POLYNOMIALS $f(X,Y,Z)$ OF LOW LND-DEGREE

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ABSTRACT. Let B be the polynomial ring in three variables over an algebraically closed field $\mathbf{k}$ of characteristic zero. We give a necessary condition that a polynomial $f \in B$ must satisfy if there exists a nonzero locally nilpotent derivation $D : B \to B$ such that $D^2(f) = 0$. In the case $f = X^a + Y^b + Z^c$, we determine the values of a, b, c for which such a derivation exists.

To my teacher, Peter Russell.

1. INTRODUCTION

Let R be an integral domain of characteristic zero. A derivation $D : R \to R$ is *locally nilpotent* if for each $x \in R$ there exists $n > 0$ such that $D^n(x) = 0$. We use the notations

$$\text{LND}(R) = \text{ set of all locally nilpotent derivations } D : R \to R$$

$$\text{KLND}(R) = \big\{ \ker(D) \mid D \in \text{LND}(R) \text{ and } D \neq 0 \big\}$$

where $\ker(D) = \big\{ x \in R \mid D(x) = 0 \big\}$.

We say that R is *rigid* if $\text{KLND}(R) = \varnothing$ (i.e., if $\text{LND}(R) = \{0\}$).

Assume that R is not rigid and consider $D \in \text{LND}(R) \setminus \{0\}$. If $f \in R \setminus \{0\}$, define $\deg_D(f)$ to be the least $i \in \mathbb{N}$ satisfying $D^{i+1}(f) = 0$; also define $\deg_D(0) = -\infty$. Then it is well-known that $\deg_D : R \to \mathbb{N} \cup \{-\infty\}$ is a degree function on R, i.e., for all $f, g \in R$ there holds: $\deg_D(f) = -\infty \Leftrightarrow f = 0$, $\deg_D(fg) = \deg_D(f) + \deg_D(g)$ and $\deg_D(f + g) \leq \max(\deg_D(f), \deg_D(g))$. Also well-known and easy to see is the fact that two elements of $\text{LND}(R) \setminus \{0\}$ determine the same degree function if and only if they have the same kernel. So each $A \in \text{KLND}(R)$ determines a degree function $\deg_A : R \to \mathbb{N} \cup \{-\infty\}$. For $f \in R$, we have $\deg_A(f) \leq 0$ if and only if $f \in A$; if $\deg_A(f) = 1$, we call f a *preslice* of A (or a preslice of D, where $D \in \text{LND}(R)$ is such that $\ker D = A$). Discussions between this author and Gene Freudenburg led to the idea that one should consider the function

$$\text{lndeg} : R \to \mathbb{N} \cup \{-\infty\}, \quad \text{lndeg}(f) = \min \big\{ \deg_A(f) \mid A \in \text{KLND}(R) \big\},$$

and that it would be interesting to characterize the elements $f \in R$ which satisfy

$$\text{lndeg}(f) \leq 1,$$

or equivalently,
$$D^2(f) = 0 \text{ for some } D \in \text{LND}(R) \setminus \{0\}.$$
We call $\text{lndeg}(f)$ the "LND-degree" of f, even though lndeg is not a degree function.

Consider the polynomial ring $B = \mathbf{k}[X, Y, Z]$, where $\mathbf{k}$ is an algebraically closed field of characteristic zero. Then Theorem 3.11, below, gives a necessary condition that $f \in B$ must satisfy if $\text{lndeg}(f) \leq 1$. That result is one of the tools used in Section 4 for answering the following question, posed to this author by Freudenburg: *Let $f = X^a + Y^b + Z^c \in B = \mathbf{k}[X, Y, Z]$; for which a, b, c does there exist $D \in \text{LND}(B) \setminus \{0\}$ satisfying $D^2(f) = 0$?* Corollary 4.8 gives the answer. Freudenburg's motivation for asking this question is the fact that its answer, combined with certain results of Freudenburg and Moser-Jauslin, implies rigidity of the integral domain $\mathbf{k}[X, Y, Z, T]/(X^a + Y^b + Z^c + T^d)$ for certain values of a, b, c, d.

Conventions. All rings are commutative and have a unity. The set of units of a ring R is denoted R^*. If $r \in R$, we denote by R_r the localization $S^{-1}R$ where $S = \{1, r, r^2, \dots\}$. If R is an integral domain, $\text{Frac}\, R$ is its field of fractions.

If A is a subring of a ring B then the notation $B = A^{[n]}$ means that B is isomorphic as an A-algebra to the polynomial ring in n variables over A. If $K/\mathbf{k}$ is a field extension, we write $K = \mathbf{k}^{(n)}$ to indicate that K is a purely transcendental extension of $\mathbf{k}$, of transcendence degree n.

2. Gradings and homogeneous derivations

In this section we recall some notations, definitions and known facts concerning gradings and homogeneous derivations.

Let R be an integral domain of characteristic zero and let $\mathfrak{g}$ be a $\mathbb{Z}$-grading of R, with notation $R = \bigoplus_{i \in \mathbb{Z}} R_i$. A derivation $D : R \to R$ is $\mathfrak{g}$-*homogeneous* if there exists $d \in \mathbb{Z}$ satisfying $D(R_i) \subseteq R_{i+d}$ for all $i \in \mathbb{Z}$. We write

$$\text{LND}(R, \mathfrak{g}) = \big\{\, D \in \text{LND}(R) \mid D \text{ is } \mathfrak{g}\text{-homogeneous} \,\big\},$$

$$\text{KLND}(R, \mathfrak{g}) = \big\{\, \ker(D) \mid D \in \text{LND}(R, \mathfrak{g}) \text{ and } D \neq 0 \,\big\}.$$

Note that if $A \in \text{KLND}(R, \mathfrak{g})$ then $A = \bigoplus_{i \in \mathbb{Z}} A_i$ where $A_i = R_i \cap A$, i.e., A inherits a grading from $(R, \mathfrak{g})$.

We often write "homogeneous" when we mean $\mathfrak{g}$-homogeneous (i.e., homogeneous with respect to $\mathfrak{g}$), and "$\deg(f)$" when we mean $\deg_{\mathfrak{g}}(f)$. We observe:

2.1. **Lemma.** *Let R be an integral domain and a finitely generated algebra over a field $\mathbf{k}$ of characteristic zero, and assume that R is not rigid. For any $\mathbb{Z}$-grading $\mathfrak{g}$ of R, the following hold.*

(1) $\text{KLND}(R, \mathfrak{g}) \neq \varnothing$

(2) *For each $A \in \text{KLND}(R)$, there exists $\mathcal{A} \in \text{KLND}(R, \mathfrak{g})$ satisfying*

$$\deg_A \bar{f} \leq \deg_A f \quad \text{for all } f \in R,$$

where $\bar{f}$ stands for the highest $\mathfrak{g}$-homogeneous component of f.

