LOCALLY NILPOTENT DERIVATIONS OVER A UFD AND AN APPLICATION TO RANK TWO LOCALLY NILPOTENT DERIVATIONS OF $k[X_1,\dots,X_n]$

DANIEL DAIGLE AND GENE FREUDENBURG

ABSTRACT. Given a UFD R containing the rational numbers, we study locally nilpotent R-derivations of the polynomial ring $R[X,Y]$; in particular, we give a generalization of Rentschler's Theorem and a criterion for the existence of a slice. These results are then applied to describe rank two locally nilpotent derivations of $\mathbf{k}[X_1,\dots,X_n]$, where $\mathbf{k}$ is a field of characteristic zero. We also give an example of a non-triangulable locally nilpotent derivation of $\mathbf{k}[X,Y,Z]$ whose set of fixed points is a line.

Let $\mathbf{k}$ be a field of characteristic zero.

It is well-known [12] that studying algebraic G_a-actions on the affine space $\mathbf{A}^n$ (over the field $\mathbf{k}$) is equivalent to studying locally nilpotent derivations $D : B \to B$, where B is the polynomial ring in n variables over $\mathbf{k}$ (abbreviated $B = \mathbf{k}^{[n]}$). Hence, much effort has gone into attempts to understand those derivations. One way to approach this problem is to classify derivations according to their rank:

Definition (See [7]**).** Let D be a $\mathbf{k}$-derivation of $B = \mathbf{k}^{[n]}$. The *rank* of D is the least integer $r \geq 0$ for which there exists a coordinate system $(X_1,\dots,X_n)$ of B satisfying $\mathbf{k}[X_{r+1},\dots,X_n] \subseteq \ker D$.

Recall that Rentschler [10] showed that every locally nilpotent derivation of $\mathbf{k}^{[2]}$ is of rank at most one. Until recently, it was not known whether, for $n \geq 3$, locally nilpotent derivations of $\mathbf{k}^{[n]}$ having maximal rank n could exist; this question was answered affirmatively in [6]. Note that $\operatorname{rank} D = n$ means that no variable of $B = \mathbf{k}^{[n]}$ is in $\ker D$.

Derivations of low rank are easier to understand: $\operatorname{rank} D = 0$ means $D = 0$, and it was shown in [7] that if $\operatorname{rank} D = 1$ then D has the form $f(X_2,\dots,X_n)\cdot \partial/\partial X_1$ for some coordinate system $(X_1,\dots,X_n)$ of B; in other words, $\operatorname{rank} D = 1$ is equivalent to the two conditions:

$$\ker D = \mathbf{k}^{[n-1]} \quad \text{and} \quad B = (\ker D)^{[1]}.$$

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As to derivations of rank two, it seems that only examples and special classes have been understood. For instance, [5] studies the class of rank two derivations of the form

$$D = p(X_2, \dots, X_n)\frac{\partial}{\partial X_1} + q(X_3, \dots, X_n)\frac{\partial}{\partial X_2}.$$

The third section of this paper is devoted to locally nilpotent derivations of $\mathbf{k}^{[n]}$ of rank at most two. In particular, 3.2 gives an explicit description of all such derivations. Note that 3.2 and 3.3 generalize the results of [5] and have the following consequence:

Corollary. *Let $D \neq 0$ be a locally nilpotent derivation of $B = \mathbf{k}^{[n]}$ of rank at most two. Then*

(1) $\ker D = \mathbf{k}^{[n-1]}$.
(2) *If D is fixed point free then $D(s) = 1$ for some $s \in B$. Consequently, D has rank one.*

What we mean, here, by a fixed point of D, is a fixed point of the corresponding G_a-action on $\mathbf{A}^n$. Hence, there do not exist *fixed point free* G_a-actions on $\mathbf{A}^n$ having rank 2. However, free actions of higher rank do exist: for example, J. Winkelman constructed a triangular G_a-action on $\mathbf{A}^4$ which is fixed point free and of rank 3 (c.f. [13]). It remains an open question whether any rank 3 algebraic G_a-action on $\mathbf{A}^3$ can be fixed point free.

We also point out our example 4.3 of a (rank two) non-triangulable locally nilpotent derivation of $\mathbf{k}[X, Y, Z]$ whose set of fixed points is a line. This gives a negative answer to the question whether every rank two locally nilpotent derivation of $\mathbf{k}[X, Y, Z]$ is of the form $f \cdot T$, where T is a triangulable derivation and $f \in \mathbf{k}[X, Y, Z]$ (note that rank three locally nilpotent derivations of $\mathbf{k}^{[3]}$, which are now known to exist, cannot be of the form $f \cdot T$). In fact, the main purpose of section four is to establish 4.3.

