TRIANGULAR DERIVATIONS OF $\mathbf{k}[X,Y,Z]$

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Abstract. Let B be the polynomial ring in three variables over a field $\mathbf{k}$ of characteristic zero. A $\mathbf{k}$-derivation $D : B \to B$ is said to be *triangular* if there exists a triple (X, Y, Z) of elements of B satisfying $B = \mathbf{k}[X, Y, Z]$, $DX \in \mathbf{k}$, $DY \in \mathbf{k}[X]$ and $DZ \in \mathbf{k}[X, Y]$. We give a new characterization of triangular derivations.

Let $B = \mathbf{k}[X_1, X_2, X_3]$ be the polynomial ring in three variables over a field $\mathbf{k}$ of characteristic zero. Recall that a $\mathbf{k}$-derivation $D : B \to B$ is said to be *triangular* if there exists a triple (X, Y, Z) of elements of B satisfying

$$B = \mathbf{k}[X, Y, Z], \quad DX \in \mathbf{k}, \quad DY \in \mathbf{k}[X] \quad \text{and} \quad DZ \in \mathbf{k}[X, Y].$$

Because of our lack of understanding of the group of automorphisms of B, it is a nontrivial problem to decide whether a given derivation is triangular. As triangular derivations are in particular locally nilpotent, one usually seeks criteria for deciding whether a given locally nilpotent derivation $D : B \to B$ is triangular. This problem was considered by several authors (cf. [1], [16], [17], [18], [10], [3]). In [3], the problem was reduced to the case where D is irreducible and a criterion was given in that case. Section 5 of the present paper gives a new criterion in the irreducible case. (Refer to 2.7 for the definition of irreducibility.)

Let us say a few words about the method of proof. Consider any pair (D, s) where $D : B \to B$ is an irreducible locally nilpotent derivation and s is a preslice of D (i.e., s is an element of B satisfying $D(s) \neq 0$ and $D^2(s) = 0$). It is known that $\ker(D)$ is a polynomial ring in two variables over $\mathbf{k}$, so the inclusion $\ker(D) \hookrightarrow B$ determines a morphism of algebraic varieties $\mathcal{Q} : \mathbb{A}^3 \to \mathbb{A}^2$. For each $\lambda \in \mathbf{k}$, let $S_\lambda \subset \mathbb{A}^3$ be the hypersurface given by the equation $s = \lambda$ and let $f_\lambda : S_\lambda \to \mathbb{A}^2$ be the composition $S_\lambda \hookrightarrow \mathbb{A}^3 \xrightarrow{\mathcal{Q}} \mathbb{A}^2$, so the pair (D, s) determines the family $(f_\lambda)_{\lambda \in \mathbf{k}}$ of morphisms. In Section 3 we show that, for general $\lambda \in \mathbf{k}$, f_λ is a birational morphism whose missing curves and fundamental points satisfy certain constraints (cf. Section 1 for definitions). The hope, then, is to use the theory of birational morphisms of surfaces for understanding the relation between the geometric properties of the surfaces S_λ and the algebraic properties of the derivation D. That analysis turns out to be feasible in the cases that we consider in sections 4 and 5. Note that, although $\mathcal{Q} : \mathbb{A}^3 \to \mathbb{A}^2$ is a well-studied morphism, it appears to be the first time that a systematic analysis

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of the geometric properties of preslices leads to answering an algebraic question about derivations.

Section 4 considers an interesting subset of $\mathbf{k}[X,Y,Z]$ whose elements we call the “weakly basic” polynomials and which includes in particular all variables of $\mathbf{k}[X,Y,Z]$. Theorem 4.2 describes what happens when a weakly basic polynomial is a preslice of a locally nilpotent derivation. In the special case where the weakly basic preslice is a variable, one obtains the results of Section 5 on triangular derivations.

Conventions. All rings are commutative and have a unity. The set of units of a ring R is denoted R^*. If $r \in R$, we denote by R_r the localization $S^{-1}R$ where $S = \{1, r, r^2, \dots\}$. If R is an integral domain, $\operatorname{Frac} R$ is its field of fractions. If E is a subset of a ring R then $V(E) = V_R(E)$ denotes the closed subset $\big\{\, \mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq E \,\big\}$ of $\operatorname{Spec} R$.

If A is a subring of a ring B then the notation $B = A^{[n]}$ means that B is isomorphic as an A-algebra to the polynomial ring in n variables over A. If $B = \mathbf{k}^{[n]}$ for some field $\mathbf{k} \subseteq B$ then by a *variable of* B we mean an element $f \in B$ for which there exist $f_2, \dots, f_n \in B$ such that $B = \mathbf{k}[f, f_2, \dots, f_n]$.

1. Birational morphisms of surfaces

The aim of this section is to prove result 1.6, which is used in the proof of Theorem 4.2. Throughout the section, $\mathbf{k}$ is an algebraically closed field of arbitrary characteristic.

1.1. **Definition.** Let Ω be a nonsingular projective algebraic surface over $\mathbf{k}$. An *SNC-divisor* of Ω is a reduced effective divisor $D = \sum_{i=1}^n D_i$ satisfying: (i) each irreducible component D_i of D is a nonsingular curve; (ii) if $i \neq j$ then $D_i \cdot D_j \leq 1$ (where $D_i \cdot D_j$ denotes the intersection number in Ω); (iii) if i, j, k are distinct then $D_i \cap D_j \cap D_k = \varnothing$. If D is an SNC-divisor of Ω then the *dual graph of* D *in* Ω is the weighted graph with vertex set $\{D_1, \dots, D_n\}$, where distinct vertices D_i and D_j are joined by an edge if and only if $D_i \cap D_j \neq \varnothing$, and where the weight of a vertex D_i is defined to be the self-intersection number of D_i in Ω. If the dual graph of D in Ω is a linear chain, i.e., has the form $\overset{x_1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{x_2}{\bullet}\!\!-\!\!\!- \ \cdots \ -\!\!\!-\!\!\overset{x_q}{\bullet}$ where $q \geq 0$, $x_i \in \mathbb{Z}$, we say that D is a *zigzag*.