(3) *For any $\mathfrak{g}$-homogeneous $h \in R$,*

$$\text{lndeg}(h) = \min \left\{ \deg_A(h) \mid A \in \text{KLND}(R, \mathfrak{g}) \right\}.$$

In particular, the following conditions are equivalent:
- *there exists $D \in \text{LND}(R) \setminus \{0\}$ satisfying $D^2(h) = 0$*
- *there exists $D \in \text{LND}(R, \mathfrak{g}) \setminus \{0\}$ satisfying $D^2(h) = 0$.*

Proof. Assertions (1) and (3) follow from (2), and (2) follows from the well-known fact that, given $D \in \text{LND}(R) \setminus \{0\}$, the "homogeneization" $\mathcal{D}$ of D is a nonzero element of $\text{LND}(R, \mathfrak{g})$ which satisfies $\deg_{\mathcal{D}} \bar{f} \leq \deg_D f$ for all $f \in R$.   □

We are particularly interested in $\mathbb{Z}$-gradings of polynomial rings. The following definitions and facts (2.2 and 2.3) can be found in Chapter 2 of [Kol10].

2.2. Let $\mathbf{k}$ be a field of characteristic zero, $B = \mathbf{k}^{[n]}$ a polynomial ring over $\mathbf{k}$ and $\mathfrak{g}$ a $\mathbb{Z}$-grading of B. A *homogeneous coordinate system* of B is an ordered n-tuple $\gamma = (X_1, \ldots, X_n)$ of elements of B such that $B = \mathbf{k}[X_1, \ldots, X_n]$ and such that each X_i is homogeneous (with respect to $\mathfrak{g}$). If there exists a homogeneous coordinate system of B, we say that the grading $\mathfrak{g}$ is *coordinatizable.*

Fix a $\mathbb{Z}$-grading $\mathfrak{g}$ of B which is coordinatizable and nontrivial. Then each homogeneous coordinate system $\gamma = (X_1, \ldots, X_n)$ of B determines an n-tuple $\alpha(\mathfrak{g}, \gamma) = (\alpha_1, \ldots, \alpha_n)$, as follows: first, let $d_i = \deg(X_i) / \gcd(\deg(X_1), \ldots, \deg(X_n))$, $1 \leq i \leq n$; then,

$$\alpha_i = \gcd \left\{ d_j \mid j \in \{1, \ldots, n\} \setminus \{i\} \right\}.$$

Note that $\alpha_1, \ldots, \alpha_n$ are pairwise relatively prime natural numbers. It can be shown that, up to permutation, the n-tuples $(\deg(X_1), \ldots, \deg(X_n))$ and $\alpha(\mathfrak{g}, \gamma)$ are independent of the choice of γ. So the cardinality of the set $\left\{ i \mid \alpha_i \neq 1 \right\}$ is completely determined by $\mathfrak{g}$; this cardinality is called the *type* of $\mathfrak{g}$. We also define the type of the trivial grading to be zero.[1] So each coordinatizable grading $\mathfrak{g}$ of B has a type, and $\text{type}\,\mathfrak{g} \in \{0, 1, \ldots, n\}$.

Also define $d(A) = \gcd \left\{ \deg(h) \mid h \text{ is a nonzero homogeneous element of } A \right\}$ for each $A \in \text{KLND}(B, \mathfrak{g})$.

2.3. Theorem (Kolhatkar)**.** *Let $\mathbf{k}$ be a field of characteristic zero, $B = \mathbf{k}^{[n]}$ a polynomial ring, and $\mathfrak{g}$ a $\mathbb{Z}$-grading of B which is coordinatizable and nontrivial. Let $\gamma = (X_1, \ldots, X_n)$ be a homogeneous coordinate system of B. Define $\alpha(\mathfrak{g}, \gamma) = (\alpha_1, \ldots, \alpha_n)$ and $\text{type}(\mathfrak{g})$ as in 2.2.*

(1) $\left\{ d(A) \mid A \in \text{KLND}(B, \mathfrak{g}) \right\} = \{\alpha_1, \ldots, \alpha_n\}$

(2) *Suppose that $A \in \text{KLND}(B, \mathfrak{g})$ and $i \in \{1, \ldots, n\}$ are such that $d(A) = \alpha_i \neq 1$. Then the set of homogeneous prime elements h of B satisfying $\deg_A(h) = 1$ is equal to $\left\{ \lambda X_i \mid \lambda \in \mathbf{k}^* \right\}$.*

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[1] The trivial grading is of course not the only one of type 0. For nontrivial $\mathfrak{g}$, we have $\text{type}(\mathfrak{g}) = 0$ if and only if $d_1, \ldots, d_n$ are $(n-1)$-wise relatively prime.

(3) *For each $A \in \mathrm{KLND}(B, \mathfrak{g})$ we have $\{\, X_i \mid i \text{ satisfies } \alpha_i \notin \{1, d(A)\} \,\} \subset A$, so in particular $|\{X_1, \ldots, X_n\} \cap A| \geq \mathrm{type}(\mathfrak{g}) - 1$.*

3. A CRITERION

The aim of this section is to prove Theorem 3.11. We first recall some known facts.

3.1. Let R be a domain containing $\mathbb{Q}$, $D : R \to R$ a nonzero locally nilpotent derivation and $A = \ker D$. The following facts are well-known, see for instance [Fre06] or [vdE00].

(a) *A is a factorially closed subring of R (that is, the conditions $x, y \in R \setminus \{0\}$ and $xy \in A$ imply $x, y \in A$). Consequently, $A^* = R^*$; if k is any field contained in R then D is a k-derivation; A is algebraically closed in R (i.e., each element of R which is a root of a nonzero polynomial $f(T) \in A[T]$ belongs to A); if R is a UFD then so is A.*

(b) *If $s \in R$ is any preslice of D, and if we write $a = Ds$, then $R_a = A_a[s] = A_a^{[1]}$.*

(c) *The transcendence degree of R over A is 1.*

We also need the following consequence of a result of Miyanishi [Miy85]:

3.2. *Let $B = \mathbf{k}^{[3]}$ where $\mathbf{k}$ be a field of characteristic zero.*

(a) *If $D \in \mathrm{LND}(B) \setminus \{0\}$, then $\ker D = \mathbf{k}^{[2]}$.*

(b) *If $D \in \mathrm{LND}(B, \mathfrak{g}) \setminus \{0\}$ where $\mathfrak{g}$ is a $\mathbb{Z}$-grading of B, then $\ker D = \mathbf{k}[f, g]$ for some $\mathfrak{g}$-homogeneous elements $f, g \in B$.*

3.3. Lemma. *Let R be a UFD containing $\mathbb{Q}$ and let $\mathfrak{g}$ be a nontrivial $\mathbb{Z}$-grading on R (say $R = \bigoplus_{i \in \mathbb{Z}} R_i$). If there exists $D \in \mathrm{LND}(R, \mathfrak{g})$ such that the grading on $\ker(D)$ is trivial, then $R = R_0[v] = R_0^{[1]}$ for some homogeneous element $v \in R$ of nonzero degree.*