All results of section three are immediate consequences of the results of the preceding section. In section two, we investigate locally nilpotent R-derivations of the polynomial ring $R[X, Y]$, where R is a UFD containing the rational numbers. In particular, we give a generalization of Rentschler's Theorem, a criterion for the existence of a slice and a criterion for triangulability over R.

1. Preliminaries

Throughout this paper, all rings are commutative and have an identity element. If A is a ring then A^* is the group of units of A; if A is an integral domain then $\operatorname{qt} A$ denotes the field of fractions of A.

If P is a polynomial in $X_1, \dots, X_n$ then P_{X_i} denotes $\partial P / \partial X_i$. If B is a ring and $f, g \in B$, then $(f, g)B$ denotes the ideal of B generated by f and g; if B is a UFD then $\gcd_B(f, g)$ denotes the greatest common divisor (in B) of f and g.

If A is a subring of B and n is a positive integer, then the notation $B = A^{[n]}$ means that B is A-isomorphic to the polynomial ring in n variables over A. Suppose that $B = A^{[n]}$. A *coordinate system of B over A* is an ordered n-tuple $(X_1, \dots, X_n)$ of elements of B satisfying $B = A[X_1, \dots, X_n]$; a *variable of B over A* is an element X of B such that $B = A[X, X_2, \dots, X_n]$ for some $X_2, \dots, X_n \in B$. If A is a field then we may simply speak of a coordinate system of B, or of a variable of B, with no mention of A; indeed, A is then uniquely determined: $A = \{0\} \cup B^*$.

A subring A of a domain B is said to be *factorially closed in B* if for all $x, y \in B$ we have $xy \in A \setminus \{0\} \Rightarrow x, y \in A$.

Let B be a domain of characteristic zero and let D be a derivation of B (i.e., a derivation from B to B). D is *locally nilpotent* if for each $b \in B$ there exists an integer $n > 0$ such that $D^n(b) = 0$. D is *irreducible* if the only principal ideal of B containing $D(B)$ is B (or equivalently, if D cannot be written as $D = \alpha D'$ with α a nonunit element of B and D' a derivation of B).

1.1. Let B be an integral domain of characteristic zero, let $D : B \to B$ be a nonzero derivation of B, and let $A = \ker D$. The following facts are mostly well-known.

(1) If D is locally nilpotent then A is a factorially closed subring of B. In particular, if D is locally nilpotent and B is a UFD then A is a UFD.

(2) Let S be a multiplicatively closed subset of $B \setminus \{0\}$, and consider the derivation $S^{-1}D$ of $S^{-1}B$. Then
   (a) $S^{-1}D$ is locally nilpotent if and only if D is locally nilpotent and $S \subseteq A$.
   (b) If $S \subseteq A$ then $\ker S^{-1}D = S^{-1}A$ and $S^{-1}A \cap B = A$.

(3) Assume that $\mathbf{Q} \subseteq B$. If D is locally nilpotent and $D(b) = 1$ for some $b \in B$, then $B = A[b] = A^{[1]}$.

(4) Assume that $\mathbf{Q} \subseteq B$. If D is locally nilpotent, choose any $b \in B$ such that $Db \neq 0$ and $D^2 b = 0$, and let $S = \{1, Db, (Db)^2, \dots\} \subset A$. Then $S^{-1}D(b/Db) = 1$ so, by (3), $S^{-1}B = (S^{-1}A)[b] = (S^{-1}A)^{[1]}$.

(5) If D is locally nilpotent, let $S = A \setminus \{0\}$, then (4) implies $S^{-1}B = (\operatorname{qt} A)^{[1]}$ and (2b) implies $\operatorname{qt} A \cap B = A$.

(6) Let $b \in B \setminus \{0\}$. The derivation bD is locally nilpotent if and only if D is locally nilpotent and $b \in A$.

(7) Let $f \in B[T] = B^{[1]}$ and let D' be the unique derivation of $B[T]$ which extends D and satisfies $D'T = f$. Then D' is locally nilpotent if and only if D is locally nilpotent and $f \in B$.

(8) Suppose that B is a UFD. Then
   (a) $D = \alpha\, D_0$ where $\alpha \in B$ and D_0 is an irreducible derivation of B. Moreover, α and D_0 are unique, up to multiplication by units of B.
   (b) If D is locally nilpotent then so is D_0, and $\alpha \in \ker D$.

