathbf{k}[f, g]$ and $\Delta_A(s) = \alpha g$ for some $\alpha \in \mathbf{k}[f] \setminus \{0\}$.

The following is essentially Theorem 2 of [16].

2.7. **Theorem and definition.** *Suppose that $A \in \text{KLND}(R)$ and $\sigma = (f, g, s) \in \Sigma_A$.*

(a) *There exists an essentially unique irreducible polynomial in two variables $\Phi(U, V) \in \mathbf{k}[U, V]$ such that $\Phi(f, s) \in gR$.*

(b) *If we define* $g' = \Phi(f,s)/g \in R$, *then there exists a unique element* $A' \in \text{KLND}(R)$ *such that* $\mathbf{k}[f,g'] \subseteq A'$. *Moreover,* $A' \neq A$.

In this situation, we define $\text{LSC}(A,\sigma) = A'$.

2.7.1. Note that in 2.7 we have $f \in A \cap A'$.

2.8. **Examples** (cf. Section 4.2 of [16]). Fix X, Y, Z such that $R = \mathbf{k}[X,Y,Z]$ and let $s = X^3 + XYZ - Y^3$.

- Let $A_1 = \mathbf{k}[H_0, H_1] \in \text{KLND}(R)$, where $H_0 = Y$ and $H_1 = X$, and let $\sigma_1 = (H_1, H_0, s)$; then $\sigma_1 \in \Sigma_{A_1}$ and $\text{LSC}(A_1, \sigma_1) = \mathbf{k}[H_1, H_2]$ where $H_2 = XZ - Y^2$.
- Let $A_2 = \mathbf{k}[H_1, H_2]$ and $\sigma_2 = (H_2, H_1, s)$, then $\sigma_2 \in \Sigma_{A_2}$ and $\text{LSC}(A_2, \sigma_2) = \mathbf{k}[H_2, H_3]$, where $H_3 = X^5 + 2X^3YZ - 2X^2Y^3 + X^2Z^3 - 2XY^2Z^2 + Y^4Z$.
- Let $A_3 = \mathbf{k}[H_2, H_3]$ and $\sigma_3 = (H_3, H_2, s)$, then $\sigma_3 \in \Sigma_{A_3}$ and $\text{LSC}(A_3, \sigma_3) = \mathbf{k}[H_3, H_4]$, where H_4 is a homogeneous polynomial of degree 13 which we will not write down explicitely.

Continuing this process, one obtains an infinite sequence $\{H_n\}_{n=0}^{\infty}$ of homogeneous polynomials of degrees $1, 1, 2, 5, 13, 34, 89, \dots$ (every other term in the Fibonacci sequence), and an infinite sequence $\{A_n\}_{n=1}^{\infty}$ of elements of $\text{KLND}(R)$, where $A_n = \mathbf{k}[H_{n-1}, H_n]$. Moreover, A_{n+1} is obtained from A_n by a local slice construction:

$$\mathbf{k}[H_0,H_1] \xrightarrow{\text{LSC}} \mathbf{k}[H_1,H_2] \xrightarrow{\text{LSC}} \mathbf{k}[H_2,H_3] \xrightarrow{\text{LSC}} \mathbf{k}[H_3,H_4] \longrightarrow \cdots$$

$$\mathbf{k}[H_{n-1}, H_n] \in \text{KLND}(R) \text{ for all } n \geq 1.$$

Remark. The sequence $\{H_n\}_{n=1}^{\infty}$ was discovered and rediscovered by several authors. We have already mentioned Freudenburg's paper [16]; there it is shown that $\mathbf{k}[H_{n-1}, H_n] \in \text{KLND}(R)$ for all $n \geq 1$. The sequence $\{H_n\}_{n=1}^{\infty}$ also appeared in unpublished work of Gizatullin, in relation with automorphisms of $\mathbf{k}^{[3]}$. Now consider the curve $V(H_n) \subset \mathbb{P}^2$ whose equation is $H_n = 0$. Then $V(H_3)$ is Yoshihara's rational quintic [33]. More generally, all $V(H_n)$ are "Kashiwara curves": in [23], they correspond to the case where the divisor Γ is a linear chain. The $V(H_n)$ are also "Orevkov curves": in [29] Orevkov defines curves $C_j \subset \mathbb{P}^2$ where $j > 0$ is either odd or a multiple of 4, and uses them to show that a certain inequality (involving degree of a rational curve and highest multiplicity of a singular point) is best possible; the $V(H_n)$ correspond exactly to the C_j with j odd, i.e., to the Orevkov curves whose complements have logarithmic Kodaira dimension $-\infty$. In my joint work [13] with Peter Russell, the curves $V(H_n)$ appear in the *basic* affine rulings of $\mathbb{P}^2$.

