CLASSIFICATION OF LINEAR WEIGHTED GRAPHS UP TO BLOWING-UP AND BLOWING-DOWN

DANIEL DAIGLE

ABSTRACT. We classify linear weighted graphs up to the blowing-up and blowing-down operations which are relevant for the study of algebraic surfaces.

The word "graph" in this text means a finite undirected graph such that no edge connects a vertex to itself and at most one edge joins any given pair of vertices. A *weighted graph* is a graph in which each vertex is assigned an integer (called its weight). Two operations are performed on weighted graphs: the blowing-up and its inverse, the blowing-down. Two weighted graphs are said to be equivalent if one can be obtained from the other by means of a finite sequence of blowings-up and blowings-down (see 1.1–1.2). These weighted graphs and operations are well known to geometers who study algebraic surfaces. Many problems in the geometry of surfaces can be formulated in graph-theoretic terms and solving these sometimes requires elaborate graph-theoretic considerations. This gives rise to a variety of questions about weighted graphs, all in connection with the equivalence relation generated by blowing-up and blowing-down.

The first four sections of the present paper classify linear chains up to equivalence, where by a "linear chain" we mean a weighted graph of the form:

$$\overset{x_1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{x_2}{\bullet}\!\!-\!\!\!-\ \cdots\ -\!\!\!-\!\!\overset{x_q}{\bullet} \qquad (x_i \in \mathbb{Z}).$$

In particular 3.2 shows that each equivalence class of linear chains contains a canonical form, unique up to an operation which we call "transposition"; and 3.4 states that two linear chains are equivalent if and only if they have the same invariants. The fourth section introduces the notion of "prime class" and uses it to express the classification in a very simple form (see 4.4).

Although this paper is motivated by the theory of algebraic surfaces, we make only one brief incursion into geometry: Section 5 recalls the geometric interpretation of weighted graphs and blowing-up, and characterizes the linear chains which occur in the context of algebraic surfaces. The rest of the paper is pure graph theory, and the results of Section 5 are not used in other parts of the paper.

The last two sections are concerned with the problem of listing all minimal weighted graphs equivalent to a given linear chain: the preliminary technical results are gathered in Section 6 and the conclusions are given in Section 7. We give a general recursive solution and, in some simple cases, an explicit solution. Incidentally, the cases that

Mathematics Subject Classification. Primary: 14J26. Secondary: 14E07, 14R05, 05C99.
Key words and phrases. Weighted graph, dual graph, blowing-up, algebraic surface.
Research supported by a grant from NSERC Canada.

we are able to describe explicitely are precisely those which are relevant for algebraic surfaces.

This paper is essentially a subset of our unpublished [1], with some improvements and clarifications. Note that [1] is more general, as it classifies weighted forests; however the classification of linear chains—arguably the most important case—is given in [1] after a relatively long route, and this is one of the reasons for writing the present paper. As we wanted this paper to be self-contained we avoided replacing proofs by references to [1], the only exception being Lemma 6.6: including that proof here would have required the addition of a substantial amount of material (some version of Section 6 of [1]); that material is needed for other purposes in [1], but not here.

Remarks. Papers [6] and [7] classify weighted forests up to an equivalence relation weaker than the one considered here (the relation is generated by blowing-up, blowing-down and other operations which are not allowed here). Result 3.2.1 of [8] classifies linear chains but, again, this is relative to a weak equivalence relation. Paper [9] uses the same equivalence relation as we do, but classifies a set of weighted trees which does not contain all linear chains.

Proposition 3.2 of [8] almost[1] implies the fact (2.26) that each linear chain is equivalent to at least one canonical chain. As we realized *a posteriori,* there is even some similarity between the cited result and our method for proving 2.26.

1. Weighted graphs

See the introduction for the definition of weighted graph. If $\mathcal{G}$ is a weighted graph, $\text{Vtx}(\mathcal{G})$ is its vertex set. If $v \in \text{Vtx}(\mathcal{G})$ then $w(v, \mathcal{G})$ denotes the weight of v in $\mathcal{G}$; $\deg(v, \mathcal{G})$ denotes the degree of v in $\mathcal{G}$, that is, the number of neighbors of v.

1.1. **Definition.** We define three types of *blowing-up* of a weighted graph $\mathcal{G}$.

(1) If v is a vertex of $\mathcal{G}$ then the *blowing-up of* $\mathcal{G}$ *at* v is the weighted graph $\mathcal{G}'$ obtained from $\mathcal{G}$ by adding one vertex e of weight -1, adding one edge joining e to v, and decreasing the weight of v by 1. (This process is called a blowing-up "at a vertex".)

(2) If $\varepsilon = \{v_1, v_2\}$ is an edge of $\mathcal{G}$ (so v_1, v_2 are distinct vertices of $\mathcal{G}$), then the *blowing-up of* $\mathcal{G}$ *at* ε is the weighted graph $\mathcal{G}'$ obtained from $\mathcal{G}$ by adding one vertex e of weight -1, deleting the edge $\varepsilon = \{v_1, v_2\}$, adding the two edges $\{v_1, e\}$ and $\{e, v_2\}$, and decreasing the weights of v_1 and v_2 by 1. (This is called a blowing-up "at an edge", or a "subdivisional" blowing-up.)

(3) The *free blowing-up of* $\mathcal{G}$ is the weighted graph $\mathcal{G}'$ obtained by taking the disjoint union of $\mathcal{G}$ and of a vertex e of weight -1.

In each of the above three cases, we call e the vertex *created* by the blowing-up. If $\mathcal{G}'$ is a blowing-up of $\mathcal{G}$ then there is a natural way to identify $\text{Vtx}(\mathcal{G})$ with a subset of $\text{Vtx}(\mathcal{G}')$ (whose complement is $\{e\}$). It is understood that, whenever a blowing-up is performed, such an injective map $\text{Vtx}(\mathcal{G}) \hookrightarrow \text{Vtx}(\mathcal{G}')$ is chosen. If $\mathcal{G}'$ is a blowing-up of $\mathcal{G}$ and $\mathcal{G}''$ is a weighted graph isomorphic to $\mathcal{G}'$, then $\mathcal{G}''$ is a blowing-up of $\mathcal{G}$.

[1]One also needs 2.20 for the proof.

1.2. **Definitions.** (1) A vertex e of a weighted graph $\mathcal{G}'$ is said to be *contractible* if the following three conditions hold: (i) e has weight -1; (ii) e has at most two neighbors; (iii) if v_1 and v_2 are distinct neighbors of e then v_1, v_2 are not neighbors of each other.

If e is a contractible vertex of $\mathcal{G}'$ then $\mathcal{G}'$ is the blowing-up of some weighted graph $\mathcal{G}$ in such a way that e is the vertex created by this process. Up to isomorphism of weighted graphs, $\mathcal{G}$ is uniquely determined by $\mathcal{G}'$ and e. We say that $\mathcal{G}$ is obtained by *blowing-down* $\mathcal{G}'$ at e. The blowing-down is the inverse operation of the blowing-up.

(2) A weighted graph is *minimal* if it does not have a contractible vertex.

(3) Two weighted graphs $\mathcal{G}$ and $\mathcal{H}$ are *equivalent* (notation: $\mathcal{G} \sim \mathcal{H}$) if one can be obtained from the other by a finite sequence of blowings-up and blowings-down.

1.3. **Definition.** Given a weighted graph $\mathcal{G}$, consider the real vector space V with basis $\text{Vtx}(\mathcal{G})$ and define a symmetric bilinear form $B_\mathcal{G} : V \times V \to \mathbb{R}$ by:

$$B_\mathcal{G}(u,v) = \begin{cases} w(u,\mathcal{G}), & \text{if } u = v \in \text{Vtx}(\mathcal{G}), \\ 1, & \text{if } u,v \in \text{Vtx}(\mathcal{G}) \text{ are distinct and joined by an edge,} \\ 0, & \text{if } u,v \in \text{Vtx}(\mathcal{G}) \text{ are distinct and not joined by an edge.} \end{cases}$$

One calls $B_\mathcal{G}$ the *intersection form* of $\mathcal{G}$. Then define the natural number $\|\mathcal{G}\| = \max_W \dim W$, where W runs in the set of subspaces of V satisfying

$$\forall_{x \in W} \quad B_\mathcal{G}(x,x) \geq 0.$$

Note that $\|\mathcal{G}\| = 0$ iff $B_\mathcal{G}$ is negative definite, in which case we say that $\mathcal{G}$ is negative definite.

1.4. **Lemma.** *For weighted graphs $\mathcal{G}$ and $\mathcal{G}'$, $\mathcal{G} \sim \mathcal{G}' \implies \|\mathcal{G}\| = \|\mathcal{G}'\|$.*

Proof. See for instance 1.14 of [8]. □

1.5. **Definition.** Consider a weighted graph $\mathcal{G}$ and its intersection form $B_\mathcal{G} : V \times V \to \mathbb{R}$ (see 1.3). Let $v_1, \dots, v_n$ be the distinct vertices of $\mathcal{G}$ (enumerated in any order) and let M be the $n \times n$ matrix representing $B_\mathcal{G}$ with respect to the basis $(v_1, \dots, v_n)$ of V. That is, $M_{ii} = w(v_i, \mathcal{G})$ and, if $i \neq j$, $M_{ij} = 1$ (resp. 0) if v_i, v_j are neighbors (resp. are not neighbors) in $\mathcal{G}$. Note that $\det(-M)$ is independent of the choice of an ordering for $\text{Vtx}(\mathcal{G})$. One defines the *determinant* of the weighted graph $\mathcal{G}$ by:

$$\det(\mathcal{G}) = \det(-M).$$

Note that $\det(\mathcal{G}) \in \mathbb{Z}$. By convention, the empty graph has determinant 1.

The following is well-known, and easily verified:

1.6. **Lemma.** *For weighted graphs $\mathcal{G}$ and $\mathcal{G}'$, $\mathcal{G} \sim \mathcal{G}' \implies \det(\mathcal{G}) = \det(\mathcal{G}')$.*

Remark. Without the minus sign in $\det(-M)$, 1.6 would only be true up to sign.

2. Finite sequences of integers

This section classifies finite sequences of integers up to the equivalence relation defined in 2.4 (which of course mimics equivalence of linear chains). From this, it will be very easy to derive, in the next section, a classification of linear chains.

The material up to 2.14 is well known when stated for linear chains. The main results of the section are 2.28 and 2.29.