1.2. **Definition.** Let U be a nonsingular algebraic surface over $\mathbf{k}$.

(1) We say that U is *completable by rational curves* if there exists an open immersion $U \hookrightarrow \Omega$ such that Ω is a nonsingular projective surface and $\Omega \setminus U$ is a union of rational curves.
(2) We say that U is *completable by a zigzag* if there exists an open immersion $U \hookrightarrow \Omega$ such that Ω is a nonsingular projective surface, $\Omega \setminus U$ is the support of an SNC-divisor D of Ω, and D is a zigzag. If in addition each irreducible component of D is a rational curve we say that U is *completable by a rational zigzag*.

(3) We say that U *has no loops at infinity* if there exists an open immersion $U \hookrightarrow \Omega$ such that Ω is a nonsingular projective surface, $\Omega \setminus U$ is the support of an SNC-divisor D of Ω, and the dual graph of D in Ω does not have simple circuits (i.e., the graph is a forest).

1.3. **Definition.** Consider a birational morphism $f : X \to Y$ where X, Y are nonsingular surfaces over $\mathbf{k}$. A *fundamental point* of f is a closed point $y \in Y$ such that $f^{-1}(y)$ contains more than one point. A curve $C \subset Y$ is called a *missing curve* of f if it is an irreducible component of the closure of $Y \setminus f(X)$ in Y. Note that f has finitely many missing curves, and that a curve $C \subset Y$ is a missing curve of f if and only if $C \cap f(X)$ is a finite set. Refer to [2] for details.

1.4. Consider a birational morphism $f : X \to Y$ where X, Y are nonsingular surfaces over $\mathbf{k}$. It is known that f factors as

$$X \hookrightarrow Y_n \xrightarrow{\pi_n} \cdots \xrightarrow{\pi_1} Y_0 = Y, \tag{1}$$

where each Y_i is a nonsingular surface, $X \hookrightarrow Y_n$ is an open immersion and $\pi_i : Y_i \to Y_{i-1}$ is the blowing-up of Y_{i-1} at a point. In particular note that f has finitely many fundamental points and that, if $y \in Y$ is a fundamental point of f, $f^{-1}(y)$ has pure dimension 1. We shall always assume that the decomposition (1) of f has been chosen so as to minimize the integer n. This has the following consequence:

(2) *Let $\pi : Y_n \to Y_0$ be the composition $\pi_1 \circ \cdots \circ \pi_n$. If P is a point of Y which is not a fundamental point of f then there exists an open neighborhood U of P in Y such that π restricts to an isomorphism $\pi^{-1}(U) \to U$.*

Choose an open immersion $Y_0 \hookrightarrow \bar{Y}_0$ such that $\bar{Y}_0$ is a nonsingular projective surface and $\bar{Y}_0 \setminus Y_0$ is the support of an SNC-divisor of $\bar{Y}_0$. We may then form the larger commutative diagram

$$\begin{tikzcd}
\bar{Y}_n \arrow[r, "\bar{\pi}_n"] & \cdots \arrow[r, "\bar{\pi}_2"] & \bar{Y}_1 \arrow[r, "\bar{\pi}_1"] & \bar{Y}_0 \\
Y_n \arrow[u, hook] \arrow[r, "\pi_n"] & \cdots \arrow[r, "\pi_2"] & Y_1 \arrow[u, hook] \arrow[r, "\pi_1"] & Y_0 \arrow[u, hook] \\
X \arrow[u, hook] \arrow[rrr, "f"] & & & Y \arrow[u, equal]
\end{tikzcd} \tag{3}$$

where each $\bar{Y}_i$ is a nonsingular projective surface, each "$\hookrightarrow$" is an open immersion and $\bar{\pi}_i : \bar{Y}_i \to \bar{Y}_{i-1}$ is the blowing-up of $\bar{Y}_{i-1}$ at a point (i.e., π_i and $\bar{\pi}_i$ are centered at the same point).

1.5. **Lemma.** *Let $f : X \to Y$ be a birational morphism where X, Y are nonsingular surfaces over $\mathbf{k}$.*

(a) *X is completable by rational curves if and only if Y is completable by rational curves and all missing curves of f are rational.*

(b) *If X has no loops at infinity and Y is affine then each missing curve of f has one place at infinity.*
(c) *Consider a decomposition* (1) *of f and let $C \subset Y$ be a missing curve of f. If X is completable by a zigzag and Y is affine then the strict transform of C on Y_n is a nonsingular curve.*

Proof. Assertions (a) and (b) (resp. (c)) follow from result 2.16 (resp. 2.17) of [2]. □

1.6. **Lemma.** *Let X, Y be nonsingular affine surfaces over* $\mathbf{k}$ *and assume that X is completable by a rational zigzag. For any birational morphism $f : X \to Y$, the following hold:*
(a) *Every missing curve of f is a rational curve with one place at infinity.*
(b) *If P is a singular point of some missing curve of f, or a common point of two missing curves, then P is a fundamental point of f.*

Proof. Assertion (a) follows from parts (a) and (b) of 1.5. We prove (b). Consider the diagram (3) in 1.4.