Proof. Note that $D \neq 0$; so R has transcendence degree 1 over $A = \ker D$. As $A \subseteq R_0 \subseteq R$ and R is not algebraic over R_0, R_0 is algebraic over A; as A is algebraically closed in R,
$$A = R_0.$$
Pick a homogeneous preslice $s' \in R$ of D. Note that $R^* \subset A$, so $s' \notin R^*$; consequently s' is a (nonempty) product of prime elements. As $\deg_D(s') = 1$, some prime factor of s' is a preslice of D. So there exists a preslice $s \in R$ of D which is prime and homogeneous. Let $\alpha = D(s) \in A \setminus \{0\}$, then $R_\alpha = A_\alpha[s] = A_\alpha^{[1]}$. If $r \in R \setminus \{0\}$ is homogeneous then $\alpha^N r = \sum_{i=0}^m a_i s^i$ for some $N, m \geq 0$, $a_i \in A = R_0$. Because $\alpha^N r$ is homogeneous, it follows that $\alpha^N r = a_i s^i$ for some $i \geq 0$. Then $s^i \mid \alpha^N r$ and $\gcd(s^i, \alpha) = 1$ imply that $s^i \mid r$, so $r = a s^i$ for some $a \in R_0$. We have shown that $R = R_0[s] = R_0^{[1]}$. $\qquad\square$

3.4. Definition. Let $R = \mathbf{k}[X_1, \ldots, X_n] = \mathbf{k}^{[n]}$, where $n \geq 2$ and $\mathbf{k}$ is a field of characteristic zero. Given $(a_1, \ldots, a_n) \in \mathbb{Z}^n$, let $\mathfrak{g}(a_1, \ldots, a_n)$ denote the $\mathbb{Z}$-grading of R obtained by stipulating that X_i is homogeneous of degree a_i. Note that $\mathfrak{g}(a_1, \ldots, a_n)$ is a coordinatizable grading and that $(X_1, \ldots, X_n)$ is a homogeneous coordinate system (cf. 2.2). We say that $\mathfrak{g}(a_1, \ldots, a_n)$ is an *admissible* grading if $\{\, i \mid a_i \neq 0 \,\}$ has cardinality at least 2.

3.5. Lemma. *Let* $\mathfrak{g} = \mathfrak{g}(a_1, \ldots, a_n)$ *be an admissible* $\mathbb{Z}$-*grading of* $R = \mathbf{k}[X_1, \ldots, X_n] = \mathbf{k}^{[n]}$, *where* $n \geq 2$ *and* $\mathbf{k}$ *is a field of characteristic zero. Then for each* $D \in \mathrm{LND}(R, \mathfrak{g})$, *the grading on* $\ker(D)$ *is nontrivial.*

Proof. By contradiction, suppose that there exists $D \in \mathrm{LND}(R, \mathfrak{g})$ such that the grading on $\ker(D)$ is trivial. Write $R = \bigoplus_{i \in \mathbb{Z}} R_i$. It follows from 3.3 that $R = R_0^{[1]}$, so in particular (i) R_0 is factorially closed in R, and (ii) R has transcendence degree 1 over R_0. By (i), all nonzero a_i have the same sign; then, by (ii), only one a_i is nonzero so $\mathfrak{g}$ is not admissible. $\square$

Until the end of this section, we assume that $\mathbf{k}$ is an algebraically closed field of characteristic zero and $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$. The following notion is needed in the proof of 3.9:

3.6. Definition. Let U be a nonsingular algebraic surface over $\mathbf{k}$. We say that U is *completable by rational curves* if there exists an open immersion $U \hookrightarrow \Omega$ such that Ω is a nonsingular projective surface and $\Omega \setminus U$ is a union of rational curves.

3.7. Remark. Let U be an open subset of a (possibly singular) surface S and assume that U is nonsingular and completable by rational curves. If $C \subset S$ is a curve lying in the complement of U then C is a rational curve.

3.8. Notations. Let R be an $\mathbb{N}$-graded ring and $h \in R$ a homogeneous element. Then we write $V(h)$ for the closed subset $\{\, \mathfrak{p} \in \mathrm{Spec}\, R \mid h \in \mathfrak{p} \,\}$ of $\mathrm{Spec}\, R$, $V_+(h)$ for the closed subset $\{\, \mathfrak{p} \in \mathrm{Proj}\, R \mid h \in \mathfrak{p} \,\}$ of $\mathrm{Proj}\, R$ and $D_+(h) = \mathrm{Proj}(R) \setminus V_+(h)$. As is well-known, if $\deg(h) > 0$ then $D_+(h) \cong \mathrm{Spec}\, R_{(h)}$ where $R_{(h)}$ is the component of degree zero of the $\mathbb{Z}$-graded localized ring R_h. If $(a, b, c) \in \mathbb{N}^3 \setminus \{(0,0,0)\}$ and $\mathbf{k}[X, Y, Z]$ is equipped with the $\mathbb{N}$-grading $\mathfrak{g}(a, b, c)$, then $\mathrm{Proj}\, \mathbf{k}[X, Y, Z]$ is denoted $\mathbb{P}(a, b, c)$.

3.9. Lemma. *Let* $(a, b, c) \in \mathbb{N}^3$ *be such that at most one of* a, b, c *is zero, and consider the* $\mathbb{N}$-*grading* $\mathfrak{g} = \mathfrak{g}(a, b, c)$ *of* $B = \mathbf{k}[X, Y, Z]$. *Let* h *be a nonconstant homogeneous element of* B *and consider the closed set* $V_+(h) \subset \mathbb{P}(a, b, c)$. *If* $D^2(h) = 0$ *for some* $D \in \mathrm{LND}(B, \mathfrak{g}) \setminus \{0\}$, *then each irreducible component of* $V_+(h)$ *is a rational curve.*

Proof. We first consider the case where $D(h) = 0$. To prove that $V_+(h) \subset \mathbb{P}(a, b, c)$ is a union of rational curves, we may assume that $\gcd(a, b, c) = 1$. Let $A = \ker(D)$, then (cf. 3.2) $A = \mathbf{k}[f, g] = \mathbf{k}^{[2]}$ where f, g are homogeneous elements of B. By 3.5, the grading on A is nontrivial; so the integer $d = \gcd(\deg f, \deg g)$ is nonzero and hence satisfies $d \geq 1$, which implies (by [Dai98, 3.10]) that $B_{(fg)} = (A_{(fg)})^{[1]}$. So for some v transcendental over $A_{(fg)}$ we have

$$(1) \qquad\qquad B_{(fg)} = (A_{(fg)})[v] = (A_{(fg)})^{[1]}.$$

Define $p = (\deg f)/d$, $q = (\deg g)/d$ and $\xi = f^q/g^p$; then $A_{(fg)} = \mathbf{k}[\xi, \xi^{-1}]$. As h is a homogeneous element of A, we have $h = \lambda f^r g^s \prod_{k=1}^m (f^q - \lambda_k g^p)$ for some $r, s, m \in \mathbb{N}$

and $\lambda, \lambda_k \in \mathbf{k}^*$, so we may write $fgh = \lambda f^i g^j \prod_{k=1}^m (\xi - \lambda_k)$ (where $i, j \in \mathbb{N}$). Let us record this as

$$fgh = f^i g^j \varphi \text{ for some } i, j \in \mathbb{N} \text{ and } \varphi \in \mathbf{k}[\xi] \setminus \{0\} \subseteq A_{(fg)} \setminus \{0\}.$$