2.1. **Notation.** If E is a set then $E^* = \cup_{n=0}^{\infty} E^n$ denotes the set of finite sequences in E, including the empty sequence $\varnothing \in E^*$. We write A^- for the *reversal* of $A \in E^*$, i.e., if $A = (a_1, \dots, a_n)$ then $A^- = (a_n, \dots, a_1)$. We shall consider $\mathbb{Z}^*$ and $\mathcal{N}^*$, where

$$\mathcal{N} = \big\{\, x \in \mathbb{Z} \mid x < -1 \,\big\}.$$

2.2. **Definition.** A *linear chain* is a weighted tree in which every vertex has degree at most two. An *admissible chain* is a linear chain in which every weight is strictly less than -1. The empty graph is an admissible chain. Given an element $X = (x_1, \dots, x_q)$ of $\mathbb{Z}^*$, the linear chain

is denoted $[x_1, \dots, x_q]$ or $[X]$. So we distinguish between the graph $[X]$ and the sequence X and we note that $[X] = [X^-]$.

2.3. **Notation.** For each $i \in \{1, \dots, r\}$, let A_i be either an integer or an element of $\mathbb{Z}^*$. We write $(A_1, \dots, A_r)$ for the concatenation of $A_1, \dots, A_r$; that is, $(A_1, \dots, A_r) \in \mathbb{Z}^*$ is a single sequence. Also, we will use superscripts to indicate repetitions. For instance, if $A = (0^3, -5, -1) \in \mathbb{Z}^*$ and $B = (-2^3, 3, -2) \in \mathbb{Z}^*$ then

$$(A, -2, B) = (0^3, -5, -1, -2, -2^3, 3, -2) = (0, 0, 0, -5, -1, -2, -2, -2, -2, 3, -2).$$

Superscripts occurring in sequences (or linear chains) should always be interpreted in this way, never as exponents.

2.4. **Definition.** If $X = (x_1, \dots, x_n) \in \mathbb{Z}^*$ and $X \neq \varnothing$, then any of the following sequences $X' \in \mathbb{Z}^*$ is called a *blowing-up* of X:

- $X' = (-1, x_1 - 1, x_2, \dots, x_n)$;
- $X' = (x_1, \dots, x_{i-1}, x_i - 1, -1, x_{i+1} - 1, x_{i+2}, \dots, x_n)$ (where $1 \le i < n$);
- $X' = (x_1, \dots, x_{n-1}, x_n - 1, -1)$.

Moreover, we regard the one-term sequence (-1) as a blowing-up of the empty sequence $\varnothing$. If X' is a blowing-up of X, we also say that X is a blowing-down of X'. Two elements of $\mathbb{Z}^*$ are said to be *equivalent* if one can be obtained from the other by a finite sequence of blowings-up and blowings-down. This defines an equivalence relation "$\sim$" on the set $\mathbb{Z}^*$ and we write $\mathbb{Z}^*/\!\sim$ for the set of equivalence classes. We also consider the partial order relation "$\le$" on the set $\mathbb{Z}^*$ which is generated by the condition:

$$X \le X' \text{ whenever } X' \text{ is a blowing-up of } X.$$

Thus a minimal element of $\mathbb{Z}^*$ is a sequence which cannot be blown-down, i.e., an element of $(\mathbb{Z} \setminus \{-1\})^*$.

The exact relation between equivalence of sequences (2.4) and equivalence of linear chains (1.2) is given by 2.6, below. But first we need to point out:

2.5. **Lemma.** *Let $\mathcal{L}$ and $\mathcal{L}'$ be equivalent linear chains. Then there exists a sequence of blowings-up and blowings-down which transforms $\mathcal{L}$ into $\mathcal{L}'$ and which has the additional property that every graph which occurs in the sequence is itself a linear chain.*

This fact is obtained in [1] as an immediate consequence of a more general result (see 3.3 of [1]). However, 2.5 is rather trivial and we leave it without proof.

2.6. **Lemma.** *Given $X, Y \in \mathbb{Z}^*$,*

(1) $X \sim Y \iff X^- \sim Y^-$

(2) $[X] \sim [Y] \iff X \sim Y$ *or* $X \sim Y^-$.

Proof. The only nontrivial claim is implication "$\Rightarrow$" in assertion (2); for proving this implication we may, by 2.5, restrict ourselves to the case where $[Y]$ is obtained from $[X]$ by blowing-up once; then it is clear that $X \sim Y$ or $X \sim Y^-$. ☐

Refer to 1.3 and 1.5 for the following:

2.7. **Definition.** Given $X \in \mathbb{Z}^*$, we define $\det(X) = \det\big([X]\big)$ and $\|X\| = \|[X]\|$.

2.8. **Lemma.** *If $X, Y \in \mathbb{Z}^*$ satisfy $X \sim Y$, then* $\det(X) = \det(Y)$ *and* $\|X\| = \|Y\|$.

Proof. Follows from 2.6, 1.4 and 1.6. ☐

By 2.8 we may define $\det(\mathcal{C})$ and $\|\mathcal{C}\|$ for any equivalence class $\mathcal{C} \in \mathbb{Z}^*/\!\sim$ (the definitions are the obvious ones).

2.9. **Notation.** Given $X = (x_1, \dots, x_n) \in \mathbb{Z}^*$, define:

$$\det_i(X) = \begin{cases} \det\big(x_{i+1}, \dots, x_n\big), & \text{if } 0 \le i < n, \\ 1, & \text{if } i = n, \\ 0, & \text{if } i > n; \end{cases}$$

$$\det_*(X) = \begin{cases} \det(x_2, \dots, x_{n-1}), & \text{if } n > 2, \\ 1, & \text{if } n = 2, \\ 0, & \text{if } n < 2. \end{cases}$$

In particular, note that $\det_0(X) = \det(X)$. The sequence X determines the ordered pair

$$\mathrm{Sub}(X) = \big(\det_1(X), \det_1(X^-)\big)$$

which is an element of the $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z}$. This gives in particular $\mathrm{Sub}(\varnothing) = (0, 0)$ and if $a \in \mathbb{Z}$, $\mathrm{Sub}\big((a)\big) = (1, 1)$. Finally, let $d = \det(X)$ and define the pair

$$\overline{\mathrm{Sub}}(X) = \big(\pi(\det_1(X)), \pi(\det_1(X^-))\big) \in \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z},$$

where $\pi : \mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ is the canonical epimorphism and where we regard $\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ as a $\mathbb{Z}$-module.

Facts 2.10–2.14 are, in one form or another, contained in [4]. We include the proofs for the reader's convenience.

2.10. **Lemma.** *If* $X = (x_1, \dots, x_n) \in \mathbb{Z}^*$ *and* $n \geq 1$ *then:*

$$\det_i(X) = (-x_{i+1}) \det_{i+1}(X) - \det_{i+2}(X) \qquad (0 \leq i < n).$$

In particular, $\det X = (-x_1)\det_1(X) - \det_2(X)$.

Proof. Let $v_1, \dots, v_n$ be the vertices of $\mathcal{G} = [X] = [x_1, \dots, x_n]$, where the labelling is such that $w(v_i, \mathcal{G}) = x_i$ and $\{v_i, v_{i+1}\}$ is an edge for every i. Let M be as in 1.5. The Laplace expansion of $\det(-M)$ along the first row gives $\det_0(X) = (-x_1)\det_1(X) - \det_2(X)$. Applying this formula to $(x_{i+1}, \dots, x_n)$ gives the desired result. □

2.11. **Lemma.** *The assignment* $X \mapsto (\det(X), \det_1(X))$ *is a well-defined bijection:*

$$\mathcal{N}^* \longrightarrow \big\{ (r_0, r_1) \in \mathbb{N}^2 \mid 0 \leq r_1 < r_0 \text{ and } \gcd(r_0, r_1) = 1 \big\}.$$

Proof. Consider $X = (-q_1, \dots, -q_n) \in \mathcal{N}^*$, where $q_i \geq 2$ for all i. Define $r_i = \det_i(X)$, then 2.10 gives

$$r_0 = q_1 r_1 - r_2 \tag{1}$$
$$r_1 = q_2 r_2 - r_3 \tag{2}$$
$$\vdots$$
$$r_{n-2} = q_{n-1} r_{n-1} - r_n \tag{3}$$
$$r_{n-1} = q_n r_n - r_{n+1} \tag{4}$$

where $r_{n+1} = 0$ and $r_n = 1$ by definition of $\det_i(X)$. Then (4) gives $r_{n-1} > r_n$ and by descending induction we get

$$0 = r_{n+1} < r_n < r_{n-1} < \cdots < r_1 < r_0 = \det(X). \tag{5}$$

Thus $0 \leq r_1 < r_0$ and $\gcd(r_0, r_1) = 1$. Moreover, we may interpret (1)–(4) together with (5) as the "outer" Euclidean algorithm of the pair (r_0, r_1), which shows that the sequence $(q_1, \dots, q_n)$, and hence X, is completely determined by $(\det(X), \det_1(X))$. Bijectivity follows from this remark. □

2.12. **Lemma.** *If* $X \in \mathbb{Z}^*$, $d = \det(X)$ *and* $(x, y) = \mathrm{Sub}(X)$, *then* $xy \equiv 1 \pmod{d}$.

Proof. The result holds trivially when $X = \varnothing$. For $X \neq \varnothing$, we prove

$$xy = 1 + d \det_*(X). \tag{6}$$

Write $X = (x_1, \dots, x_n)$. We leave the cases $n = 1, 2$ to the reader. Assume that $n > 2$ and that (6) holds for the shorter sequence $(x_2, \dots, x_n)$; i.e., we are assuming that

$$\det_2(X) \det_*(X) = 1 + x\delta, \tag{7}$$

where $\delta = \det_*(x_2, \dots, x_n)$. We obtain $d = -x_1 x - \det_2(X)$ by 2.10, so

$$\det_2(X) = -x_1 x - d. \tag{8}$$

Applying 2.10 to $(x_1, \dots, x_{n-1})$ gives $y = -x_1 \det_*(X) - \delta$ and hence

$$\delta = -x_1 \det_*(X) - y. \tag{9}$$

Substituting (8) and (9) in (7) yields the desired conclusion (6). ☐

2.13. **Lemma.** *Suppose that $A, B \in \mathbb{Z}^*$ satisfy $A \sim B$ and let $d = \det(A) = \det(B)$. Then there exists $(x, y) \in \mathbb{Z}^2$ such that*

$$\operatorname{Sub}(A) = \operatorname{Sub}(B) + d(x, y). \tag{10}$$

Proof. Note that $\det(A) = \det(B)$ by 2.8. Since $A \sim B$, performing a certain sequence of blowings-up and blowings-down on A produces B; if the same sequence of operations is performed on $(0, A)$ then (obviously) we obtain (x, B) for some $x \in \mathbb{Z}$, which shows that $(0, A) \sim (x, B)$. By the same argument, $(A, 0) \sim (B, y)$ for some $y \in \mathbb{Z}$. By 2.10 we have $\det(0, A) = -\det_1(A)$ and $\det(x, B) = -xd - \det_1(B)$; since $(0, A) \sim (x, B)$ implies $\det(0, A) = \det(x, B)$, we obtain

$$\det_1(A) = \det_1(B) + dx.$$

Similarly, we have $(0, A^-) = (A, 0)^- \sim (B, y)^- = (y, B^-)$, so $\det(0, A^-) = \det(y, B^-)$ and consequently $\det_1(A^-) = \det_1(B^-) + dy$. So (x, y) satisfies (10). ☐

2.14. **Corollary.** *If $A, B \in \mathbb{Z}^*$ and $A \sim B$, then $\overline{\operatorname{Sub}}(A) = \overline{\operatorname{Sub}}(B)$.*

Proof. Obvious consequence of 2.13. ☐

We shall now develop a classification of sequences up to equivalence. Sequences of the form $(0^{2i}, A)$ (see 2.3 for notations) play an important role in that classification.