Let $C \subset Y$ be a missing curve of f. If some singular point of C is not a fundamental point of f then, by (2), the strict transform of C on Y_n has a singular point, which contradicts part (c) of 1.5. So every singular point of C_j is a fundamental point of f.

Now suppose that C_1, C_2 are distinct missing curves of f and that P is a point of $C_1 \cap C_2$ (in $Y_0 = Y$) but not a fundamental point of f. For each $j = 1, 2$, let $\bar{C}_j$ be the closure of C_j in $\overline{Y}_0$ and let $\hat{C}_j$ be the strict transform of $\bar{C}_j$ in $\overline{Y}_n$. Then, by (2),

$$\hat{C}_1 \cap \hat{C}_2 \cap Y_n \neq \varnothing. \tag{4}$$

Since Y is affine, $\bar{Y}_0 \setminus Y_0$ is a connected divisor and each $\bar{C}_j$ meets $\overline{Y}_0 \setminus Y_0$; it follows that $\bar{Y}_n \setminus Y_n$ is a connected divisor and that each $\hat{C}_j$ meets $\bar{Y}_n \setminus Y_n$. Since $\overline{Y}_n \setminus Y_n$, $\hat{C}_1$ and $\hat{C}_2$ are all included in $\overline{Y}_n \setminus X$, it follows from (4) that X has a loop at infinity, which contradicts the assumption that X is completable by a zigzag. So each point belonging to two missing curves is a fundamental point of f. □

2. Preliminaries on locally nilpotent derivations

This section gathers some definitions and known results on locally nilpotent derivations.

Let R be a ring. A derivation $D : R \to R$ is *locally nilpotent* if for each $x \in R$ there exists $n > 0$ such that $D^n(x) = 0$. We use the notations

$$\text{LND}(R) = \text{ set of all locally nilpotent derivations } D : R \to R$$
$$\text{KLND}(R) = \big\{ \ker(D) \mid D \in \text{LND}(R) \text{ and } D \neq 0 \big\}$$

where $\ker(D) = \big\{ x \in R \mid D(x) = 0 \big\}$. By a *preslice* of D we mean an element $s \in R$ satisfying $Ds \neq 0$ and $D^2 s = 0$; it is clear that if D is locally nilpotent and $D \neq 0$ then D admits a preslice.

2.1. Let R be a domain containing $\mathbb{Q}$, $D : R \to R$ a nonzero locally nilpotent derivation and $A = \ker D$. The following facts are well-known, see for instance [11] or [9].

(a) *A is a factorially closed subring of R (that is, the conditions $x, y \in R \setminus \{0\}$ and $xy \in A$ imply $x, y \in A$); consequently, $A^* = R^*$ and if k is any field contained in R then D is a k-derivation.*

(b) *If $s \in R$ is any preslice of D, and if we write $a = Ds$, then $R_a = A_a[s] = {A_a}^{[1]}$. In particular, if s is any element of R satisfying $Ds \in R^*$, then $R = A[s] = A^{[1]}$.*

(c) *Define $\deg_D(x) = \max\big\{\, n \in \mathbb{N} \mid D^n x \neq 0 \,\big\}$ for $x \in R \setminus \{0\}$, and $\deg_D(0) = -\infty$. Then the map $\deg_D : R \to \mathbb{N} \cup \{-\infty\}$ is a degree function, i.e., the following hold for all $x, y \in R$: (i) $\deg_D x = -\infty \Leftrightarrow x = 0$; (ii) $\deg_D(xy) = \deg_D x + \deg_D y$; (iii) $\deg_D(x+y) \leq \max(\deg_D x, \deg_D y)$.*

2.2. **Theorem** (Miyanishi). *Let $\mathbf{k}$ be a field of characteristic zero and $B = \mathbf{k}^{[3]}$. For each $A \in \text{KLND}(B)$, $A = \mathbf{k}^{[2]}$.*

2.3. **Theorem** (Kaliman). *Let $\mathbf{k}$ be a field of characteristic zero and $f \in B = \mathbf{k}^{[3]}$. If $\mathbf{k}(f) \otimes_{\mathbf{k}[f]} B = \mathbf{k}(f)^{[2]}$ then f is a variable of B.*

Results 2.2 and 2.3 were proved in [14] and [12] respectively, under the assumption that $\mathbf{k} = \mathbb{C}$. Then the general cases follow from [13] and Lefschetz Principle arguments, see [7] for details. The next statement is part (1) of [4, 3.8] (with a slightly different notation); it is used in the proof of 5.1:

2.4. **Lemma.** *Let $B = \mathbf{k}^{[3]}$ where $\mathbf{k}$ is a field of characteristic zero. Suppose that $D \in \text{LND}(B)$ satisfies $D(V) \in \mathbf{k}[f] \setminus \{0\}$, for some variable V of B and some variable f of $\ker D$. Then $B = \mathbf{k}[f, V]^{[1]}$.*

We recall the proof of the following fact:

2.5. **Lemma.** *Let $\mathbf{k}$ be a field of characteristic zero and $B = \mathbf{k}^{[3]}$. Let $0 \neq D : B \to B$ be a locally nilpotent derivation, $A = \ker D$ and $f \in A$.*

(a) *If f is a variable of B then f is a variable of A.*

(b) *If f is a variable of A and there exists $s \in B$ such that $Ds \in \mathbf{k}[f] \setminus \{0\}$, then f is a variable of B.*

Proof. If f is a variable of B then $\mathbf{k}[f] \subset A \subset B = \mathbf{k}[f]^{[2]}$, where all rings are UFDs and where A has transcendence degree one over $\mathbf{k}[f]$; as is well-known, it follows that $A = \mathbf{k}[f]^{[1]}$, which proves (a).