This implies that $A_{(fgh)} = A_{(fg)}[1/\varphi]$ and $B_{(fgh)} = B_{(fg)}[1/\varphi]$ so, by (1),

$$B_{(fgh)} = B_{(fg)}[1/\varphi] = A_{(fg)}[v, 1/\varphi] = A_{(fgh)}[v].$$

Note that v is transcendental over $A_{(fgh)}$, because it is transcendental over $A_{(fg)}$ and $A_{(fgh)} = A_{(fg)}[1/\varphi]$ is a localization of $A_{(fg)}$. Consequently,

$$B_{(fgh)} = (A_{(fgh)})^{[1]} = \mathbf{k}[\xi, \xi^{-1}, \varphi^{-1}]^{[1]}.$$

We have $\deg(fgh) > 0$, because the grading on A is not trivial. It follows that the open subset $D_+(fgh)$ of $\mathbb{P}(a, b, c)$ is isomorphic to $\operatorname{Spec} B_{(fgh)}$, which is isomorphic to $C_0 \times \mathbb{A}^1_{\mathbf{k}}$, where $C_0 = \operatorname{Spec} \mathbf{k}[\xi, \xi^{-1}, \varphi(\xi)^{-1}]$ is an affine line minus finitely many points; so $D_+(fgh)$ is isomorphic to $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ minus finitely many projective lines; in particular, $D_+(fgh)$ is completable by rational curves. If $C \subset \mathbb{P}(a, b, c)$ is an irreducible component of $V_+(h)$, then C lies in the complement of $D_+(fgh)$ and hence is a rational curve by 3.7. Thus $V_+(h)$ is a union of rational curves.

Next, we consider the case where $D(h) \neq 0$ and $D^2(h) = 0$. Noting that $\deg_D(h) = 1$, we see that $h = h_0 h_1$ where $h_0 \in A$, h_1 is an irreducible element of B, $D(h_1) \neq 0$ and $D^2(h_1) = 0$, and of course h_0, h_1 are homogeneous. The first part of the proof implies that $V_+(h_0) \subset \mathbb{P}(a, b, c)$ is a union of rational curves. It is well-known that if s is an irreducible element of B satisfying $\Delta^2(s) = 0$ and $\Delta(s) \neq 0$ for some $\Delta \in \operatorname{LND}(B)$, then $V(s) \subset \operatorname{Spec}(B)$ is a rational surface;[2] thus $V(h_1) \subset \operatorname{Spec}(B)$ is a rational surface. As $V(h_1) \subset \operatorname{Spec}(B)$ is a cone over $V_+(h_1) \subset \mathbb{P}(a, b, c)$, it is birational to $V_+(h_1) \times \mathbb{P}^1$; so $V_+(h_1)$ is a rational curve. So $V_+(h) = V_+(h_0) \cup V_+(h_1)$ is a union of rational curves. $\square$

3.10. Notation. Let $f \in B = \mathbf{k}[X, Y, Z]$ be a nonconstant polynomial. Given $(a, b, c) \in \mathbb{N}^3 \setminus \{(0, 0, 0)\}$, let $\Gamma(f; a, b, c) = V_+(h) \subset \mathbb{P}(a, b, c)$ where h denotes the highest $\mathfrak{g}(a, b, c)$-homogeneous component of f.

3.11. Theorem. *Let $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$ where $\mathbf{k}$ is an algebraically closed field of characteristic zero, and let $f \in B$ be a nonconstant polynomial such that $D^2(f) = 0$ for some $D \in \operatorname{LND}(B) \setminus \{0\}$. Then, for each choice of $(a, b, c) \in \mathbb{N}^3$ such that at most one of a, b, c is zero, each irreducible component of $\Gamma(f; a, b, c)$ is a rational curve.*

Proof. Let (a, b, c) be as in the statement, let $\mathfrak{g} = \mathfrak{g}(a, b, c)$ and let h be the highest $\mathfrak{g}$-homogeneous component of f. If $D \in \operatorname{LND}(B) \setminus \{0\}$ satisfies $D^2(f) = 0$, then by 2.1 there exists $\tilde{D} \in \operatorname{LND}(B, \mathfrak{g}) \setminus \{0\}$ satisfying $\tilde{D}^2(h) = 0$, so the result follows from 3.9. $\square$

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[2]The composite $V(s) \hookrightarrow \operatorname{Spec} B \to \operatorname{Spec}(\ker \Delta) = \mathbb{A}^2_{\mathbf{k}}$ is a birational morphism.

4. An application

The aim of this section is to prove that $P(3)$ is true, where $P(n)$ is the following assertion:

4.1. Statement $P(n)$: *Let $\mathbf{k}$ be a field of characteristic zero, $B = \mathbf{k}[X_1, \ldots, X_n] = \mathbf{k}^{[n]}$ and $f = X_1^{a_1} + \cdots + X_n^{a_n} \in B$ where $(a_1, \ldots, a_n) \in \mathbb{N}^n$. If $\min(a_1, \ldots, a_n) \geq 2$ and at most one a_i is equal to 2, then no element D of $\mathrm{LND}(B) \setminus \{0\}$ satisfies $D^2(f) = 0$.*

To be precise, "$P(n)$ is true" means that the assertion is true for all fields $\mathbf{k}$ of characteristic zero. However, we observe that it's enough to prove $P(n)$ under the assumption that $\mathbf{k}$ is algebraically closed, so we are free to make this assumption whenever it is convenient. It is trivial that $P(1)$ is true. We prove $P(2)$ and $P(3)$ below, and also show that if $P(n-1)$ is true then $P(n)$ is at least partially true.

4.2. Let $\mathbf{k}$ be a field of characteristic zero and $n \geq 2$. Let the notation be as in 4.1 and suppose that $\min(a_1, \ldots, a_n) \geq 2$ and that at most one a_i is equal to 2. Let $M = \mathrm{lcm}(a_1, \ldots, a_n)$, let $a_i^* = M/a_i$ $(1 \leq i \leq n)$ and note that $\gcd(a_1^*, \ldots, a_n^*) = 1$. Let $\mathfrak{g}$ be the $\mathbb{N}$-grading of B defined by stipulating that X_i is homogeneous of degree a_i^*, and note that f is $\mathfrak{g}$-homogeneous of degree M. Thanks to 2.1 we know that, in order to prove $P(n)$,

it's enough to show that no $D \in \mathrm{LND}(B, \mathfrak{g}) \setminus \{0\}$ satisfies $D^2(f) = 0$.