2.15. **Lemma.** *Let $i \in \mathbb{N}$ and $A \in \mathbb{Z}^*$.*

(1) $\det(0^{2i}, A) = (-1)^i \det(A)$
(2) $\operatorname{Sub}(0^{2i}, A) = (-1)^i \operatorname{Sub}(A)$.

Proof. We may assume that $i > 0$, then 2.10 gives

$$\det(0^{2i}, A) = 0 \det_1(0^{2i}, A) - \det_2(0^{2i}, A) = -\det(0^{2i-2}, A)$$

and assertion (1) follows by induction. We also have:

$$\begin{aligned} \det_1(0^{2i}, A) = \det(0^{2i-2}, 0, A) &\overset{(1)}{=} (-1)^{i-1} \det(0, A) \\ &\overset{2.10}{=} (-1)^{i-1}(0 \det(A) - \det_1(A)) = (-1)^i \det_1(A) \end{aligned} \tag{11}$$

so, to prove (2), there remains only to show that

$$\det_1\big((0^{2i}, A)^-\big) = (-1)^i \det_1(A^-). \tag{12}$$

If $A = \varnothing$ then (12) reads $\det(0^{2i-1}) = 0$, which is true by assertion (1). So we may assume that $A = (a_1, \dots, a_n)$ with $n \geq 1$, in which case

$$\begin{aligned} \det_1\big((0^{2i}, A)^-\big) &= \det(a_{n-1}, \dots, a_1, 0^{2i}) = \det(0^{2i}, a_1, \dots, a_{n-1}) \\ &\overset{(1)}{=} (-1)^i \det(a_1, \dots, a_{n-1}) = (-1)^i \det(a_{n-1}, \dots, a_1) = (-1)^i \det_1(A^-). \end{aligned}$$

So (12) holds and assertion (2) follows from (11) and (12). $\square$

2.16. **Lemma.** *If $i \in \mathbb{N}$ and $A \in \mathbb{Z}^*$, then $\|(0^{2i}, A)\| = i + \|A\|$.*

Proof. This is an exercise in diagonalization. It suffices to prove that $\|(0,0,A)\| = 1 + \|A\|$ for every $A \in \mathbb{Z}^*$. This is obvious if $A = \varnothing$, so assume that $A \neq \varnothing$ and write $A = (a_1, \dots, a_n)$. Consider the linear chain

$$\mathcal{L} = [0,0,A] = \overset{0}{\underset{u_1}{\bullet}} \!\!-\!\!-\!\! \overset{0}{\underset{u_2}{\bullet}} \!\!-\!\!-\!\! \overset{a_1}{\underset{v_1}{\bullet}} \!\!-\!\!-\; \cdots \;-\!\!-\!\! \overset{a_n}{\underset{v_n}{\bullet}}$$

and let V be the real vector space with basis $\mathrm{Vtx}(\mathcal{L})$. Then the matrix representing $B_{\mathcal{L}}$ with respect to the basis $(u_1, u_2, v_1 - u_1, v_2, \dots, v_n)$ of V is:

$$\left(\begin{array}{cc|ccc} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & M & \\ 0 & 0 & & & \end{array}\right) \tag{13}$$

where M is the $n \times n$ matrix given by $M_{ii} = a_i$, $M_{ij} = 1$ if $|i-j| = 1$ and $M_{ij} = 0$ if $|i-j| > 1$, that is, M is the matrix representing the intersection form of the linear chain $[A]$. Now $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ can be diagonalized to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and we conclude that a diagonal matrix congruent to (13) has $1 + \|A\|$ nonnegative entries on its main diagonal, i.e., $\|\mathcal{L}\| = 1 + \|A\|$. $\square$

2.17. **Lemma.** *Let $a, b, x \in \mathbb{Z}$ and $A, B \in \mathbb{Z}^*$. Then*

$$(A, a, 0, b, B) \sim (A, a - x, 0, b + x, B) \sim (A, 0, 0, a + b, B).$$

Proof. $(A, a, 0, b, B) \sim (A, a-1, -1, -1, b, B) \sim (A, a-1, 0, b+1, B)$, from which the result follows. $\square$

2.18. **Lemma.** *Let $n \in \mathbb{N}$ and $A, B, C \in \mathbb{Z}^*$. Then $(A, B, 0^{2n}, C) \sim (A, 0^{2n}, B, C)$.*

Proof. If $A, C \in \mathbb{Z}^*$ and $b \in \mathbb{Z}$ then by 2.17

$$(A, b, 0, 0, C) \sim (A, b - b, 0, 0 + b, C) = (A, 0, 0, b, C),$$

from which the result follows. $\square$

2.19. **Lemma.** *Let $n \in \mathbb{N}$, $x, y \in \mathbb{Z}$ and $A \in \mathbb{Z}^*$. Then $(0^{2n+1}, x, A) \sim (0^{2n+1}, y, A)$.*

Proof. We first consider the case $n = 0$: $(0, x, A) \sim (-1, -1, x-1, A) \sim (0, x-1, A)$, from which we deduce $(0, x, A) \sim (0, y, A)$. Now the general case:

$$(0^{2n+1}, x, A) \overset{2.18}{\sim} (0, x, 0^{2n}, A) \overset{(n=0)}{\sim} (0, y, 0^{2n}, A) \overset{2.18}{\sim} (0^{2n+1}, y, A).$$

$\square$

2.20. **Lemma.** *Let $n \in \mathbb{N}$ and $A, B \in \mathbb{Z}^*$. Then:*

$$A \sim B \quad \Longrightarrow \quad (0^{2n}, A) \sim (0^{2n}, B).$$

Remark. We will see in 2.30 that the converse of 2.20 is also true.

Proof of 2.20. We may assume that $n \geq 1$. If $A \sim B$ then performing a certain sequence of blowings-up and blowings-down on A produces B; if the same sequence of operations is performed on $(0^{2n}, A) = (0^{2n-1}, 0, A)$, then we obtain $(0^{2n-1}, x, B)$ for some $x \in \mathbb{Z}$, i.e., only the rightmost zero in 0^{2n} is affected. So

$$(0^{2n}, A) \sim (0^{2n-1}, x, B) \overset{2.19}{\sim} (0^{2n-1}, 0, B) = (0^{2n}, B).$$

□

2.21. **Definition.** Let $B = (b_1, \dots, b_n) \in \mathbb{Z}^*$.

(1) Given $x \in \mathbb{Z}$, define ${}_xB = (0, x, B) = (0, x, b_1, \dots, b_n) \in \mathbb{Z}^*$ and $B_x = (B, x, 0) = (b_1, \dots, b_n, x, 0) \in \mathbb{Z}^*$.
(2) Suppose that $B \neq \varnothing$. Given $i \in \{1, \dots, n\}$ and $x, y \in \mathbb{Z}$ such that $x + y = b_i$, define $B_{(i;x,y)} = (b_1, \dots, b_{i-1}, x, 0, y, b_{i+1}, \dots, b_n) \in \mathbb{Z}^*$.

2.22. **Definition.** Given a minimal element $M = (m_1, \dots, m_k)$ of $\mathbb{Z}^*$, let $M^{\oplus}$ be the set of sequences $Z \in \mathbb{Z}^*$ which can be constructed in one of the following ways.

(1) Pick $x \in \mathbb{Z}$ and let Z be the unique minimal sequence such that $Z \leq {}_xM$.
(2) Pick $x \in \mathbb{Z}$ and let Z be the unique minimal sequence such that $Z \leq M_x$.
(3) Assuming that $M \neq \varnothing$, pick $j \in \{1, \dots, k\}$ and $x, y \in \mathbb{Z}$ such that $x + y = m_j$ and let Z be the unique minimal sequence such that $Z \leq M_{(j;x,y)}$.
(4) Pick $M' = (\mu_1, \dots, \mu_\ell)$ such that $M' \geq M$ and exactly one term μ_j is equal to -1; pick $x, y \in \mathbb{Z} \setminus \{-1\}$ such that $x + y = -1$ and let $Z = M'_{(j;x,y)}$.

Note that each element of $M^{\oplus}$ is a minimal element of $\mathbb{Z}^*$.

2.23. **Lemma.** *If M is a minimal element of $\mathbb{Z}^*$ and $Z \in M^{\oplus}$ then $Z \sim (0, 0, M)$. Moreover,* $\det Z = -\det M$ *and* $\|Z\| = \|M\| + 1$.

Proof. By definition 2.22 of $M^{\oplus}$, one of the following holds:

$$Z \leq {}_xM, \quad Z \leq M_x, \quad Z \leq M_{(j;x,y)} \quad \text{or} \quad Z = M'_{(j;x,y)} \text{ where } M' \sim M.$$

Consequently, one of the following holds:

$$Z \sim {}_xM, \quad Z \sim M_x \quad \text{or} \quad Z \sim M'_{(j;x,y)} \text{ where } M' \sim M.$$

By 2.19, ${}_xM = (0, x, M) \sim (0, 0, M)$. Since $X \sim Y$ implies $X^- \sim Y^-$, we also have $M_x = \big({}_x(M^-)\big)^- \sim (0, 0, M^-)^- = (M, 0, 0) \sim (0, 0, M)$ by 2.18.