Suppose that f is a variable of A and that $s \in B$ is such that $Ds \in \mathbf{k}[f] \setminus \{0\}$. Let g be such that $A = \mathbf{k}[f, g]$ and let $S = \mathbf{k}[f] \setminus \{0\}$, then $S^{-1}D \in \text{LND}(S^{-1}B)$ satisfies $(S^{-1}D)(s) \in (S^{-1}B)^*$, so 2.1(b) implies that $S^{-1}B = \big(\mathbf{k}(f)[g]\big)[s] = \big(\mathbf{k}(f)[g]\big)^{[1]}$, so $S^{-1}B = \mathbf{k}(f)^{[2]}$, so f is a variable of B by 2.3. $\square$

2.6. **Lemma.** *Let $\mathbf{k}$ be a field of characteristic zero and $B = \mathbf{k}^{[3]}$. Let $A \in \text{KLND}(B)$ and consider the morphism $\mathfrak{Q} : \operatorname{Spec} B \to \operatorname{Spec} A$ determined by the inclusion $A \hookrightarrow B$. Then every nonempty fiber of $\mathfrak{Q}$ has pure dimension one.*

Statement 2.6 easily follows from the fact that A is factorially closed in B. Although 2.6 is all that is needed here, one should note that stronger results are known regarding $\mathcal{Q}$, for instance one knows that $\mathcal{Q}$ is surjective, that B is faithfully flat over A, that the general fiber of $\mathcal{Q}$ is an affine line, etc.

2.7**. Definition.** Let R be a ring. A derivation $D : R \to R$ is *irreducible* if R is the only principal ideal of R which contains $D(R)$. It is easy to see that if R is a UFD of characteristic zero and $A \in \text{KLND}(R)$ then there exists an irreducible locally nilpotent derivation $\Delta_A : R \to R$ such that $\ker(\Delta_A) = A$, and Δ_A is unique up to multiplication by a unit. This defines the notation Δ_A. Also, one can easily show that the set of locally nilpotent derivations of R with kernel A is $\{ a\Delta_A \mid a \in A \setminus \{0\} \}$.

2.8**. Definition.** Let R be a ring and $D : R \to R$ a locally nilpotent derivation. Then the closed subset $V(D(R)) = \{ \mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq D(R) \}$ of $\operatorname{Spec} R$ is denoted $\operatorname{Fix}(D)$.

2.9. It is clear that if R is a UFD of characteristic zero and $D : R \to R$ is a locally nilpotent derivation then D is irreducible if and only if the codimension of $\operatorname{Fix}(D)$ in $\operatorname{Spec}(R)$ is strictly greater than 1.

The following is a variation on a known theme, see for instance [15, Lemma 1.1].

2.10**. Lemma.** *Let R be a $\mathbb{Q}$-algebra with finitely many minimal prime ideals, let $\mathfrak{p}$ be a minimal prime ideal of R and let $D : R \to R$ be a derivation. Then $D(\mathfrak{p}) \subseteq \mathfrak{p}$.*

Proof. Let η be the nilradical of R. Consider the ring homomorphism $\varepsilon : R \to R[[t]]$, $\varepsilon(x) = \sum_{n=0}^{\infty} \frac{D^n(x)}{n!} t^n$, where t is an indeterminate. If $x \in \eta$ then $\varepsilon(x)$ is a nilpotent element of $R[[t]]$, so each coefficient of the power series $\varepsilon(x)$ belongs to η, so in particular $D(x) \in \eta$. So, without using the assumption that R has finitely many minimal prime ideals, we have shown:

$$D(\eta) \subseteq \eta. \tag{5}$$

Now let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the distinct minimal prime ideals of R and let us show that $D(\mathfrak{p}_1) \subseteq \mathfrak{p}_1$. Note that $\eta = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$, so if $n = 1$ then we are done by (5).

Assume that $n > 1$ and consider $x \in \mathfrak{p}_1$. Pick $y \in (\mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_n) \setminus \mathfrak{p}_1$, then $xy \in \eta$, so (5) gives $D(xy) \in \eta$, so

$$D(x)y^2 + xyD(y) = y(D(x)y + xD(y)) = yD(xy) \in \eta.$$

As $xyD(y) \in \eta$, it follows that $D(x)y^2 \in \eta \subseteq \mathfrak{p}_1$ and hence that $D(x) \in \mathfrak{p}_1$, showing that $D(\mathfrak{p}_1) \subseteq \mathfrak{p}_1$. □