One can also show that, up to automorphism of $\mathbb{P}^2$, the $V(H_n)$ are precisely the curves $C \subset \mathbb{P}^2$ whose complement $\mathbb{P}^2 \setminus C$ is completable by a rational zigzag, or equivalently, whose complement $\mathbb{P}^2 \setminus C$ has trivial Makar-Limanov invariant.

All this shows that $\{H_n\}_{n=1}^{\infty}$ is indeed a remarkable sequence of polynomials!

As we have already said, one hopes to describe $\text{KLND}(R)$ in terms of the local slice construction. One aspect of such a description would be:

2.9. **Conjecture.** *Given any $A, A' \in$ KLND(R), there exists a finite sequence of local slice constructions which transforms A into A'.*

A weaker version of 2.9 asks for the existence of a finite sequence of "operations" which transforms A into A', where the allowed operations are local slice constructions and $\mathbf{k}$-automorphisms of R; this appeared as a question in [16].

Two special cases of Conjecture 2.9 have now been proved; in both cases, assume that $\mathbf{k}$ is algebraically closed:

2.9.1 (cf. [8]). *If $A, A' \in$ KLND$(R, \mathfrak{g})$ for some positive grading $\mathfrak{g}$ of R, there exists a finite sequence of local slice constructions which transforms A into A'.*

2.9.2 (cf. 1.13 of [7]). *Let A, A' be elements of* KLND(R) *such that $A \cap A' \neq \mathbf{k}$. Then there exists a finite sequence of local slice constructions which transforms A into A'.*

Result 2.9.2 is a corollary of 2.14 (below) and of [6]. It was obtained as a byproduct of an effort to answer the following question:

(†) *Which polynomials $f(X, Y, Z)$ are annihilated by at least two independent locally nilpotent derivations?*

(Here the word "independent" means that the derivations are nonzero and have distinct kernels.) It appears to this author that answering the above question is an essential step in the classification of locally nilpotent derivations of $\mathbf{k}^{[3]}$. So let us now discuss this question.

Basic elements

Until the end of this section, we assume that $\mathbf{k}$ is algebraically closed (and has characteristic zero, as before). Let $R = \mathbf{k}^{[3]}$. The following discussion is based on [7].

2.10. **Definition.** An element of R is said to be *basic* if it is irreducible and belongs to at least two elements of KLND(R).

One can show that if A, A' are distinct elements of KLND(R) such that $A \cap A'$ contains a nonconstant polynomial, then $A \cap A' = \mathbf{k}[f]$ where f is basic. So question (†) will be fully answered if we can understand basic elements of R.

It is immediate that all variables of R are basic. Moreover, one can show that each basic element is a "good field generator," i.e.,

2.11 (cf. 1.5 of [7]). *If f is a basic element of R then there exist $g, h \in R$ such that $\mathbf{k}(f, g, h)$ is the field of fractions of R.*

As the set of basic elements includes all variables and is included in the set of good field generators, it seems legitimate to state that *the concept of basic element is a natural generalization of that of variable.*

2.12. **Example.** Let $R = \mathbf{k}[X, Y, Z]$, choose a nonconstant $\varphi(Z) \in \mathbf{k}[Z]$ and let $f = XY - \varphi(Z)$; then f is an irreducible element of R and $\mathbf{k}[X, f]$, $\mathbf{k}[Y, f]$ are two elements of KLND(R); thus f is a basic element of R.

2.13. **Example.** Let $P(X,Y,Z) \in R = \mathbf{k}[X,Y,Z]$ be a variable of R over $\mathbf{k}[X]$ such that $\deg P(0,0,Z) > 0$, and define $f = P(X, XY, Z)$. Then one can show that f is a basic element of R.

The following result characterizes basic elements in terms of their generic fiber.