Let $M' = (b_1, \dots, b_m)$ be any nonempty sequence equivalent to M and let $j \in \{1, \dots, m\}$ and $x, y \in \mathbb{Z}$ be such that $x + y = b_j$; then

$$\begin{aligned} M'_{(j;x,y)} &= (b_1, \dots, b_{j-1}, x, 0, y, b_{j+1}, \dots, b_m) \overset{2.17}{\sim} (b_1, \dots, b_{j-1}, 0, 0, x + y, b_{j+1}, \dots, b_m) \\ &= (b_1, \dots, b_{j-1}, 0, 0, b_j, b_{j+1}, \dots, b_m) \overset{2.18}{\sim} (0, 0, M') \overset{2.20}{\sim} (0, 0, M). \end{aligned}$$

Thus $Z \sim (0, 0, M)$ whenever $Z \in M^{\oplus}$. By 2.15 and 2.16 we get $\det Z = -\det M$ and $\|Z\| = \|M\| + 1$. □

2.24. **Proposition.** *Let Z be a minimal element of $\mathbb{Z}^*$ such that $\|Z\| > 0$ and $Z \neq (0)$. Then $Z \in M^{\oplus}$ for some minimal element M of $\mathbb{Z}^*$.*

Proof. Assume that $Z = (z_1, \dots, z_n)$ is minimal, $\|Z\| > 0$ and $Z \neq (0)$. In particular, $\|Z\| > 0$ implies that $z_i \geq -1$ for some i; so by minimality of Z there exists i such that $z_i \geq 0$. If $z_i = 0$ for some i, we distinguish three cases:

- (i) If $z_1 = 0$ then, since $Z \neq (0)$, we have $Z = (0, x, M) = {}_xM$ for some $M \in \mathbb{Z}^*$ and $x \in \mathbb{Z}$; then M is minimal and $Z \in M^{\oplus}$.
- (ii) If $z_n = 0$ then, similarly, $Z = M_x$ for some $M \in \mathbb{Z}^*$ and $x \in \mathbb{Z}$; then M is minimal and $Z \in M^{\oplus}$.
- (iii) If $z_i = 0$ for some i such that $1 < i < n$, then $Z = (z_1, \dots, z_{i-1}, 0, z_{i+1}, \dots, z_n) = B_{(i-1; z_{i-1}, z_{i+1})}$ where $B = (z_1, \dots, z_{i-2}, z_{i-1} + z_{i+1}, z_{i+2}, \dots, z_n)$. If B is minimal then $Z \in M^{\oplus}$ where $M = B$. If B is not minimal then its $(i-1)$-th term $(z_{i-1} + z_{i+1})$ is the only one which is equal to -1; we have $B \geq M$ for some minimal M, then $Z \in M^{\oplus}$.

From now-on, assume that $z_j \neq 0$ for all $j \in \{1, \dots, n\}$. Then $z_i > 0$ for some i and we have four cases:

- (iv) If $Z = (p)$ where $p > 0$, then $Z \leq (0, -1, -2^{p-1}) = {}_{-1}M$ where $M = (-2^{p-1})$ is minimal; then $Z \in M^{\oplus}$.
- (v) If $z_1 > 0$ and $n > 1$ then $Z \leq (0, -1, -2^{z_1 - 1}, z_2 - 1, z_3, \dots, z_n) = {}_{-1}M$ where $M = (-2^{z_1-1}, z_2 - 1, z_3, \dots, z_n)$ is minimal; then $Z \in M^{\oplus}$.
- (vi) If $z_n > 0$ and $n > 1$ then $Z \leq (z_1, \dots, z_{n-2}, z_{n-1} - 1, -2^{z_n - 1}, -1, 0) = M_{-1}$ where $M = (z_1, \dots, z_{n-2}, z_{n-1} - 1, -2^{z_n-1})$ is minimal; then $Z \in M^{\oplus}$.
- (vii) If $z_i > 0$ and $1 < i < n$ then $Z \leq (z_1, \dots, z_{i-1}, 0, -1, -2^{z_i-1}, z_{i+1}-1, z_{i+2}, \dots, z_n) = M_{(i-1; z_{i-1}, -1)}$, where $M = (z_1, \dots, z_{i-2}, z_{i-1} - 1, -2^{z_i-1}, z_{i+1} - 1, z_{i+2}, \dots, z_n)$ is minimal; then $Z \in M^{\oplus}$.

□

2.25. **Definition.** An element C of $\mathbb{Z}^*$ is a *canonical sequence* if it has the form

$$C = (0^r, A), \text{ where } r \in \mathbb{N},\ A \in \mathbb{N}^* \text{ and if } A \neq \varnothing \text{ then } r \text{ is even.}$$

We now proceed to show that each element of $\mathbb{Z}^*$ is equivalent to a unique canonical sequence. The proof consists of 2.26 and 2.27, below.

2.26. **Lemma.** *Every element of $\mathbb{Z}^*$ is equivalent to a canonical sequence.*

Proof. It suffices to show that every minimal element Z of $\mathbb{Z}^*$ is equivalent to a canonical sequence. We proceed by induction on $\|Z\|$. If $\|Z\| = 0$ then $Z \in \mathbb{N}^*$, so Z itself is canonical. If $\|Z\| > 0$ then, by 2.24, either $Z = (0)$ or $Z \in M^{\oplus}$ for some minimal element M of $\mathbb{Z}^*$. In the first case, Z is canonical and we are done. In the second case, 2.23 gives $\|M\| < \|Z\|$ so we may assume by induction that M is equivalent to a canonical sequence C; then $Z \sim (0, 0, M) \sim (0, 0, C)$ by 2.23 and 2.20, and clearly $(0, 0, C)$ is canonical. □

2.27. **Lemma.** *Let $L \in \mathbb{Z}^*$, let $n = \|L\|$ and let d be the absolute value of* $\det(L)$.
If $(0^r, A)$ (where $r \in \mathbb{N}$ and $A \in \mathcal{N}^*$) is a canonical sequence equivalent to L, then:

(a) *If $d = 0$ then $r = 2n - 1$ and $A = \varnothing$.*
(b) *If $d \neq 0$ then $r = 2n$ and A is the unique element of $\mathcal{N}^*$ which satisfies:*

$$\det(A) = d \quad \textit{and} \quad \overline{\mathrm{Sub}}(A) = (-1)^n\,\overline{\mathrm{Sub}}(L).$$

In particular, r and A are uniquely determined by L.

Proof. The claim that r and A are uniquely determined by L is obvious in case (a), and follows from 2.11 in case (b). Consider any canonical sequence $(0^r, A)$ equivalent to L; we have $r \in \mathbb{N}$, $A \in \mathcal{N}^*$, and if $A \neq \varnothing$ then r is even. To prove (a) and (b), it suffices to show:

(a′) *If r is odd then $d = 0$ and $r = 2n - 1$.*
(b′) *If r is even then* $\det(A) = d \neq 0$, $r = 2n$ *and* $\overline{\mathrm{Sub}}(A) = (-1)^n\,\overline{\mathrm{Sub}}(L)$.

If r is odd then $A = \varnothing$; writing $r = 2i + 1$, we get $\pm d = \det(L) = \det(0^{2i+1}) = \det(0^{2i}, 0) = (-1)^i \det(0) = 0$ by 2.15 and $n = \|L\| = \|(0^{2i+1})\| = i + \|(0)\| = i + 1$ by 2.16. This proves (a′).

If r is even then 2.16 gives $n = \|L\| = \|(0^r, A)\| = \frac{r}{2} + \|A\| = \frac{r}{2}$, so $r = 2n$. Then 2.15 gives $\pm d = \det(L) = \det(0^{2n}, A) = (-1)^n \det(A)$; since $\det(A) > 0$ by 2.11, we obtain $\det(A) = d \neq 0$. Since $(0^{2n}, A) \sim L$, 2.13 implies that there exist $(u, v) \in \mathbb{Z}^2$ such that $\mathrm{Sub}(0^{2n}, A) = \mathrm{Sub}(L) + d(u, v)$. On the other hand, 2.15 gives $\mathrm{Sub}(A) = (-1)^n \mathrm{Sub}(0^{2n}, A)$, so

$$\mathrm{Sub}(A) = (-1)^n\big(\mathrm{Sub}(L) + d(u, v)\big).$$

It follows that $\overline{\mathrm{Sub}}(A) = (-1)^n\,\overline{\mathrm{Sub}}(L)$ and that (b′) is true. □

As an immediate consequence of 2.26 and 2.27, we obtain the fundamental result:

2.28. **Theorem.** *Each element of $\mathbb{Z}^*$ is equivalent to a unique canonical sequence.*

2.29. **Corollary.** *For $L, L' \in \mathbb{Z}^*$, the following are equivalent:*

(1) $L \sim L'$
(2) $\|L\| = \|L'\|$, $\det(L) = \det(L')$ *and* $\overline{\mathrm{Sub}}(L) = \overline{\mathrm{Sub}}(L')$.

Proof. (1) implies (2) by 2.8 and 2.14, and (2) implies (1) by 2.27. □

Remark. Note the following consequence of 2.27:

If $L \in \mathbb{Z}^*$ and $\det(L) = 0$ *then $L \sim (0^{2i+1})$ for some $i \in \mathbb{N}$.*

Remark. One can state some variants of 2.29, for instance:

- Suppose that $L, L' \in \mathbb{Z}^*$ satisfy $\det(L) = 0 = \det(L')$. Then
$$L \sim L' \iff \|L\| = \|L'\|.$$
- Suppose that $L, L' \in \mathbb{Z}^*$ satisfy $\det(L) = d = \det(L')$ and $\|L\| = \|L'\|$. Then
$$L \sim L' \iff \det{}_1(L) \equiv \det{}_1(L') \pmod{d}.$$

We may now prove the converse of 2.20.

2.30. **Corollary.** *Let $n \in \mathbb{N}$ and $A, B \in \mathbb{Z}^*$. Then:*

$$A \sim B \quad \iff \quad (0^{2n}, A) \sim (0^{2n}, B).$$

Proof. Implication "$\implies$" is 2.20. Conversely, if $(0^{2n}, A) \sim (0^{2n}, B)$ then:

$$\begin{aligned} \|A\| &\overset{2.16}{=} -n + \|(0^{2n}, A)\| \overset{2.29}{=} -n + \|(0^{2n}, B)\| \overset{2.16}{=} \|B\|, \\ \det A &\overset{2.15}{=} (-1)^n \det(0^{2n}, A) \overset{2.29}{=} (-1)^n \det(0^{2n}, B) \overset{2.15}{=} \det B, \\ \overline{\mathrm{Sub}}(A) &\overset{2.15}{=} (-1)^n\, \overline{\mathrm{Sub}}(0^{2n}, A) \overset{2.29}{=} (-1)^n\, \overline{\mathrm{Sub}}(0^{2n}, B) \overset{2.15}{=} \overline{\mathrm{Sub}}(B), \end{aligned}$$

so we obtain $A \sim B$ by 2.29. □

2.31. **Definition.** Let $C = (0^r, A) \in \mathbb{Z}^*$ be a canonical sequence (where $r \in \mathbb{N}$ and $A \in \mathcal{N}^*$). The *transpose* C^t of C is defined by $C^t = (0^r, A^-)$. Note that C^t is a canonical sequence.