2.11**. Corollary.** *Let R be a finitely generated algebra over a field K of characteristic zero and $D : R \to R$ a locally nilpotent derivation. If for each minimal prime ideal $\mathfrak{p}$ of R we have*

$$\dim(R/\mathfrak{p}) = 1 \quad \textit{and} \quad D(R) \not\subseteq \mathfrak{p},$$

then each element of $\ker D$ *is algebraic over K.*

Proof. Let $x \in \ker D$. If $\mathfrak{p}$ is a minimal prime ideal of R then $D(\mathfrak{p}) \subseteq \mathfrak{p}$ by 2.10, so D determines a locally nilpotent derivation $\bar{D}$ of $R/\mathfrak{p}$. Moreover, $\bar{D} \neq 0$ since we have $D(R) \not\subseteq \mathfrak{p}$ by assumption. As $R/\mathfrak{p}$ is a one-dimensional integral domain and a finitely generated K-algebra, $\ker \bar{D}$ is algebraic over K, so the element $x+\mathfrak{p}$ of $R/\mathfrak{p}$ is algebraic over K. So there exists a nonzero polynomial $F_{\mathfrak{p}}(T) \in K[T]$ such that $F_{\mathfrak{p}}(x) \in \mathfrak{p}$, and this is true for each $\mathfrak{p} \in M$ where M is the set of minimal prime ideals of R. Define $F(T) = \prod_{\mathfrak{p}\in M} F_{\mathfrak{p}}(T) \in K[T] \setminus \{0\}$, then $F(x)$ belongs to all minimal prime ideals of R. Thus $F(x)$ is nilpotent and hence x is a root of $F(T)^n \in K[T] \setminus \{0\}$ for n large enough. So x is algebraic over K. $\square$

2.12**. Example.** In 2.11, D is not necessarely a K-derivation. For instance, let X, Y, Z be indeterminates over $\mathbb{Q}$, $K = \mathbb{Q}(X)$ and $R = K[Y,Z]/(Y^2)$. The derivation $Y\frac{\partial}{\partial X} + \frac{\partial}{\partial Z}$ of $K[Y,Z]$ maps the ideal (Y^2) into itself and so determines a derivation $D : R \to R$. Then D and R satisfy the hypothesis of 2.11 but D is not a K-derivation.

3. Family of birational morphisms determined by a preslice

Let $B = \mathbf{k}^{[3]}$ where $\mathbf{k}$ is an algebraically closed field of characteristic zero.

Throughout this section we fix a pair (D,s) where $D : B \to B$ is an irreducible locally nilpotent derivation and $s \in B$ is a preslice of D. This pair determines a family $(f_\lambda)_{\lambda\in\mathbf{k}}$ of morphisms which we define in 3.1. The purpose of this section is to study the properties of $(f_\lambda)_{\lambda\in\mathbf{k}}$.

3.1**. Definition.** Let $A = \ker D$ and let $\mathfrak{Q} : \operatorname{Spec} B \to \operatorname{Spec} A$ be the morphism determined by $A \hookrightarrow B$. For each $\lambda \in \mathbf{k}$, consider the closed subscheme S_λ of $\operatorname{Spec} B = \mathbb{A}^3$ determined by the ideal $(s-\lambda)B$ of B and define $f_\lambda : S_\lambda \to \operatorname{Spec} A$ to be the composite $S_\lambda \hookrightarrow \operatorname{Spec} B \xrightarrow{\mathfrak{Q}} \operatorname{Spec} A$. Recall that $\operatorname{Spec} A = \mathbb{A}^2$, by 2.2.

3.2**. Lemma.** *For general $\lambda \in \mathbf{k}$, $s - \lambda$ is irreducible in B.*

Proof. Let $A = \ker D$ and $\varphi = Ds \in A \setminus \{0\}$, and consider the prime factorization $\varphi = \prod_{i=1}^N p_i^{e_i}$ of φ in A, where the e_i are positive integers and the p_i are prime elements of A no two of which are associates. Suppose that $\lambda_1, \dots, \lambda_n$ are distinct elements of $\mathbf{k}$ such that $s-\lambda_i$ is reducible for each i. Then (for each i) we have $s-\lambda_i = a_i s_i$ for some $a_i, s_i \in B\setminus\mathbf{k}$ satisfying $\deg_D(a_i) \le \deg_D(s_i)$. As $1 = \deg_D(s-\lambda_i) = \deg_D(a_i)+\deg_D(s_i)$, we have $\deg_D(a_i) = 0$ and hence $a_i \in A \setminus \mathbf{k}$. Thus $\varphi = D(s-\lambda_i) = a_i D(s_i)$ shows that $a_i \mid \varphi$. Moreover, if $i \neq j$ then $\gcd(a_i, a_j) = 1$ since $a_i s_i - a_j s_j = \lambda_j - \lambda_i \in \mathbf{k}^*$. Thus $a_1, \dots, a_n \in A \setminus \mathbf{k}$ are pairwise relatively prime divisors of φ and consequently $n \le N$. $\square$

3.3**. Proposition.** *Let $\{Q_1, \dots, Q_m\}$ be a finite set of closed points of* $\operatorname{Spec} A$*. For general $\lambda \in \mathbf{k}$, the morphism $f_\lambda : S_\lambda \to$* $\operatorname{Spec} A$ *(cf.* 3.1*) has the following properties.*

(a) *S_λ is a nonsingular irreducible affine surface*
(b) *$f_\lambda : S_\lambda \to$* $\operatorname{Spec} A$ *is a birational morphism*

(c) *the missing curves of* f_λ *are precisely the irreducible components of the closed subset* $V(\varphi)$ *of* $\operatorname{Spec} A$, *where* $\varphi = Ds$

(d) *no element of* $\{Q_1, \dots, Q_m\}$ *is a fundamental point of* f_λ.