2.14. **Theorem** (cf. 1.10 and 1.11 of [7]). *Let $R = \mathbf{k}^{[3]}$ where $\mathbf{k}$ is an algebraically closed field of characteristic zero. For an element $f \in R$, the following conditions are equivalent:*

(a) *f is a basic element of R*
(b) *the $\mathbf{k}(f)$-algebra $\mathbf{k}(f) \otimes_{\mathbf{k}[f]} R$ is isomorphic to $\mathbf{k}(f)[U,V,W]\,/(UV - P(W))$ for some nonconstant polynomial $P(W) \in \mathbf{k}(f)[W]$, where U,V,W are independent indeterminates over $\mathbf{k}(f)$.*

Moreover, if f satisfies the above conditions then it also satisfies:

(c) *For general $\lambda \in \mathbf{k}$, the hypersurface "$f = \lambda$" in $\mathbb{A}^3$ is isomorphic to a hypersurface with equation $xy = \varphi_\lambda(z)$, for some nonconstant polynomial $\varphi_\lambda(z) \in \mathbf{k}[z]$.*

Remark. In statement (c), there does not necessarely exist an automorphism of $\mathbb{A}^3$ which maps one hypersurface onto the other.

2.15. **Example.** Let $f = H_3$, where $H_3 \in R = \mathbf{k}[X,Y,Z]$ is defined in 2.8, i.e.,

$$f = X^5 + 2X^3YZ - 2X^2Y^3 + X^2Z^3 - 2XY^2Z^2 + Y^4Z.$$

As f is irreducible and belongs to each of $\mathbf{k}[H_2,H_3], \mathbf{k}[H_3,H_4] \in \text{KLND}(R)$, f is a basic element of R and so satisfies 2.14(a). Therefore, f must satisfy conditions (b–c) of 2.14. Regarding (b), one can show that there is an isomorphism of $\mathbf{k}(f)$-algebras,

$$\mathbf{k}(f) \otimes_{\mathbf{k}[f]} R \quad \cong \quad \mathbf{k}(f)[U,V,W]/(UV - W^5 - f^3).$$

Regarding (c) one can show that, for each $\lambda \in \mathbf{k}^*$, the hypersurface $f = \lambda$ in $\mathbb{A}^3$ is isomorphic to the hypersurface $xy = z^5 + 1$, but that no automorphism of $\mathbb{A}^3$ maps one hypersurface onto the other.

More generally, each H_n is a basic element of R and, for each $\lambda \in \mathbf{k}^*$, the hypersurface $H_n = \lambda$ in $\mathbb{A}^3$ is isomorphic to the hypersurface $xy = z^{a_n} + 1$, where $a_n = \deg H_n$, and if $n \geq 3$ then no automorphism of $\mathbb{A}^3$ maps one hypersurface onto the other.

Let us also recall that it was once hoped that every element of $\text{KLND}(R)$ contained a variable of R, because that would have enabled one to classify all locally nilpotent derivations of R; however Freudenburg [17] exhibited a kernel which did not contain a variable (namely, $\mathbf{k}[H_2,H_3]$ in 2.8). In this regard, we propose:

2.16. **Conjecture.** *Each element of $\text{KLND}(R)$ contains a basic element of R.*

It is shown in [8] that Conjecture 2.16 is true in the homogeneous case. In fact the basic elements are well understood in the homogeneous case, and we conclude this text with a brief (and very incomplete) remark to this effect.

Homogeneous basic elements

Let $\mathfrak{g}$ be a positive grading of R (cf. 2.4). By a $\mathfrak{g}$-*basic* element of R, we mean a $\mathfrak{g}$-homogeneous irreducible element $f \in R$ which belongs to at least two elements of $\text{KLND}(R, \mathfrak{g})$. Then [8] gives, for each $\mathfrak{g}$, the complete list of $\mathfrak{g}$-basic elements of R. This list is obtained as a corollary to a classification of a certain type of curve on the weighted projective plane $\mathbb{P}_{\mathfrak{g}} = \text{Proj}(R, \mathfrak{g})$.

It is worthwile to give that list in the special case where $\mathfrak{g}$ is the standard grading of $R = \mathbf{k}[X, Y, Z]$, i.e., each of X, Y, Z is homogeneous of degree 1:

2.17. **Theorem** (cf. [8]). *Let $\mathfrak{g}$ be the standard grading of R. Up to an automorphism of the graded ring $(R, \mathfrak{g})$ (i.e., a linear automorphism), the $\mathfrak{g}$-basic elements of R are the terms of the sequence $\{H_n\}_{n=1}^{\infty}$ defined in* 2.8.

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