2.32. **Lemma.** *Let $X \in \mathbb{Z}^*$. If C is the unique canonical sequence equivalent to X, then C^t is the unique canonical sequence equivalent to X^-.*

Proof. Since $X^- \sim C^-$, it suffices to show that $C^- \sim C^t$. Write $C = (0^r, A)$ with $A \in \mathcal{N}^*$. If r is odd then $A = \varnothing$ and the result holds trivially. Assume that r is even, then:

$$C^- = (0^r, A)^- = (A^-, 0^r) \overset{2.18}{\sim} (0^r, A^-) = C^t.$$

□

3. Classification of linear chains

This section reformulates 2.28 and 2.29 in terms of linear chains.

3.1. **Definition.** By a *canonical chain*, we mean a linear chain of the form $[L]$ where $L \in \mathbb{Z}^*$ is a canonical sequence. The *transpose* $\mathscr{L}^t$ of a canonical chain $\mathscr{L}$ is defined by:

$$\mathscr{L}^t = [L^t]$$

where $L \in \mathbb{Z}^*$ is a canonical sequence satisfying $\mathscr{L} = [L]$ and where L^t was defined in 2.31. Note that $\mathscr{L}^t$ is a canonical chain.

Remark. The linear chain $\mathscr{L}^t$ is well-defined even when L is not uniquely determined by $\mathscr{L}$. Indeed, if L and L' are distinct canonical sequences such that $[L] = \mathscr{L} = [L']$ then $L \in \mathcal{N}^*$ and $L' = L^- = L^t$, so $[L^t] = [L^-] = \mathscr{L}$ and $[(L')^t] = [L] = \mathscr{L}$, so $\mathscr{L}^t$ is well-defined and equal to $\mathscr{L}$.

Concretely, a linear chain is canonical if it is $[0^r]$ with r odd or if it has the form:

(14) $$\underbrace{\overset{0}{\bullet} - \cdots - \overset{0}{\bullet}}_{r \text{ vertices}} - \overset{a_1}{\bullet} - \cdots - \overset{a_n}{\bullet} \qquad (r \geq 0 \text{ is even, } n \geq 0 \text{ and } \forall_i\ a_i \leq -2).$$

The transpose of $[0^r]$ is the same graph $[0^r]$ and the transpose of (14) is:

$$\underbrace{\overset{0}{\bullet} \text{---} \cdots \text{---} \overset{0}{\bullet}}_{r \text{ vertices}} \text{---} \overset{a_n}{\bullet} \text{---} \cdots \text{---} \overset{a_1}{\bullet}\ .$$

As a corollary to the classification of sequences, we obtain:

3.2. **Theorem.** *Every linear chain is equivalent to a canonical chain. Moreover, if $\mathcal{L}$ and $\mathcal{L}'$ are canonical chains then*

$$\mathcal{L} \sim \mathcal{L}' \iff \mathcal{L}' \in \{\mathcal{L}, \mathcal{L}^t\}.$$

Proof. In view of 2.6, this is a corollary to 2.28 and 2.32. □

For the next result, we need:

3.3. **Definition.** Let $\mathcal{L}$ be a linear chain. Define a subset $\mathrm{Sub}(\mathcal{L})$ of $\mathbb{Z}$ as follows: choose $L \in \mathbb{Z}^*$ such that $\mathcal{L} = [L]$, let $(x, y) = \mathrm{Sub}(L) \in \mathbb{Z} \times \mathbb{Z}$ and set

$$\mathrm{Sub}(\mathcal{L}) = \{x, y\}.$$

We also define the subset $\overline{\mathrm{Sub}}(\mathcal{L})$ of $\mathbb{Z}/d\mathbb{Z}$, where $d = \det(\mathcal{L})$, by taking the image of $\mathrm{Sub}(\mathcal{L})$ via the canonical epimorphism $\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$.

3.4. **Corollary.** *For linear chains $\mathcal{L}$ and $\mathcal{L}'$, the following are equivalent:*

(1) $\mathcal{L} \sim \mathcal{L}'$
(2) $\|\mathcal{L}\| = \|\mathcal{L}'\|$, $\det \mathcal{L} = \det \mathcal{L}'$ *and* $\overline{\mathrm{Sub}}(\mathcal{L}) \cap \overline{\mathrm{Sub}}(\mathcal{L}') \neq \varnothing$
(3) $\|\mathcal{L}\| = \|\mathcal{L}'\|$, $\det \mathcal{L} = \det \mathcal{L}'$ *and* $\overline{\mathrm{Sub}}(\mathcal{L}) = \overline{\mathrm{Sub}}(\mathcal{L}')$.

Proof. Follows immediately from 2.29. Note that the condition $\overline{\mathrm{Sub}}(\mathcal{L}) \cap \overline{\mathrm{Sub}}(\mathcal{L}') \neq \varnothing$ is equivalent to $\overline{\mathrm{Sub}}(\mathcal{L}) = \overline{\mathrm{Sub}}(\mathcal{L}')$ by 2.12. □

4. Prime classes of sequences

Let $\mathbb{Z}^*/{\sim}$ denote the set of equivalence classes of sequences. Given $\mathcal{C} \in \mathbb{Z}^*/{\sim}$, let $\min \mathcal{C} = \{ M \in \mathcal{C} \mid M \text{ is a minimal element of } \mathbb{Z}^* \}$ denote the set of minimal elements of $\mathcal{C}$ (see 2.4 for the notion of minimal sequence).

Recall that if $M \in \mathbb{Z}^*$ is a minimal sequence, then $M^\oplus \subset \mathbb{Z}^*$ is a nonempty set of minimal sequences (see 2.22 for the definition).

4.1. **Definition.** For each $\mathcal{C} \in \mathbb{Z}^*/{\sim}$ we define an element $\mathcal{C}^\oplus$ of $\mathbb{Z}^*/{\sim}$ in two ways:

- Pick a minimal element M of $\mathcal{C}$, pick $X \in M^\oplus$ and let $\mathcal{C}^\oplus$ be the class of X.
- Pick any $X \in \mathcal{C}$ and let $\mathcal{C}^\oplus$ be the class of $(0, 0, X)$.

The two definitions are equivalent (and logically sound) by 2.23 and 2.30, and 2.30 also gives:

$$\mathcal{C} \longmapsto \mathcal{C}^\oplus \quad \text{is an injective map from } \mathbb{Z}^*/{\sim} \text{ to itself.} \tag{15}$$

Note that $\|\mathcal{C}^\oplus\| = 1 + \|\mathcal{C}\|$ by 2.16. We call $\mathcal{C}^\oplus$ the *successor* of $\mathcal{C}$. If $\mathcal{C} = \mathcal{C}_1^\oplus$ for some $\mathcal{C}_1$ then $\mathcal{C}_1$ is unique by (15); in this case we say that "$\mathcal{C}$ has a predecessor" and we call $\mathcal{C}_1$ the *predecessor* of $\mathcal{C}$.

Remark. The symbol $U^\oplus$ has two meanings, depending on the nature of U:

- If $U \in \mathbb{Z}^*$ is a minimal sequence, then $U^\oplus$ is the set of minimal sequences defined in 2.22;
- if $U \in \mathbb{Z}^*/\!\sim$ is an equivalence class of sequences, then $U^\oplus \in \mathbb{Z}^*/\!\sim$ is another equivalence class of sequences, as defined in 4.1.

4.2. **Lemma.** *For an element $\mathcal{C}$ of $\mathbb{Z}^*/\!\sim$, the following are equivalent:*

(1) $\min \mathcal{C}$ *is a singleton*
(2) $\min \mathcal{C}$ *is a finite set*
(3) *the canonical element of* $\mathcal{C}$ *is either* (0) *or an element of* $\mathcal{N}^*$
(4) $\mathcal{C}$ *does not have a predecessor.*

Proof. Note that $\neg(2) \Rightarrow \neg(1)$ is trivial; we prove $\neg(1) \Rightarrow \neg(4) \Rightarrow \neg(3) \Rightarrow \neg(2)$.

If $\|\mathcal{C}\| = 0$ then the canonical element of $\mathcal{C}$ is a sequence $X \in \mathcal{N}^*$; clearly, X is then the unique minimal element of $\mathcal{C}$, so the condition $\|\mathcal{C}\| = 0$ implies (1).

Hence, if (1) is false then $\|\mathcal{C}\| > 0$; since $\min \mathcal{C}$ has more than one element, we may pick a minimal $X \in \mathcal{C}$ such that $X \neq (0)$; then 2.24 gives $X \in M^\oplus$ for some minimal element M of $\mathbb{Z}^*$. Thus (4) is false.

If (4) is false then we may consider the canonical element C of the predecessor of $\mathcal{C}$; then $(0, 0, C)$ is the canonical element of $\mathcal{C}$, so (3) is false.

If (3) is false then the canonical element $(0^r, A)$ of $\mathcal{C}$ (where $r \in \mathbb{N}$ and $A \in \mathcal{N}^*$) satisfies $r \geq 2$. By 2.19, $(0, x, 0^{r-2}, A) \in \min \mathcal{C}$ for every $x \in \mathbb{Z} \setminus \{-1\}$, so (2) is false. $\square$

4.3. **Definition.** A *prime class* is an element $\mathcal{C}$ of $\mathbb{Z}^*/\!\sim$ which satisfies conditions (1)–(4) of Lemma 4.2.

Remark. All prime classes are known explicitely, by condition (3) of 4.2.

We now give a remarkably simple formulation of the classification of linear chains. Given $\mathcal{C} \in \mathbb{Z}^*/\!\sim$ and $n \in \mathbb{N}$, let $\mathcal{C}^{\oplus n} \in \mathbb{Z}^*/\!\sim$ be the equivalence class of $(0^{2n}, X)$, where X is an arbitrary element of $\mathcal{C}$. Thus $\mathcal{C}^{\oplus 0} = \mathcal{C}$, $\mathcal{C}^{\oplus 1} = \mathcal{C}^\oplus$, $\mathcal{C}^{\oplus 2} = (\mathcal{C}^\oplus)^\oplus$, etc. Then we note that 2.28 implies:

4.4. **Corollary.** *If $\mathcal{P} \subset \mathbb{Z}^*/\!\sim$ denotes the set of prime classes, then the map*

$$\begin{array}{rcl} \mathcal{P} \times \mathbb{N} & \longrightarrow & \mathbb{Z}^*/\!\sim \\ (\mathcal{C}, n) & \longmapsto & \mathcal{C}^{\oplus n} \end{array}$$

is bijective.