Proof. By 3.2, S_λ is an irreducible and reduced affine surface; since $\operatorname{char} \mathbf{k} = 0$, the general fibers of $s : \mathbb{A}^3 \to \mathbb{A}^1$ are smooth and assertion (a) is clear. We show that assertions (b) and (c) hold for any $\lambda \in \mathbf{k}$ satisfying the two conditions:

$$s - \lambda \text{ is irreducible} \tag{6}$$

$$\text{no irreducible component of } \operatorname{Fix}(D) \text{ is included in } S_\lambda. \tag{7}$$

Fix λ satisfying these conditions. By 2.1 we have $B \subseteq A_\varphi[s]$. As $A_\varphi[s] = (A_\varphi)^{[1]}$, we may consider the A_φ-homomorphism $e_\lambda : A_\varphi[s] \to A_\varphi$ which maps s to λ. By (6), the kernel of the composite $B \hookrightarrow A_\varphi[s] \xrightarrow{e_\lambda} A_\varphi$ is $(s - \lambda)B$ so we have the commutative diagram:

$$\begin{tikzcd} A \arrow[r, hook] & B \arrow[r, hook] \arrow[d] & A_\varphi[s] \arrow[d, "e_\lambda"] \\ & B/(s-\lambda) \arrow[r] & A_\varphi \end{tikzcd}$$

where the composite homomorphism from A to A_φ is the inclusion map. So we have $A \subseteq B/(s - \lambda) \subseteq A_\varphi$, from which we obtain

$$S_\lambda \xrightarrow{f_\lambda} \operatorname{Spec} A \text{ is birational} \qquad \text{and} \qquad \operatorname{Spec}(A) \setminus \operatorname{im}(f_\lambda) \subseteq V_A(\varphi).$$

As the union of the missing curves is (by definition) the closure of $\operatorname{Spec}(A) \setminus \operatorname{im}(f_\lambda)$, all missing curves are included in $V_A(\varphi)$. Let $P \in A$ be an irreducible factor of φ and consider the curve $C = V_A(P)$ in $\operatorname{Spec} A$. To prove (c), there remains to show:

$$C \text{ is a missing curve of } f_\lambda. \tag{8}$$

Consider the ideal $I = (P, s - \lambda)B$ of B and note that $f_\lambda^{-1}(C) = V_B(I)$, where we regard $f_\lambda^{-1}(C)$ as a subset of $\operatorname{Spec} B$ via the inclusions $f_\lambda^{-1}(C) \subset S_\lambda \subset \operatorname{Spec} B$. If $I = B$ then $f_\lambda^{-1}(C) = \varnothing$, so (8) is proved. So from now-on we may assume that $I \neq B$. As A is factorially closed in B (cf. 2.1), P is irreducible in B. Thus $V_B(P)$ and $V_B(s - \lambda)$ are distinct irreducible surfaces in $\operatorname{Spec} B = \mathbb{A}^3$ and it follows that $V_B(P, s - \lambda)$ has pure dimension one; so each one of the minimal prime over ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ of I (in B) has height 2.

If for some i we have $D(B) \subseteq \mathfrak{p}_i$ then $V_B(\mathfrak{p}_i) \subseteq \operatorname{Fix}(D)$; as $V_B(\mathfrak{p}_i)$ is a curve and $\operatorname{Fix}(D)$ has dimension at most one (because D is irreducible, cf. 2.9), it follows that $V_B(\mathfrak{p}_i)$ is an irreducible component of $\operatorname{Fix}(D)$; as $\mathfrak{p}_i \supseteq I \supseteq (s - \lambda)B$, we obtain $V_B(\mathfrak{p}_i) \subset S_\lambda$, which contradicts (7). So $D(B) \not\subseteq \mathfrak{p}_i$, i.e., for each i we have

$$\dim(B/\mathfrak{p}_i) = 1 \text{ and } D(B) \not\subseteq \mathfrak{p}_i. \tag{9}$$

Note that D maps the ideal I into itself. Indeed, if $b, b' \in B$ then

$$D(Pb + (s - \lambda)b') = PD(b) + b'D(s - \lambda) + (s - \lambda)D(b')$$

belongs to $I = (P, s-\lambda)B$ because $D(s-\lambda) = D(s) = \varphi \in PB$. So D determines a locally nilpotent derivation $\bar{D} : R \to R$, where $R = B/I$, and (by (9)) for each minimal prime ideal $\mathfrak{q}$ of R we have

$$\dim(R/\mathfrak{q}) = 1 \quad \text{and} \quad \bar{D}(R) \not\subseteq \mathfrak{q}.$$

By 2.11, it follows that each element of $\ker \bar{D}$ is algebraic over $\mathbf{k}$; if we write $A = \mathbf{k}[u,v]$, then $\bar{u}$ and $\bar{v}$ are algebraic over $\mathbf{k}$ (where for $b \in B$, $\bar{b} = b + I \in R$); so there exist $\alpha(T), \beta(T) \in \mathbf{k}[T] \setminus \{0\}$ such that $\alpha(u), \beta(v) \in I$. Then

$$f_\lambda^{-1}(C) = V_B(I) \subseteq V_B(\alpha(u), \beta(v)) = \mathfrak{Q}^{-1}\big(V_A(\alpha(u), \beta(v))\big)$$

and consequently $f_\lambda\big(f_\lambda^{-1}(C)\big)$ is included in the finite set $V_A(\alpha(u), \beta(v))$. This proves (8), so (c) is proved.

Consider the closed subset $\Gamma = \mathfrak{Q}^{-1}(\{Q_1, \dots, Q_m\})$ of $\operatorname{Spec} B$ and note that the set $\Lambda = \big\{ \lambda \in \mathbf{k} \mid S_\lambda \text{ contains an irreducible component of } \Gamma \big\}$ is finite. Let $\lambda \in \mathbf{k} \setminus \Lambda$. As Γ is a finite union of irreducible curves by 2.6, $f_\lambda^{-1}(\{Q_1, \dots, Q_m\}) = S_\lambda \cap \Gamma$ is a finite set and consequently no fundamental point of f_λ belongs to $\{Q_1, \dots, Q_m\}$. □

4. Weakly basic elements of $\mathbf{k}^{[3]}$

In this section $B = \mathbf{k}^{[3]}$ where $\mathbf{k}$ is any field of characteristic zero.