Observe that $\mathfrak{g}$ is a coordinatizable grading of B and that $\gamma = (X_1, \ldots, X_n)$ is a homogeneous coordinate system of B; so the concepts of 2.2 can be used here. In particular, we have the n-tuple $\alpha(\mathfrak{g}, \gamma) = (\alpha_1, \ldots, \alpha_n)$, defined by

$$\alpha_i = \gcd \left\{ a_j^* \mid j \in \{1, \ldots, n\} \setminus \{i\} \right\},$$

and we have type $\mathfrak{g} \in \{0, 1, \ldots, n\}$ defined by $\mathrm{type}(\mathfrak{g}) = |\{ i \mid \alpha_i \neq 1 \}|$. It is convenient to define

$$\mathrm{cotype}(a_1, \ldots, a_n) = \mathrm{type}\,\mathfrak{g}.$$

4.3. Lemma. *Let $n \geq 2$ and assume that $P(n-1)$ is true. Then the following assertions are also true (where the notations and assumptions are as in 4.2):*

    (1) *If $D \in \mathrm{LND}(B) \setminus \{0\}$ satisfies $D(X_i) = 0$ for some $i \in \{1, \ldots, n\}$, then $D^2(f) \neq 0$.*

    (2) *$P(n)$ is true for all $(a_1, \ldots, a_n)$ satisfying $\mathrm{cotype}(a_1, \ldots, a_n) > 1$.*

Proof. Consider $D \in \mathrm{LND}(B) \setminus \{0\}$ satisfying $D(X_i) = 0$ for some $i \in \{1, \ldots, n\}$, and let us show that $D^2(f) \neq 0$. Without loss of generality, we may assume that $D(X_n) = 0$. Let $K = \mathbf{k}(X_n)$, $\mathcal{B} = K[X_1, \ldots, X_{n-1}] = K^{[n-1]}$ and $F = X_1^{a_1} + \cdots + X_{n-1}^{a_{n-1}} \in \mathcal{B}$. Since $D(X_n) = 0$, D extends to $\mathcal{D} \in \mathrm{LND}(\mathcal{B}) \setminus \{0\}$. If $D^2(f) = 0$ then $\mathcal{D}^2(f) = 0$, so $\mathcal{D}^2(F) = 0$, which contradicts the assumption that $P(n-1)$ is true. So $D^2(f) \neq 0$ and assertion (1) is proved.

Suppose that $\mathrm{cotype}(a_1, \ldots, a_n) > 1$. We have to show that $D^2(f) \neq 0$ for all $D \in \mathrm{LND}(B, \mathfrak{g}) \setminus \{0\}$, so let us consider such a D. By Theorem 2.3,

$$|\{X_1, \ldots, X_n\} \cap \ker D| \geq \mathrm{type}(\mathfrak{g}) - 1,$$

and since $\text{type}(\mathfrak{g}) - 1 = \text{cotype}(a_1, \ldots, a_n) - 1 > 0$ we have $D(X_i) = 0$ for some i, so we are done by the first assertion. $\qquad\square$

4.4. Lemma. $P(2)$ *is true.*

Proof. As we already noted, we may assume that $\mathbf{k}$ is algebraically closed. Let the notations and assumptions be as in 4.2, with $n = 2$. In view of 4.3 (and since $P(1)$ is true), we may assume that $\text{cotype}(a_1, a_2) \leq 1$. As $\alpha_1 = a_2^* = M/a_2$ and $\alpha_2 = a_1^* = M/a_1$, this assumption implies that one of M/a_1, M/a_2 is equal to 1, so one of a_1, a_2 divides the other. Without loss of generality, we may assume that $a_1 \mid a_2$. Then for some $m \geq 1$ we have $f = X_1^{a_1} + X_2^{ma_1} = \prod_{i=1}^{a_1} g_i$, where $g_i = X_1 + \lambda_i X_2^m$ for all $i = 1, \ldots, a_1$, and where $\lambda_1, \ldots, \lambda_{a_1}$ are distinct elements of $\mathbf{k}^*$.

Now suppose that there exists $D \in \text{LND}(B, \mathfrak{g})$ such that $D^2(f) = 0$, and let $A = \ker D$. It is not possible to have $\deg_D(g_i) = 0$ (i.e., $g_i \in A$) for two values of i, because that would imply $A = \mathbf{k}[X_1, X_2]$, a contradiction. As $1 = \deg_D(f) = \sum_{i=1}^{a_1} \deg_D(g_i)$ and $a_1 \geq 2$, it follows that $a_1 = 2$ and that $\{\deg_D(g_1), \deg_D(g_2)\} = \{0, 1\}$. Then

$$1 = \deg_D(g_1 - g_2) = \deg_D\left((X_1 + \lambda_1 X_2^m) - (X_1 + \lambda_2 X_2^m)\right) = m \deg_D(X_2),$$

so $m = 1$ and hence $(a_1, a_2) = (2, 2)$, a contradiction. So D does not exist. $\qquad\square$

4.5. Lemma. *Let S be a nonsingular surface over an algebraically closed field $\mathbf{k}$ of characteristic zero, let $C \subset S$ be a curve and let $P \in S$ be a point. Recall that the completion of the local ring $\mathcal{O}_{S,P}$ is isomorphic to $\mathbf{k}[[x, y]] = \mathbf{k}^{[[2]]}$ and suppose that the local equation of C at P is of the form*

$$x^a + y^b + \sum_{i,j} c_{ij} x^i y^j \in \mathbf{k}[[x, y]],$$

where $a, b \geq 0$ and where $bi + aj > ab$ for all i, j such that $c_{ij} \neq 0$. Then

$$\sum_Q \mu(Q)(\mu(Q) - 1) = ab - a - b + \gcd(a, b),$$

where Q (in the left hand side) runs in the set of all singular points of C infinitely near P, and where $\mu(Q)$ is the multiplicity of the singular point Q of C.

The above fact is well-known. Note that it remains valid when $P \notin C$, in which case the left-hand-side is the empty sum and the right-hand-side adds up to zero, because $\min(a, b) = 0$.

4.6. Lemma. *Let $\mathbf{k}$ be an algebraically closed field of characteristic zero and consider $F = Y^M + X^N + X^n \in \mathbf{k}[X, Y] = \mathbf{k}^{[2]}$, where $M \geq 2$ and $N > n \geq 0$. Let $C \subset \mathbb{A}_{\mathbf{k}}^2$ be the zero-set of F. Then C is rational if and only if one of the following conditions is satisfied:*

(a) $F = Y^2 + X^{2k+2} + X^{2k}$, $k \geq 0$
(b) $F = Y^M + X^{kM+1} + X^{kM}$, $k \geq 0$, $M \geq 2$
(c) $F = Y^M + X^{kM} + X^{kM-1}$, $k \geq 1$, $M \geq 2$.

Proof. The closure $\bar{C} \subset \mathbb{P}^2$ of $C \subset \mathbb{A}^2$ is the zero-set of the homogeneization $F^* \in$ $\mathbf{k}[X,Y,Z]$ of F. We compute the geometric genus g of $\bar{C}$.