5. Geometric weighted graphs

We recall the classical notion of the "dual graph" of a divisor on an algebraic surface. Then we characterize the linear chains which can arise as dual graphs.

5.1. **Definition.** Let S be a nonsingular projective algebraic surface (over some algebraically closed field). By an *SNC-divisor* of S we mean a reduced effective divisor of

S, say $D = \sum_{i=1}^n C_i$ where $C_1, \dots, C_n$ are distinct irreducible curves on S, satisfying the following conditions:

- each C_i is a nonsingular curve;
- if $i \neq j$ then the intersection number $C_i \cdot C_j$ is 0 or 1;
- if i, j, k are distinct then $C_i \cap C_j \cap C_k = \varnothing$.

Given an SNC-divisor $D = \sum_{i=1}^n C_i$ of S, one defines a weighted graph $\mathcal{G}(D, S)$ by stipulating that:

- the vertices of $\mathcal{G}(D, S)$ are $C_1, \dots, C_n$;
- distinct vertices C_i, C_j are joined by an edge if and only if $C_i \cap C_j \neq \varnothing$;
- the weight of the vertex C_i is the self-intersection number C_i^2 of the curve C_i.

The weighted graph $\mathcal{G}(D, S)$ is called *the dual graph of D in S*.

Remark. Let D be an SNC-divisor of a nonsingular projective surface S, let $\pi : S' \to S$ be the blowing-up of S at a point $P \in S$ and let D' be the unique SNC-divisor of S' whose support is equal to $\pi^{-1}\big(\{P\} \cup \operatorname{supp}(D)\big)$. Then $\mathcal{G}(D', S')$ is a blowing-up of $\mathcal{G}(D, S)$. The exceptional curve $E = \pi^{-1}(P)$ is an irreducible component of D' and hence is a vertex of $\mathcal{G}(D', S')$; in the terminology of 1.1, E is in fact the vertex which is created by the blowing-up of $\mathcal{G}(D, S)$. Moreover, if we write $D = \sum_{i=1}^n C_i$ then:

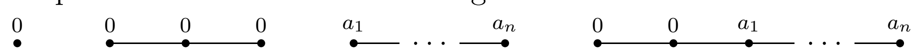
- if P belongs to exactly one irreducible component C_i of D, then $\mathcal{G}(D', S')$ is the blowing-up of $\mathcal{G}(D, S)$ at the vertex C_i;
- if P belongs to C_i and C_j where $i \neq j$ (so $C_i \cap C_j = \{P\}$), then $\mathcal{G}(D', S')$ is the blowing-up of $\mathcal{G}(D, S)$ at the edge $\{C_i, C_j\}$;
- if $P \notin \operatorname{supp}(D)$ then $\mathcal{G}(D', S')$ is the free blowing-up of $\mathcal{G}(D, S)$.

Remark. If D' is an SNC-divisor of a nonsingular projective surface S' and E is a contractible vertex of $\mathcal{G}(D', S')$ then we may blow-down the graph $\mathcal{G}(D', S')$ at the vertex E, and this graph-theoretic operation can be realized geometrically if and only if the curve E is rational. Indeed, if E is a rational curve then it can be shrunk to a smooth point; what we mean by this is that there exists a nonsingular projective surface S such that the blowing-up of S at a suitable point $P \in S$ is S' and the exceptional curve is E; if $\pi : S' \to S$ is this blowing-up morphism, then there is a unique SNC-divisor D of S satisfying $\pi\big(\operatorname{supp} D'\big) = \{P\} \cup \operatorname{supp} D$. Then $\mathcal{G}(D, S)$ is the blowing-down of $\mathcal{G}(D', S')$ at the vertex E.

5.2**. Definition.** A weighted graph $\mathcal{G}$ is said to be *geometric* if it is isomorphic to $\mathcal{G}(D, S)$ for some pair (D, S), where:

- S is a smooth projective algebraic surface over an algebraically closed field
- D is an SNC-divisor of S and every irreducible component of D is a rational curve.

Proposition 5.4, below, states (in particular) that a linear chain is geometric if and only if it is equivalent to one of the following:

(16) 

where in the last two graphs n is any nonnegative integer and $a_1, \dots, a_n$ are any integers satisfying $a_i \le -2$ for all i. This claim, and more generally the fact that the first three conditions of 5.4 are equivalent, was at least partially known prior to this work (compare for instance with 3.2.4 of [8]), but we don't know a suitable reference so we shall give a proof. The main novelty in 5.4 is the observation that conditions (1) and (4) are equivalent, which can be paraphrased as follows:

The prime classes and their immediate successors give exactly the set of geometric linear chains.

Also note that the weighted graphs pictured in (16) are canonical chains, as defined in 3.1. So, by 3.2, we immediately know when two such chains are equivalent.

The following is needed for proving 5.4:

5.3. **Lemma.** *Let $\mathcal{G}$ be a geometric weighted graph.*

(1) $\|\mathcal{G}\| \le 1$ *or* $\det(\mathcal{G}) = 0$.
(2) *If $\mathcal{G}' \sim \mathcal{G}$ then $\mathcal{G}'$ is geometric.*
(3) *Every induced subgraph of $\mathcal{G}$ is geometric.*
(4) *Let $\mathcal{G}'$ be a weighted graph with the same underlying graph as $\mathcal{G}$ and such that $w(v, \mathcal{G}') \le w(v, \mathcal{G})$ holds for every vertex v. Then $\mathcal{G}'$ is geometric.*

Note that a subgraph G' of a graph G is "induced" if every edge of G which has its two endpoints in G' is an edge of G'. Lemma 5.3 is well known (the first assertion is a consequence of the Hodge Index Theorem, see for instance [8]; (2) and (3) are trivial and (4) follows from (2) and (3)).

5.4. **Proposition.** *For a linear chain $\mathcal{L}$, the following conditions are equivalent:*

(1) *$\mathcal{L}$ is geometric*
(2) $\|\mathcal{L}\| \le 1$ *or* $\mathcal{L} \sim [0, 0, 0]$
(3) *$\mathcal{L}$ is equivalent to one of the following:*

$$[0],\ [0,0,0],\ [A] \text{ or } [0,0,A] \text{ (for some } A \in \mathcal{N}^*)$$

(4) *Let $X \in \mathbb{Z}^*$ be such that $\mathcal{L} = [X]$; then the equivalence class of X is either a prime class or the successor of a prime class.*

Proof. By 4.2, (3) is equivalent to (4); we prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Suppose that $\mathcal{L}$ is geometric and that $\det(\mathcal{L}) = 0$. By 2.27, $\mathcal{L} \sim [0^{2n+1}]$ for some $n \in \mathbb{N}$; by parts (2) and (3) of 5.3, it follows that $[0^{2n+1}]$ is geometric and then that $[0^{2n}]$ is geometric. We have $\det[0^{2n}] \neq 0$ and $\|[0^{2n}]\| = n$ by 2.15 and 2.16, so $n \le 1$ by part (1) of 5.3. Thus:

If $\mathcal{L}$ is geometric and $\det \mathcal{L} = 0$ *then $\mathcal{L}$ is equivalent to* $[0]$ *or* $[0, 0, 0]$.

The fact that (1) implies (2) follows from this and part (1) of 5.3.

Consider a canonical linear chain $[0^r, A]$ equivalent to $\mathcal{L}$ (with $r \in \mathbb{N}$, $A \in \mathcal{N}^*$ and r is even if $A \neq \varnothing$). If (2) holds then $r < 4$, so $[0^r, A]$ is one of the chains displayed in assertion (3). So (2) implies (3).

To show that (3) implies (1), we have to check that each of $[0]$, $[0, 0, 0]$, $[A]$, $[0, 0, A]$ (where $A \in \mathcal{N}^*$) is geometric; by part (3) of 5.3, it suffices to prove that $[0, 0, 0]$ and

$[0,0,A]$ are geometric, where we may assume that $A \neq \varnothing$. Considering a pair of lines in $\mathbb{P}^2$ shows that $[1,1]$ is geometric; so $[0,0,0] \sim [1,1]$ is geometric. Let $n \geq 1$ be such that $A = (a_1, \dots, a_n)$. If $n = 1$ then $[0,0,A]$ is geometric by applying part (4) of 5.3 to $[0,0,A]$ and $[0,0,0]$; if $n > 1$ then $[0,0,-1,-2^{n-2},-1] \sim [0,0,0]$ is geometric and, by part (4) of 5.3 applied to $[0,0,A]$ and $[0,0,-1,-2^{n-2},-1]$, $[0,0,A]$ is geometric. □

6. Description of certain sets of sequences

Given a minimal element M of $\mathbb{Z}^*$, a set $M^{\oplus} \subset \mathbb{Z}^*$ of minimal sequences was defined in 2.22. This notion was used in proving the classification results of Section 2 and in discussing the concepts of prime class and successor in Section 4. The aim of this section is to solve:

Problem 1. *Given a minimal element M of $\mathbb{Z}^*$, describe the set $M^{\oplus}$.*

Actually we are more interested in:

Problem 2. *List all minimal weighted graphs equivalent to a given linear chain,*

but we will see in Section 7 that solving Problem 1 is necessary in order to solve Problem 2.

Remark. In the present section and the next one we prove some mathematical results and then claim that those results solve certain problems. Such claims are useful for psychological reasons, but the extent to which the results are indeed satisfactory solutions to the problems is partly a matter of interpretation, because the problems are stated in imprecise terms: what do we mean by "describing" the set $M^{\oplus}$, or by "listing" all minimal weighted graphs equivalent to a given one?

6.1. **Definition.** Let $Z, Z' \in \mathbb{Z}^*$. We say that Z can be $(+,-)$-*contracted* to Z' (resp. $(-,+)$-*contracted,* $(+,+)$-*contracted*) if there exists a sequence of blowings-down which transforms Z into Z' and such that no blowing-down is performed at the leftmost (resp. rightmost, leftmost or rightmost) term of a sequence.

For instance, let $Z = (0,-3,-1,-2)$; then Z can be $(+,-)$-contracted to $Z' = (1)$, but Z cannot be $(-,+)$-contracted or $(+,+)$-contracted to Z'.