It is convenient to introduce the following term:

4.1. **Definition.** An element $f \in B$ is *weakly basic*[1] if there exist infinitely many $\lambda \in \mathbf{k}$ such that $f - \lambda$ is an irreducible element of B and such that the ring $B_\lambda = B/(f-\lambda)B$ satisfies $\mathrm{ML}(B_\lambda) = \mathbf{k}$.

In the above definition the symbol $\mathrm{ML}(R)$ stands for the Makar-Limanov invariant of the ring R, i.e., $\mathrm{ML}(R) = \displaystyle\bigcap_{D \in \mathrm{LND}(R)} \ker(D)$.

In the present section we consider the situation where a weakly basic element of B is a preslice of a locally nilpotent derivation. Noting that all variables of B are weakly basic, it will then be interesting to consider the special case where a variable of B is a preslice; this special case is studied in the next section.

4.2. **Theorem.** *Let $D : B \to B$ be an irreducible locally nilpotent derivation and $s \in B$ such that $Ds \neq 0$ and $D^2 s = 0$. If s is a weakly basic element of B then there exist X, Y, Z such that*

$$B = \mathbf{k}[X,Y,Z], \quad DX = 0 \quad \textit{and} \quad Ds \in \mathbf{k}[X].$$

Proof. We first consider the case where $\mathbf{k}$ is algebraically closed. Let $A = \ker D$ and, for each $\lambda \in \mathbf{k}$, let $B_\lambda = B/(s-\lambda)$. Consider $S_\lambda \cong \operatorname{Spec} B_\lambda$ and $f_\lambda : S_\lambda \to \operatorname{Spec} A$ defined in 3.1. Let $C_1, \dots, C_q$ be the distinct irreducible components of $V_A(Ds) \subset \operatorname{Spec} A$. Note that if $q = 0$ then $Ds \in \mathbf{k}^*$ so for any X, Y satisfying $A = \mathbf{k}[X,Y]$ we have $B = \mathbf{k}[X,Y,s]$ by 2.1, so the desired conclusion follows (with $Z = s$). From now-on,

[1] See Remark 4.3.

assume that $q > 0$. Let $\mathcal{E} = \{Q_1, \dots, Q_m\}$ be the finite set of closed points of $\operatorname{Spec} A$ which consists of all singular points of each C_i and of all intersection points $C_i \cap C_j$ $(i \neq j)$. By 3.3, the following conditions hold for general $\lambda \in \mathbf{k}$:

(10) S_λ is a smooth irreducible surface and $f_\lambda : S_\lambda \to \operatorname{Spec} A$ is a birational morphism

(11) $C_1, \dots, C_q$ are the missing curves of f_λ

(12) no fundamental point of f_λ belongs to $\mathcal{E}$.

Since s is a weakly basic element of B, there exist infinitely many $\lambda \in \mathbf{k}$ which satisfy all of (10–12) and moreover $\mathrm{ML}(B_\lambda) = \mathbf{k}$. For a smooth affine surface $S = \operatorname{Spec}(R)$ over $\mathbf{k}$, it is well-known that the condition $\mathrm{ML}(R) = \mathbf{k}$ implies that S is completable by a rational zigzag (cf. for instance [8]). Thus there exist infinitely many $\lambda \in \mathbf{k}$ which satisfy all of (10–12) and moreover:

(13) S_λ is completable by a rational zigzag.

For such a λ, 1.6 implies that $\mathcal{E}$ is included in the set of fundamental points of f_λ; by (12), it follows that $\mathcal{E} = \varnothing$, i.e., the curves $C_1, \dots, C_q$ are smooth and pairwise disjoint. Result 1.6 also implies that each C_j is a rational curve with one place at infinity, i.e., C_j is an affine line; let $u_1, \dots, u_q$ be irreducible elements of A such that $C_j = V_A(u_j)$, then each u_j is a variable of A by the Abhyankar-Moh-Suzuki Theorem. Write $X = u_1$ and choose V such that $A = \mathbf{k}[X, V]$.

We claim that $u_j \in \mathbf{k}[X]$ for all $j = 1, \dots, q$. Indeed, let $j \in \{2, \dots, q\}$ and consider $u_j = P(X, V) \in \mathbf{k}[X, V]$. As $V_A(X, P(X, V)) = C_1 \cap C_j = \varnothing$, we have $P(0, V) = \mu \in \mathbf{k}^*$. As X divides $P(X, V) - P(0, V) = u_j - \mu$ (which is irreducible), we have $u_j \in \mathbf{k}[X]$ (for each j). Since $Ds = c \prod_{j=1}^q u_j^{e_j}$ for some integers $e_j > 0$ and some $c \in \mathbf{k}^*$, $Ds \in \mathbf{k}[X]$. We have shown that if $\mathbf{k}$ is algebraically closed then:

(14) there exists a variable X of $\ker D = \mathbf{k}^{[2]}$ such that $Ds \in \mathbf{k}[X] \setminus \{0\}$.