Consider first the case where $M < N$. Then $\deg(F) = N$ and $F^* = Y^M Z^{N-M} + X^N + X^n Z^{N-n}$, so the singular points of $\bar{C}$ belong to $\{P_0, P_\infty\}$ where $P_0 = (0:0:1)$ and $P_\infty = (0:1:0)$. The local equations f_0 and f_∞ of $\bar{C}$ at P_0 and P_∞ respectively are:

$$f_0 = F = Y^M + X^N + X^n \in \mathbf{k}[[X,Y]], \qquad\qquad (a,b) = (M,n),$$
$$f_\infty = F^*(X,1,Z) = Z^{N-M} + X^N + X^n Z^{N-n} \in \mathbf{k}[[X,Z]], \quad (a,b) = (N, N-M),$$

where the pairs (a,b) of Lemma 4.5 are also indicated. Set $d_0 = \gcd(M,n)$ and $d_\infty = \gcd(N, N-M) = \gcd(M,N)$, then by 4.5 and the genus formula we get

$$2g = (N-1)(N-2) - \big(Mn - M - n + d_0\big) - \big(N(N-M) - N - (N-M) + d_\infty\big)$$
$$= (M-1)(N-n) + 2 - d_0 - d_\infty.$$

So, in the case where $M < N$, we obtain:

$$(2) \qquad 2g = (M-1)(N-n) + 2 - d_0 - d_\infty, \quad d_0 = \gcd(M,n), \quad d_\infty = \gcd(M,N).$$

We claim that (2) continues to be valid in the case where $M \geq N$. Indeed, in this case we have $\deg(F) = M$ and $F^* = Y^M + X^N Z^{M-N} + X^n Z^{M-n}$, so $\mathrm{Sing}(\bar{C}) \subseteq \{P_0, P_\infty\}$ where $P_0 = (0:0:1)$ and $P_\infty = (1:0:0)$. The local equations f_0 and f_∞ of $\bar{C}$ at P_0 and P_∞ respectively are:

$$f_0 = F = Y^M + X^N + X^n \in \mathbf{k}[[X,Y]], \qquad\qquad (a,b) = (M,n),$$
$$f_\infty = F^*(1,Y,Z) = Y^M + Z^{M-N} + Z^{M-n} \in \mathbf{k}[[Y,Z]], \quad (a,b) = (M, M-N),$$

where the pairs (a,b) of Lemma 4.5 are also indicated. Set $d_0 = \gcd(M,n)$ and $d_\infty = \gcd(M, M-N) = \gcd(M,N)$, then by 4.5 and the genus formula we get

$$2g = (M-1)(M-2) - \big(Mn - M - n + d_0\big) - \big(M(M-N) - M - (M-N) + d_\infty\big)$$
$$= (M-1)(N-n) + 2 - d_0 - d_\infty,$$

which is identical to (2) (and the values of d_0 and d_∞ also agree with (2)). So (2) is valid in all cases.

Using (2), it is easy to check that if one of conditions (a–c) holds (see the statement of the Lemma) then $g = 0$, so C is rational.

Conversely, suppose that $g = 0$; using (2), we show that one of conditions (a–c) must hold. Note that d_0, d_∞ satisfy (in particular) $d_0 \mid M$ and $d_\infty \mid M$, so $\max(d_0, d_\infty) \leq M$.

If $N - n \geq 3$ then $0 = 2g \geq 3(M-1) + 2 - d_0 - d_\infty = (M-1) + (M-d_0) + (M-d_\infty) \geq M - 1$, so $M \leq 1$, a contradiction. So we must have $N - n \in \{1, 2\}$.

If $N - n = 2$ then $0 = 2g = 2(M-1) + 2 - d_0 - d_\infty = (M - d_0) + (M - d_\infty)$ implies that $d_0 = M = d_\infty$, which implies that $M \mid n$ and $M \mid N$; then M divides the number $N - n = 2$, and since $M \geq 2$ we get $M = 2$. As $M = 2$ divides n and $N = n + 2$, condition (a) holds.

If $N - n = 1$ then $0 = 2g = (M - 1) + 2 - d_0 - d_\infty$, so $d_0 + d_\infty = M + 1$. This implies that $\max(d_0, d_\infty) > M/2$, so $\max(d_0, d_\infty) = M$. As one of d_0, d_∞ is equal to M, we have $M \mid n$ or $M \mid N$; in the former case (b) holds, and in the latter case (c) holds. $\square$

4.7. Proposition. $P(3)$ *is true.*

Proof. We may assume that $\mathbf{k}$ is algebraically closed. Let $n = 3$ and let the notations and assumptions be as in 4.2. The first part of the proof consists in showing that if (a_1, a_2, a_3) satisfies the hypothesis of $P(3)$ and $\operatorname{cotype}(a_1, a_2, a_3) < 3$, then one of the following conditions must hold:[3]

(i) the curve $C_{(a_1, a_2, a_3)} = V_+(f) \subset \mathbb{P}(a_1^*, a_2^*, a_3^*)$ is not rational
(ii) $\operatorname{cotype}(a_1, a_2, a_3) = 2$
(iii) up to permutation, (a_1, a_2, a_3) is (b, a, a) for some $a \geq 3$ and $b \geq 2$ such that $\gcd(a, b) = 1$.

Suppose that (a_1, a_2, a_3) satisfies the hypothesis of $P(3)$ and $\operatorname{cotype}(a_1, a_2, a_3) < 3$. This condition on the cotype means that $\gcd(a_i^*, a_j^*) = 1$ for some $i, j \in \{1, 2, 3\}$, so, replacing if necessary (a_1, a_2, a_3) by a permutation of it, we may arrange that

$$(3) \qquad\qquad\qquad \gcd(a_2^*, a_3^*) = 1.$$

Observe that $a_2^* \mid a_3$. Indeed, by definition of a_i^* we have $a_i a_i^* = M$ for all i, where $M = \operatorname{lcm}(a_1, a_2, a_3)$. So $a_2 a_2^* = a_3 a_3^*$, and we get $a_2^* \mid a_3$ by (3). The integer $a_3/a_2^* = a_2/a_3^*$ plays a role in the analysis below, and it is convenient to give it a name. Define

$$\rho = a_3/a_2^* = a_2/a_3^* \in \mathbb{N} \setminus \{0\}.$$

Then we note that

$$(4) \qquad\qquad (a_1, a_2, a_3) = (a_1, \rho a_3^*, \rho a_2^*) \quad \text{and} \quad M = \rho a_2^* a_3^*.$$