6.2. **Definition.** We define two subsets of $\mathbb{Z}^*$:

$$\mathcal{M} = \left\{ (x_1, \dots, x_n) \in \mathbb{Z}^* \mid n \geq 1,\ x_1 \neq -1 \text{ and } \forall_{i>1}\, x_i \leq -2 \right\}$$
$$\mathcal{M}^- = \left\{ X^- \mid X \in \mathcal{M} \right\}$$

and, given $\alpha, \beta \in \mathbb{Z}$, four subsets of $\mathbb{Z}^* \times \mathbb{Z}^*$:

$$E = \left\{ (X,Y) \in \mathcal{N}^* \times \mathcal{N}^* \mid (X,-1,Y) \sim \varnothing \right\}$$
$${}^{\alpha}E = \left\{ (X,Y) \in \mathcal{M} \times \mathcal{N}^* \mid (X,-1,Y) \text{ can be } (+,-)\text{-contracted to } (\alpha) \right\}$$
$$E^{\alpha} = \left\{ (X,Y) \in \mathcal{N}^* \times \mathcal{M}^- \mid (X,-1,Y) \text{ can be } (-,+)\text{-contracted to } (\alpha) \right\}$$
$${}^{\alpha}E^{\beta} = \left\{ (X,Y) \in \mathcal{M} \times \mathcal{M}^- \mid (X,-1,Y) \text{ can be } (+,+)\text{-contracted to } (\alpha,\beta) \right\}.$$

Note that in the above, "(X,Y)" is an ordered pair of sequences (i.e., we don't concatenate X and Y), whereas "$(X,-1,Y)$" is the sequence obtained by concatenating X, -1 and Y.

The first step in solving Problem 1 is to describe the four subsets E, ${}^{\alpha}E$, E^{α} and ${}^{\alpha}E^{\beta}$ of $\mathbb{Z}^* \times \mathbb{Z}^*$, for any choice of $\alpha, \beta \in \mathbb{Z}$. This is achieved by Lemma 6.6, below.

If $x \in \mathbb{R}$, let $\lceil x \rceil$ denote the least integer n such that $x \le n$.

6.3. **Lemma.** *$X \mapsto (\det X, \det_1 X)$ is a bijection from $\mathcal{M}$ to*

$$\big\{\, (r_0, r_1) \in \mathbb{Z}^2 \;\mid\; r_1 > 0,\ \gcd(r_0,r_1) = 1 \text{ and } \lceil \tfrac{r_0}{r_1} \rceil \neq 1 \,\big\} \tag{17}$$

and $X \mapsto (\det(X^-), \det_1(X^-))$ is a bijection from $\mathcal{M}^-$ to (17).

Proof. It is well known that $\gcd\big(\det(X), \det_1(X)\big) = 1$ holds for every $X \in \mathbb{Z}^*$. Consider an element $X = (-q, N)$ of $\mathbb{Z}^* \setminus \{\varnothing\}$, where $q \in \mathbb{Z}$ and $N \in \mathbb{Z}^*$. By 2.10,

$$\det(X) = q \det(N) - \det_1(N).$$

If $N \in \mathcal{N}^*$ then by 2.11 we have $0 \le \det_1(N) < \det(N)$, so $q = \lceil \frac{\det X}{\det N} \rceil$; thus $\mathcal{M}$ is mapped into the set (17). If (r_0, r_1) belongs to the set (17), there is a unique pair $(q, r_2) \in \mathbb{Z}^2$ such that $r_0 = qr_1 - r_2$ and $0 \le r_2 < r_1$; by 2.11, a unique $N \in \mathcal{N}^*$ satisfies $\det(N) = r_1$ and $\det_1(N) = r_2$; then $(-q, N) \in \mathcal{M}$ and this defines a map from the set (17) to $\mathcal{M}$. It is clear that the two maps are inverse of each other, so the first assertion is proved. The second assertion follows from the first. □

Elaborating the above proof yields the following fact, which gives a concrete description of the inverse of the bijection $X \mapsto (\det X, \det_1 X)$ given in Lemma 6.3.

6.4. **Lemma.** *If (r_0, r_1) belongs to the set (17) then the unique $X \in \mathcal{M}$ satisfying $(\det X, \det_1 X) = (r_0, r_1)$ is the sequence $X = (-q_1, \dots, -q_k)$ determined by the "outer" Euclidean algorithm of the pair (r_0, r_1):*

$$\begin{aligned} r_0 &= q_1 r_1 - r_2 \\ &\ \vdots \\ r_{k-2} &= q_{k-1} r_{k-1} - r_k \\ r_{k-1} &= q_k r_k - 0 \end{aligned}$$

where $q_i, r_i \in \mathbb{Z}$ and $r_1 > \cdots > r_k = 1$.

6.5. **Definition.** Given $\alpha, \beta \in \mathbb{Z}$, define the following subsets of $\mathbb{N}^3$:

$$\begin{aligned} P &= \big\{\, (n,p,c) \in \mathbb{N}^3 \;\mid\; 1 \le p \le c \text{ and } \gcd(c,p) = 1 \,\big\} \\ {}^{\alpha}P = P^{\alpha} &= \big\{\, (n,p,c) \in \mathbb{N}^3 \;\mid\; 1 \le p \le c,\ \gcd(p,c) = 1 \text{ and } \lceil \tfrac{c}{nc+p} \rceil \neq \alpha + 1 \,\big\} \\ {}^{\alpha}P^{\beta} &= \big\{\, (n,p,c) \in \mathbb{N}^3 \;\mid\; 1 \le p \le c,\ \gcd(p,c) = 1,\ \lceil \tfrac{c}{nc+p} \rceil \neq \alpha + 1 \text{ and } n \neq \beta \,\big\} \end{aligned}$$

and define four maps

(1) $f : P \to E$, $(n,p,c) \mapsto (X,Y)$
(2) ${}^{\alpha}f : {}^{\alpha}P \to {}^{\alpha}E$, $(n,p,c) \mapsto (X,Y)$

(3) $f^\alpha : P^\alpha \to E^\alpha$, $(n,p,c) \mapsto (X,Y)$
(4) ${}^\alpha f^\beta \ : \ {}^\alpha P^\beta \to {}^\alpha E^\beta$, $(n,p,c) \mapsto (X,Y)$

by declaring in each case that (X,Y) is the unique pair of sequences satisfying:

(1′) $(X,Y) \in \mathcal{N}^* \times \mathcal{N}^*$ and:

$$\det(X) = nc + p \qquad \det(Y^-) = c$$
$$\det_1(X) \equiv -c \pmod{nc+p} \qquad \det_1(Y^-) = c - p$$

(2′) $(X,Y) \in \mathcal{M} \times \mathcal{N}^*$ and:

$$\det(X) = c - \alpha(nc+p) \qquad \det(Y^-) = c$$
$$\det_1(X) = nc + p \qquad \det_1(Y^-) = c - p$$

(3′) $(X,Y) \in \mathcal{N}^* \times \mathcal{M}^-$ and:

$$\det(X) = c \qquad \det(Y^-) = c - \alpha(nc+p)$$
$$\det_1(X) = c - p \qquad \det_1(Y^-) = nc + p$$

(4′) $(X,Y) \in \mathcal{M} \times \mathcal{M}^-$ and:

$$\det(X) = c - \alpha(nc+p) \qquad \det(Y^-) = (n-\beta)c + p$$
$$\det_1(X) = nc + p \qquad \det_1(Y^-) = c.$$

6.6. **Lemma.** *The maps f, ${}^\alpha f$, f^α and ${}^\alpha f^\beta$ are well-defined and bijective.*

For the proof of 6.6, refer to Lemma 7.4.1 of [1].

6.7. **Example.** To describe ${}^{-2}E^{-3}$, we first note that

$$\begin{aligned} {}^{-2}P^{-3} &= \big\{ (n,p,c) \in \mathbb{N}^3 \mid 1 \le p \le c,\ \gcd(p,c) = 1,\ \lceil \tfrac{c}{nc+p} \rceil \neq -1 \text{ and } n \neq -3 \big\} \\ &= \big\{ (n,p,c) \in \mathbb{N}^3 \mid 1 \le p \le c,\ \gcd(p,c) = 1 \big\}. \end{aligned}$$

Then the desired description of ${}^{-2}E^{-3}$ is given by the bijection ${}^{-2}f^{-3} \ : \ {}^{-2}P^{-3} \to {}^{-2}E^{-3}$, where by definition ${}^{-2}f^{-3}(n,p,c)$ is the unique element (X,Y) of $\mathcal{M} \times \mathcal{M}^-$ satisfying:

$$\det(X) = c + 2(nc+p) \qquad \det(Y^-) = (n+3)c + p$$
$$\det_1(X) = nc + p \qquad \det_1(Y^-) = c.$$

For any given element (n,p,c) of ${}^{-2}P^{-3}$, the actual sequences X and Y can be obtained (if desired) from these equalities via the outer Euclidean algorithm, see Lemma 6.4.

Remark. The description of E, ${}^\alpha E$, E^α and ${}^\alpha E^\beta$ given by 6.6 is explicit to some extent, but not fully explicit. However we think that knowing the determinants of sequences is often more useful than knowing the sequences themselves, so the present form of 6.6 is probably more useful than would be a fully explicit description. Similar comments apply to 6.8 and 6.9, below.

The next two results solve Problem 1 (6.8 solves the case where M is the empty sequence and 6.9 solves all other cases). Note that the solution is expressed in terms of the sets E, ${}^\alpha E$, E^α and ${}^\alpha E^\beta$, which are described by 6.6.

6.8. **Proposition.** *The elements of* $\varnothing^{\oplus}$ *are:*

(i) (1)
(ii) $(0,x)$ *where* $x \in \mathbb{Z} \setminus \{-1\}$
(iii) $(x,0)$ *where* $x \in \mathbb{Z} \setminus \{-1\}$
(iv) $(X,x,0,-1-x,Y)$, *where* $x \in \mathbb{Z} \setminus \{-1,0\}$ *and* $(X,Y) \in E$.