Now drop the assumption that $\mathbf{k}$ is algebraically closed and let $\bar{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$. Let $\bar{B} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} B = \bar{\mathbf{k}}^{[3]}$ and let $\bar{D} : \bar{B} \to \bar{B}$ be the extension of D. It is well-known that $\bar{D}$ is an irreducible locally nilpotent derivation of $\bar{B}$ and (obviously) $\bar{D}(s) \neq 0$ and $\bar{D}^2(s) = 0$. By 3.2, $s - \lambda$ is irreducible in $\bar{B}$ for almost all $\lambda \in \bar{\mathbf{k}}$, so there exist infinitely many $\lambda \in \mathbf{k}$ such that $s - \lambda$ is irreducible in $\bar{B}$ and $\mathrm{ML}(B/(s-\lambda)) = \mathbf{k}$. As $\bar{\mathbf{k}} \otimes_{\mathbf{k}} B/(s-\lambda) \cong \bar{B}/(s-\lambda)$, it follows that $\mathrm{ML}(\bar{B}/(s-\lambda)) = \bar{\mathbf{k}}$, i.e., s is a weakly basic element of $\bar{B}$. Then $s \in \bar{B}$ and $\bar{D}$ satisfy the hypothesis of 4.2, so (14) implies that there exists a variable u of $\ker \bar{D} = \bar{\mathbf{k}}^{[2]}$ such that $Ds = \bar{D}s \in \bar{\mathbf{k}}[u] \setminus \{0\}$. As $\ker \bar{D} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} \ker D$ we may apply the following fact, whose proof is left to the reader:

> *Let $K \subseteq L$ be fields of characteristic zero, X, Y indeterminates over L and $\varphi \in K[X,Y] \subseteq L[X,Y]$. If $\varphi \in L[u]$ for some variable u of $L[X,Y]$, then $\varphi \in K[u']$ for some variable u' of $K[X,Y]$.*

This implies that there exists a variable X of $\ker D = \mathbf{k}^{[2]}$ such that $Ds \in \mathbf{k}[X] \setminus \{0\}$. Then X is a variable of B by 2.5, so the desired conclusion follows. □

4.3. **Remark.** One can also define a notion of a "basic element of B", as follows. First note the following fact, due to Freudenburg and this author:

> *Suppose that A, A' are distinct elements of* KLND(B) *such that $A \cap A' \neq \mathbf{k}$. Then $A \cap A' = \mathbf{k}^{[1]}$, $A = (A \cap A')^{[1]}$ and $A' = (A \cap A')^{[1]}$.*

It is then natural to ask: which polynomials $f \in B$ are such that $\mathbf{k}[f] = A \cap A'$ for some $A, A' \in$ KLND(B)? We give a name to these polynomials:

> An element $f \in B$ is said to be *basic* if there exist $A, A' \in$ KLND(B) such that $A \cap A' = \mathbf{k}[f]$.

It can be shown that basic elements are weakly basic; the converse is not known, but we don't expect it to be true. It can also be shown that the element s in Theorem 4.2 must be basic, but is not necessarely a variable of B.

Basic elements were studied in [5] under the assumption that $\mathbf{k}$ is algebraically closed. See also [6] for related results.

5. Triangular derivations of $\mathbf{k}^{[3]}$

We continue to assume that $B = \mathbf{k}^{[3]}$ where $\mathbf{k}$ is any field of characteristic zero.

5.1. **Theorem.** *Let $D : B \to B$ be an irreducible locally nilpotent derivation and assume that some variable Y of B satisfies $DY \neq 0$ and $D^2Y = 0$. Then there exist X, Z such that*

$$B = \mathbf{k}[X,Y,Z], \qquad DX = 0, \quad DY \in \mathbf{k}[X], \quad DZ \in \mathbf{k}[X,Y].$$

Proof. As Y is a variable of B, and hence a weakly basic element of B, result 4.2 implies that there exists a variable X of B such that $DY \in \mathbf{k}[X]$ and $DX = 0$. Then (2.5) X is also a variable of $\ker D$, so:

$$DY \in \mathbf{k}[X] \setminus \{0\} \text{ for some variable } X \text{ of } \ker D.$$

Then 2.4 implies that $B = \mathbf{k}[X,Y]^{[1]}$, so there exists Z such that $B = \mathbf{k}[X,Y,Z]$. As is well-known, the fact that D is locally nilpotent and maps $\mathbf{k}[X,Y]$ into itself implies that $DZ \in \mathbf{k}[X,Y]$, which proves the assertion. □

Several authors have studied locally nilpotent derivations D of $\mathbf{k}[X_1,\dots,X_n] = \mathbf{k}^{[n]}$ satisfying $D^2(X_i) = 0$ for all $i \in \{1,\dots,m\} \subseteq \{1,\dots,n\}$. Theorem 5.1 settles the case $(m,n) = (1,3)$ of that question. The cases $(m,n) = (2,3), (3,3)$ are described in [19]. Also note that 5.1 is an improvement of [4, 3.9], whose proof depended heavily on the extra hypotheses of homogeneity. Theorem 5.1 immediately leads to:

5.2. **Corollary.** *For an irreducible locally nilpotent derivation $D : B \to B$, the following conditions are equivalent:*

(a) *D is triangular*
(b) *some variable V of B satisfies $DV \neq 0$ and $D^2V = 0$.*

Proof. By 5.1, (b) implies (a). Conversely, if (a) holds then there exists a triple (X, Y, Z) satisfying $B = \mathbf{k}[X, Y, Z]$, $DX \in \mathbf{k}$, $DY \in \mathbf{k}[X]$ and $DZ \in \mathbf{k}[X, Y]$. If $DX \neq 0$ (resp. $DX = 0 \neq DY$, $DX = 0 = DY$) then (b) is satisfied with $V = X$ (resp. $V = Y$, $V = Z$). $\square$

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