By (3), we may choose $r, s \in \mathbb{N} \setminus \{0\}$ such that

$$(5) \qquad\qquad\qquad a_2^* r - a_3^* s = -1.$$

Consider the submodule W of $\mathbb{Z}^3$ defined by

$$W = \left\{ (i_1, i_2, i_3) \in \mathbb{Z}^3 \mid a_1^* i_1 + a_2^* i_2 + a_3^* i_3 = 0 \right\}$$

and the elements $w_1 = (1, ra_1^*, -sa_1^*)$ and $w_2 = (0, a_3^*, -a_2^*)$ of W. We claim that $\{w_1, w_2\}$ is a basis of the $\mathbb{Z}$-module W. Linear independence is clear, so it suffices to check that $\{w_1, w_2\}$ generates W. Let $w = (i_1, i_2, i_3) \in W$. As $\{w_1, w_2\}$ is a basis of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} W$, there exist $q_1, q_2 \in \mathbb{Q}$ such that $w = q_1 w_1 + q_2 w_2$. Then $q_1 = i_1 \in \mathbb{Z}$, so $q_2 w_2 = w - q_1 w_1 \in \mathbb{Z}^3$; it follows that $q_2 a_2^*$ and $q_2 a_3^*$ are integers, so $q_2 \in \mathbb{Z}$ by (3). So $\{w_1, w_2\}$ is indeed a basis of W. Now define $u, v \in B_{(X_3)}$ by

$$u = X_1 X_2^{ra_1^*} X_3^{-sa_1^*} \quad \text{and} \quad v = X_2^{a_3^*} X_3^{-a_2^*}.$$

_____
[3]In fact this implication is also the proof of Corollary 4.9.

Consider a monomial $\mathcal{M} = X_1^{i_1} X_2^{i_2} X_3^{i_3} \in \mathbf{k}[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$ satisfying $(i_1, i_2, i_3) \in W$; then we have $(i_1, i_2, i_3) = aw_1 + bw_2$ for some $a, b \in \mathbb{Z}$, so $\mathcal{M} = u^a v^b \in \mathbf{k}[u^{\pm 1}, v^{\pm 1}]$. As each element of $B_{(X_3)}$ is a linear combination (over $\mathbf{k}$) of monomials $X_1^{i_1} X_2^{i_2} X_3^{i_3}$ satisfying (in particular) $(i_1, i_2, i_3) \in W$, we have

(6) $$\mathbf{k}[u, v] \subseteq B_{(X_3)} \subseteq \mathbf{k}[u^{\pm 1}, v^{\pm 1}].$$

In particular, the elements $X_1^{a_1}/X_3^{a_3}$ and $X_2^{a_2}/X_3^{a_3}$ of $B_{(X_3)}$ can be expressed in terms of u, v as

(7) $$\frac{X_1^{a_1}}{X_3^{a_3}} = \frac{u^{a_1}}{v^{a_3 r}} \quad \text{and} \quad \frac{X_2^{a_2}}{X_3^{a_3}} = v^\rho.$$

As $\deg(X_3) = a_3^* > 0$, the open subset $D_+(X_3)$ of $\mathbb{P}(a_1^*, a_2^*, a_3^*)$ is isomorphic to $\operatorname{Spec} B_{(X_3)}$ and a birational morphism $\beta : D_+(X_3) \to \mathbb{A}^2$ is determined by the inclusion $\mathbf{k}[u, v] \hookrightarrow B_{(X_3)}$. The irreducible curve $C_{(a_1, a_2, a_3)} \subset \mathbb{P}(a_1^*, a_2^*, a_3^*)$ meets the open set $D_+(X_3)$ and we may consider the closure Γ of $\beta\big(C_{(a_1, a_2, a_3)} \cap D_+(X_3)\big)$ in $\mathbb{A}^2$. It follows from (6) that $C_{(a_1, a_2, a_3)} \cap D_+(X_3)$ is not contracted by β, so $\Gamma \subset \mathbb{A}^2$ is an irreducible curve which is birational to $C_{(a_1, a_2, a_3)}$. The equation of Γ can be obtained by dividing $X_1^{a_1} + X_2^{a_2} + X_3^{a_3} = 0$ by $X_3^{a_3}$, using (7) and clearing denominators; this gives:

(8) $\Gamma \subset \mathbb{A}^2$ is the zero-set of the polynomial $u^{a_1} + v^{a_3 r + \rho} + v^{a_3 r} \in \mathbf{k}[u, v]$.

In view of Lemma 4.6, if Γ is rational then one of the following conditions holds:

(a) $a_1 = 2$, $a_3 r$ is even and $\rho = 2$
(b) $a_1 \mid a_3 r$ and $\rho = 1$
(c) $a_1 \mid (a_3 r + \rho)$ and $\rho = 1$.

In case (a), we have $a_1 = 2$ and $\rho = 2$, so (4) gives $(a_1, a_2, a_3) = (2, 2a_3^*, 2a_2^*)$ and $M = 2a_2^* a_3^*$. Since at most one a_i is equal to 2, it follows that $\min(a_2^*, a_3^*) > 1$. As $a_1^* = M/a_1 = a_2^* a_3^*$, we have $(a_1^*, a_2^*, a_3^*) = (a_2^* a_3^*, a_2^*, a_3^*)$ and consequently $(\alpha_1, \alpha_2, \alpha_3) = (1, a_3^*, a_2^*)$. This implies that type $\mathfrak{g} = 2$, because $\min(a_2^*, a_3^*) > 1$. So in case (a) we have $\operatorname{cotype}(a_1, a_2, a_3) = 2$, i.e., condition (ii) holds.

If (b) or (c) holds then $\rho = 1$, so (4) gives $(a_1, a_2, a_3) = (a_1, a_3^*, a_2^*)$ and $M = a_2^* a_3^*$. Because $\min(a_1, a_2, a_3) > 1$, it follows that $\min(a_2^*, a_3^*) > 1$. Also, "(b) or (c)" implies that a_1 divides one of the following numbers:

$$a_3 r = a_2^* r = a_3^* s - 1 \quad \text{or} \quad a_3 r + \rho = a_3 r + 1 = a_2^* r + 1.$$

In the first (resp. second) case, a_1 is relatively prime to a_3^* (resp. a_2^*). As a_1 divides $M = a_2^* a_3^*$ and is relatively prime to one of a_2^*, a_3^*, it follows that a_1 divides one of a_2^*, a_3^*. Consider i, j such that $\{i, j\} = \{2, 3\}$ and $a_1 \mid a_i^*$. Then $a_1^* = M/a_1 = a_2^* a_3^*/a_1 = (a_i^*/a_1) a_j^*$ and consequently

$$\alpha_1 = \gcd(a_2^*, a_3^*) = 1,$$
$$\alpha_i = \gcd(a_1^*, a_j^*) = \gcd\big((a_i^*/a_1) a_j^*, a_j^*\big) = a_j^* > 1,$$
$$\alpha_j = \gcd(a_1^*, a_i^*) = \gcd\big((a_i^*/a_1) a_j^*, a_i^*\big) = \gcd\big((a_i^*/a_1), a_i^*\big) = a_i^*/a_1.$$

If $a_1 \neq a_i^*$ then $\alpha_i \neq 1$ and $\alpha_j \neq 1$, so type $\mathfrak{g} = 2$ and hence condition (ii) holds.

If $a_1 = a_i^*$ then $(a_1, a_2, a_3) = (a_i^*, a_3^*, a_2^*)$, so condition (iii) holds.

We have shown that if (i) does not hold then (ii) or (iii) holds, so the first part of the proof is complete.