6.9. **Proposition.** *If* $M=(m_1,\dots,m_k) \neq \varnothing$ *is a minimal element of* $\mathbb{Z}^*$, *the elements of* $M^{\oplus}$ *are:*

(1) (a) $(0,x,m_1,\dots,m_k)$, *for all* $x \in \mathbb{Z} \setminus \{-1\}$
  (b) *the unique minimal sequence obtained by blowing-down* $(0,-1,m_1,\dots,m_k)$
(2) (a) $(m_1,\dots,m_k,x,0)$, *for all* $x \in \mathbb{Z} \setminus \{-1\}$
  (b) *the unique minimal sequence obtained by blowing-down* $(m_1,\dots,m_k,-1,0)$
(3) *For each* $j \in \{1,\dots,k\}$,
  (a) $(m_1,\dots,m_{j-1},x,0,m_j-x,m_{j+1},\dots,m_k)$, *for all* $x \in \mathbb{Z} \setminus \{-1,m_j+1\}$
  (b) *the unique minimal sequence obtained by blowing-down*
$$(m_1,\dots,m_{j-1},-1,0,m_j+1,m_{j+1},\dots,m_k)$$
  (c) *the unique minimal sequence obtained by blowing-down*
$$(m_1,\dots,m_{j-1},m_j+1,0,-1,m_{j+1},\dots,m_k)$$
(4) (a) $(X,x,0,-1-x,Y,m_2,\dots,m_k)$, *for all* $x \in \mathbb{Z}\setminus\{-1,0\}$ *and all* $(X,Y) \in E^{m_1}$
  (b) $(m_1,\dots,m_{i-1},X,x,0,-1-x,Y,m_{i+2},\dots,m_k)$, *for all* $x \in \mathbb{Z} \setminus \{-1,0\}$, *all* $(X,Y) \in {}^{m_i}E^{m_{i+1}}$ *and all* i *such that* $1 \le i < k$
  (c) $(m_1,\dots,m_{k-1},X,x,0,-1-x,Y)$, *for all* $x \in \mathbb{Z} \setminus \{-1,0\}$ *and all* $(X,Y) \in {}^{m_k}E$.

Proof of 6.8 *and* 6.9. Both results follow immediately from definitions 2.22 (of $M^{\oplus}$) and 6.2 (of E, ${}^{\alpha}E$, E^{α} and ${}^{\alpha}E^{\beta}$). ☐

6.10. **Example.** Let $M=(-2,-3)$. By 6.9, the elements of $M^{\oplus}$ are:

(1a)	$(0,x,-2,-3)$, for all $x \in \mathbb{Z} \setminus \{-1\}$
(1b)	$(2,-2)$
(2a)	$(-2,-3,x,0)$, for all $x \in \mathbb{Z} \setminus \{-1\}$
(2b)	$(-2,-2,1)$
($3_{j=1}$ a)	$(x,0,-2-x,-3)$, for all $x \in \mathbb{Z} \setminus \{-1\}$
($3_{j=1}$ b,c)	$(2,-2)$
($3_{j=2}$ a)	$(-2,x,0,-3-x)$, for all $x \in \mathbb{Z} \setminus \{-1,-2\}$
($3_{j=2}$ b)	$(2,-2)$
($3_{j=2}$ c)	$(-2,-2,1)$
(4a)	$(X,x,0,-1-x,Y,-3)$ for all $x \in \mathbb{Z} \setminus \{-1,0\}$ and $(X,Y) \in E^{-2}$
(4b)	$(X,x,0,-1-x,Y)$ for all $x \in \mathbb{Z} \setminus \{-1,0\}$ and $(X,Y) \in {}^{-2}E^{-3}$
(4c)	$(-2,X,x,0,-1-x,Y)$ for all $x \in \mathbb{Z} \setminus \{-1,0\}$ and $(X,Y) \in {}^{-3}E$.

7. Minimal sequences and minimal linear chains

We are interested in the second problem stated in Section 6, which we repeat for the reader's convenience:

Problem 2. *List all minimal weighted graphs equivalent to a given linear chain.*

It is well-known that any minimal weighted graph equivalent to a linear chain is itself a linear chain. Consequently, if $X \in \mathbb{Z}^*$, and if $\mathcal{X} \subset \mathbb{Z}^*$ is the set of minimal sequences equivalent to X, then $\big\{ [Y] \mid Y \in \mathcal{X} \big\}$ is the set of minimal weighted graphs equivalent to $[X]$. So the above problem reduces to:

Problem 3. *Given $X \in \mathbb{Z}^*$, list all minimal sequences equivalent to X.*

Apparently, very little is known about these problems. One notable exception is [5], which can be interpreted as solving Problem 3 for $X = (1)$. This section is a modest contribution to solving Problem 3; in particular, result 7.1 gives a recursive solution.

The notations $\mathbb{Z}^*/{\sim}$ and $\min(\mathcal{C})$ are defined before 4.1.

7.1. **Proposition.** *If $\mathcal{C} \in \mathbb{Z}^*/{\sim}$ then* $\min\big(\mathcal{C}^{\oplus}\big) = \displaystyle\bigcup_{M \in \min \mathcal{C}} M^{\oplus}$.

Proof. The inclusion "$\supseteq$" is trivial by definition 4.1 of $\mathcal{C}^{\oplus}$. Consider $Z \in \min(\mathcal{C}^{\oplus})$. Since $\mathcal{C}^{\oplus}$ has a predecessor we have $(0) \notin \mathcal{C}^{\oplus}$ by 4.2 and hence $Z \neq (0)$; we also have $\|Z\| = \|\mathcal{C}^{\oplus}\| = 1 + \|\mathcal{C}\| > 0$, so 2.24 gives $Z \in M^{\oplus}$ for some minimal element M of $\mathbb{Z}^*$. We have $M \in \mathcal{C}$ by uniqueness of the predecessor of $\mathcal{C}^{\oplus}$, so $Z \in \cup_{M \in \min \mathcal{C}} M^{\oplus}$. $\square$

Together with 6.8 and 6.9, and keeping in mind 4.4, this gives substantial information about Problem 3. Note in particular that 7.1 immediately implies:

7.2. **Corollary.** *Suppose that $\mathcal{C} \in \mathbb{Z}^*/{\sim}$ is the successor of a prime class $\mathcal{C}_*$, and let M be the unique minimal element of $\mathcal{C}_*$ (see 4.2). Then* $\min \mathcal{C} = M^{\oplus}$.

7.3. **Example.** Let $\mathcal{C}$ denote the equivalence class of the sequence (1). Then $\mathcal{C} = \mathcal{C}_{\varnothing}^{\oplus}$, where $\mathcal{C}_{\varnothing}$ is the equivalence class of the empty sequence $\varnothing$. Since $\mathcal{C}_{\varnothing}$ is a prime class and its unique minimal element is $\varnothing$, we have $\min \mathcal{C} = \varnothing^{\oplus}$ by 7.2 so, by 6.8, the minimal elements of $\mathcal{C}$ are:

- (1)
- $(0, x)$ where $x \in \mathbb{Z} \setminus \{-1\}$
- $(x, 0)$ where $x \in \mathbb{Z} \setminus \{-1\}$
- $(X, x, 0, -1-x, Y)$, where $x \in \mathbb{Z} \setminus \{-1, 0\}$ and $(X, Y) \in E$.

See 6.6 for a description of E.

Remark. The result contained in 7.3 first appeared in [5] (with a different formulation) and was later reproved by several authors.

7.4. **Example.** Let $\mathcal{C} \in \mathbb{Z}^*/{\sim}$ be the equivalence class of $(0,0,0)$. Then $\mathcal{C} = \mathcal{C}_0^{\oplus}$, where $\mathcal{C}_0$ is the equivalence class of the sequence (0). By 4.2, $\mathcal{C}_0$ is a prime class and its unique minimal element is (0); so 7.2 gives $\min \mathcal{C} = (0)^{\oplus}$ and, by 6.9, the complete list of minimal elements of $\mathcal{C}$ is:

- $(1, 1)$
- $(0, x, 0)$ where $x \in \mathbb{Z} \setminus \{-1\}$
- $(x, 0, -x)$ where $x \in \mathbb{Z} \setminus \{1, -1\}$
- $(X, x, 0, -1 - x, Y)$, where $x \in \mathbb{Z} \setminus \{-1, 0\}$ and $(X, Y) \in E^0 \cup {}^0E$.

See 6.6 for a description of E^0 and ${}^0E$.

By Section 6, we know how to describe $M^\oplus$ for any given $M \in \min \mathbb{Z}^*$. So by 7.2:

We can list the minimal elements of any class $\mathcal{C} \in \mathbb{Z}^*/\sim$ which is either a prime class or the successor of a prime class.

In other words, we can solve problems 2 and 3 exactly in the cases which are relevant for the study of algebraic surfaces (see Section 5). For the other cases, one would have to describe $M^\oplus$ for infinitely many M, and we don't know how to do that. To illustrate this point, consider the following:

7.5. **Example.** Let $\mathcal{C} \in \mathbb{Z}^*/\sim$ be the equivalence class of $(0, 0, 1)$. Then $\mathcal{C} = \mathcal{C}_1^\oplus$, where $\mathcal{C}_1$ is the equivalence class of (1). In view of the description of $\min(\mathcal{C}_1)$ given in 7.3, 7.1 tells us that $\min(\mathcal{C})$ is the following union:

$$(1)^\oplus \quad \cup \bigcup_{x \in \mathbb{Z}\setminus\{-1\}} (0, x)^\oplus \quad \cup \bigcup_{x \in \mathbb{Z}\setminus\{-1\}} (x, 0)^\oplus \quad \cup \bigcup_{\substack{x \in \mathbb{Z} \setminus \{-1, 0\} \\ (X, Y) \in E}} (X, x, 0, -1 - x, Y)^\oplus.$$

Here we don't know how to describe the last union, even though we can describe $(X, x, 0, -1 - x, Y)^\oplus$ for any given choice of $x \in \mathbb{Z} \setminus \{-1, 0\}$ and $(X, Y) \in E$.

References

[1] D. Daigle, *Classification of weighted graphs up to blowing-up and blowing-down*, electronic publication (arXiv:math.AG/0305029), 2003.

[2] D. Daigle and P. Russell, *Affine rulings of normal rational surfaces*, Osaka J. Math. **38** (2001), 37–100.

[3] D. Daigle and P. Russell, *On log $\mathbb{Q}$-homology planes and weighted projective planes*, Canad. J. Math. **56** (2004), 1145–1189.

[4] F. Hirzebruch, *Über vierdimensionale Riemannsche Flächen mehrdeutiger Funktionen von zwei komplexen Veränderlichen*, Math. Ann. **126** (1953), 1–22.

[5] J. Morrow, *Minimal normal compactifications of $\mathbb{C}^2$*, Rice U. studies **59** (1973), 97–111.

[6] W. Neumann, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc. **268** (1981), 299–344.

[7] W. Neumann, *On bilinear forms represented by trees*, Bull. Austral. Math. Soc. **40** (1989), 303–321.

[8] K.P. Russell, *Some formal aspects of the theorems of Mumford-Ramanujam*, Algebra, Arithmetic and Geometry, Mumbai 2000, Tata Institute of Fundamental Research, Narosa Publishing, 2002, 557–584.

[9] A. R. Shastri, *Divisors with Finite Local Fundamental Group on a Surface*, Algebraic Geometry, Bowdoin 1985, Proceedings of Symposia in Pure Mathematics, vol. 46, American Mathematical Society, 1987, 467–481.

Department of Mathematics and Statistics, University of Ottawa, Ottawa, Canada K1N 6N5

E-mail address: `ddaigle@uottawa.ca`