To prove the Proposition, we have to show that no $D \in \mathrm{LND}(B, \mathfrak{g})$ satisfies $D^2(f) = 0$. This follows from Lemma 3.9 or Theorem 3.11 in case (i), and from 4.3 and 4.4 in case (ii). There only remains to show that D doesn't exist in case (iii); for this, we use Theorem 2.3 and some results in the classification of homogeneous locally nilpotent derivations of $\mathbf{k}^{[3]}$.

Assume that $(a_1, a_2, a_3) = (b, a, a)$ for some $a \geq 3$ and $b \geq 2$ such that $\gcd(a, b) = 1$. Then $(a_1^*, a_2^*, a_3^*) = (a, b, b)$, so the grading $\mathfrak{g}$ is defined by $\deg(X_1) = a$ and $\deg(X_2) = b = \deg(X_3)$. Also note that $(\alpha_1, \alpha_2, \alpha_3) = (b, 1, 1)$, so type $\mathfrak{g} = 1$ and $\alpha_1 = b \neq 1$.

Let $D \in \mathrm{LND}(B, \mathfrak{g}) \setminus \{0\}$ and $A = \ker D$. Let us show that $D^2(f) \neq 0$. By Lemma 4.3, we may assume that $X_1 \notin A$.

By 2.3(1), we have $d(A) \in \{b, 1\}$. If $d(A) = 1$ then $\alpha_1 = b \notin \{1, d(A)\}$, so $X_1 \in A$ by 2.3(3), contradicting our assumption. So in fact we have $d(A) = b = \alpha_1 \neq 1$. Then 2.3(2) implies that the set of homogeneous prime elements h of B satisfying $\deg_D(h) = 1$ is equal to $\{\lambda X_1 \mid \lambda \in \mathbf{k}^*\}$. As $f \neq \lambda X_1$, $\deg_D(f) \neq 1$ and there only remains to show that $f \notin A$. For this we use the following facts, which follow respectively from results 3.4.3 and 3.4.2 of [Dai07]:

(9)   Let u, v be $\mathfrak{g}$-homogeneous elements of B such that $\mathbf{k}[X_1, u, v] = \mathbf{k}[X_1, X_2, X_3]$. Then $u, v \in \mathbf{k}[X_2, X_3]$ are linearly independent linear forms.

(10) $A = \mathbf{k}[u, u^e v + \psi(u, X_1)]$ for some pair (u, v) as in (9), some $e \in \mathbb{N}$ and some $\psi(u, X_1) \in \mathbf{k}[u, X_1]$ such that $u^e v + \psi(u, X_1)$ is $\mathfrak{g}$-homogeneous and irreducible.

Let us write $A = \mathbf{k}[u, \theta]$, where $\theta = u^e v + \psi(u, X_1)$. Note that $\deg(u) = b$ and $\deg(\theta) = (e + 1)b$, while $\deg(f) = ab$. By contradiction, assume that $f \in A$. As f is an element of A which is both homogeneous and irreducible, one of the following conditions must hold for some $\lambda, \mu \in \mathbf{k}^*$:

  (a) $f = \lambda u$
  (b) $f = \lambda \theta$
  (c) $f = \lambda u^{e+1} + \mu \theta$

Condition (a) cannot hold because $\deg(f) \neq \deg(u)$, so (b) or (c) holds, so $e = a - 1$ and $f = \lambda u^a + \mu \theta$ for some $\lambda, \mu \in \mathbf{k}$. Setting $X_1 = 0$ in this equality of polynomials yields $X_2^a + X_3^a = \lambda u^a + \mu(u^{a-1} v + \psi(u, 0))$, where $\psi(u, 0) = \nu u^a$ for some $\nu \in \mathbf{k}$. Then $u^{a-1} \mid X_2^a + X_3^a$; as there can be no multiple factor in the prime factorization of $X_2^a + X_3^a$, we obtain $a \leq 2$, which contradicts one of our assumptions. This contradiction shows that $f \notin A$, so the proof is complete. $\square$

It is easy to see that if $\mathbf{k}$ is algebraically closed, $n \geq 3$ and $(a_1, \ldots, a_n)$ does not satisfy the hypothesis of $P(n)$ then $\mathrm{lndeg}(f) = 0$. So the above result implies:

4.8. Corollary. *Let* $f = X_1^{a_1} + X_2^{a_2} + X_3^{a_3} \in B = \mathbf{k}[X_1, X_2, X_3] = \mathbf{k}^{[3]}$, *where* $\mathbf{k}$ *is an algebraically closed field of characteristic zero and* $(a_1, a_2, a_3) \in \mathbb{N}^3$. *Then the following are equivalent:*

(a) $\mathrm{lndeg}(f) = 0$
(b) $\mathrm{lndeg}(f) \leq 1$
(c) $\min(a_1, a_2, a_3) \leq 1$ *or* $|\{\, i \mid a_i = 2 \,\}| > 1$.

4.9. Corollary. *Let* $f = X_1^{a_1} + X_2^{a_2} + X_3^{a_3} \in B = \mathbf{k}[X_1, X_2, X_3] = \mathbf{k}^{[3]}$, *where* $\mathbf{k}$ *is an algebraically closed field of characteristic zero. Suppose:*

(a) $\min(a_1, a_2, a_3) \geq 2$ *and at most one of* a_1, a_2, a_3 *is equal to* 2
(b) a_1, a_2, a_3 *are distinct or* $\gcd(a_1, a_2, a_3) > 1$
(c) $\mathrm{cotype}(a_1, a_2, a_3) \leq 1$.

Then the curve $C_{(a_1,a_2,a_3)} = V_+(f) \subset \mathbb{P}(a_1^*, a_2^*, a_3^*)$ *is not rational. Moreover, if* g *is any element of* B *satisfying* $\deg_{\mathfrak{g}}(g) < \deg_{\mathfrak{g}}(f)$ *(where* $\mathfrak{g}$ *is defined in 4.2), then no* $D \in \mathrm{LND}(B) \setminus \{0\}$ *satisfies* $D^2(f + g) = 0$.

Proof. Suppose that (a_1, a_2, a_3) satisfies (a–c). Then, by the proof of Proposition 4.7, one of the following conditions holds:

(i) the curve $C_{(a_1,a_2,a_3)}$ is not rational
(ii) $\mathrm{cotype}(a_1, a_2, a_3) = 2$
(iii) up to permutation, (a_1, a_2, a_3) is (b, a, a) for some $a \geq 3$ and $b \geq 2$ such that $\gcd(a, b) = 1$.

Condition (ii) does not hold by (c), and condition (iii) does not hold by (b), so $C_{(a_1,a_2,a_3)}$ is not rational. By Theorem 3.11, it follows that no $D \in \mathrm{LND}(B) \setminus \{0\}$ satisfies $D^2(f + g) = 0$. $\qquad\square$

Finally, we note that the fact that $P(3)$ is true gives us partial information about $P(4)$, by Lemma 4.3. In particular:

4.10. Corollary. $P(4)$ *is true whenever* $\mathrm{cotype}(a_1, a_2, a_3, a_4) > 1$.

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