Here are some references for the above facts. For (1), see 1.2 of [3]; for (2), see the proposition in [8]; (3) is Proposition 2.1 of [14]; (4) is mentioned in the introduction to section 2 of [4]; (5) follows from (4); (6) is well-known; (7) can be found in [7]; part (a) of (8) is an easy exercise, and (b) follows from (6).

1.2 (Rentschler's Theorem). *If L is a field of characteristic zero and D is a nonzero, locally nilpotent derivation of $L[X,Y] = L^{[2]}$ then there exist P, Q such that $L[X,Y] = L[P,Q]$ and $\ker D = L[P]$. Moreover, there exists $\alpha \in L[P]$ such that*

$$Dh = \alpha \begin{vmatrix} P_X & P_Y \\ h_X & h_Y \end{vmatrix} \qquad \text{for all } h \in L[X,Y].$$

Remark. Although we will refer to the above statement as "Rentschler's Theorem", the actual theorem of Rentschler [10] is, in a sense, stronger than this one. Indeed, Rentschler proved that the automorphism $X \mapsto P$, $Y \mapsto Q$ was tame, and obtained as a corollary that every L-automorphism of $L[X,Y]$ is tame. Note, also, that we chose to describe D in terms of a jacobian determinant (as opposed to the usual $D = \alpha\, \partial/\partial Q$), because this form seems to be more convenient for the purpose of generalization. A quick proof of 1.2 is given in the next section.

2. Locally nilpotent derivations over a UFD

Throughout this section, R denotes a UFD which contains $\mathbf{Q}$, $B = R[X,Y] = R^{[2]}$ and $K = \operatorname{qt} R$.

Our first aim is to describe all locally nilpotent R-derivations of B, and this is accomplished by 2.4; note, also, that 2.4 may be regarded as a generalization of Rentschler's theorem. Then we address the questions of existence of a slice, and of triangulability; results 2.5 and 2.8 answer these questions.

Proposition 2.1. *If $D \neq 0$ is a locally nilpotent R-derivation of $B = R[X,Y]$ then* $\ker D = R^{[1]}$.

Remark. The above statement, as well as its proof below, remains true if we replace the assumption $\mathbf{Q} \subseteq R$ by the weaker assumption that R is a UFD of characteristic zero.

Proof. Write $A = \ker D$. Since R is a UFD so is B and, by part (1) of 1.1, so is A. Hence $R \subset A \subset B = R^{[2]}$ are UFD's and A has transcendence degree one over R by part (5) of 1.1. Using 3.4 of [11], we conclude that $A = R^{[1]}$. □

We now give a quick proof of Rentschler's Theorem.

Proof of 1.2. Let $A = \ker D$. By 2.1, $A = L[P] = L^{[1]}$ for some P. Note that

(1) If $S = A \setminus \{0\}$ then $S^{-1}L[X,Y] = L(P)^{[1]}$, by part (5) of 1.1.
(2) $L(P) \cap L[X,Y] = L[P]$, again by part (5) of 1.1.
(3) $L[X,Y]$ is geometrically factorial over L, since it is a polynomial algebra.

By 2.4.2 of [11] we obtain $L[X,Y] = L[P]^{[1]}$, i.e., $L[X,Y] = L[P,Q]$ for some Q.

Regarding D as an extension of the zero derivation of $L[P]$, part (7) of 1.1 implies that $DQ \in L[P]$. Let $\alpha = DQ/\delta$, where $\delta \in L^*$ is the jacobian determinant of (P,Q) with respect to (X,Y). Then a straightforward calculation gives the desired expression for Dh. $\square$

Definition 2.2. Given $P \in B$, define an R-derivation $\Delta_P : B \to B$ by

$$\Delta_P = -P_Y \frac{\partial}{\partial X} + P_X \frac{\partial}{\partial Y}, \quad \text{or equivalently} \quad \Delta_P(h) = \begin{vmatrix} P_X & P_Y \\ h_X & h_Y \end{vmatrix} \quad \text{for all } h \in B.$$

Remark. If $P_1, P_2 \in B$ then $\Delta_{P_1} = \Delta_{P_2} \Leftrightarrow P_1 - P_2 \in R$.

Proposition 2.3. *If $P \in B$ is a variable of $K[X,Y]$ such that $\gcd_B(P_X, P_Y) = 1$, then the R-derivation $\Delta_P : B \to B$ satisfies:*

(1) *Δ_P is locally nilpotent and irreducible;*
(2) *$\ker \Delta_P = R[P]$;*
(3) *$\Delta_P(B)$ contains a nonzero element of R.*

Proof. Write $D = \Delta_P$. To show that D is locally nilpotent, consider the K-derivation $S^{-1}D$ of $K[X,Y]$ where $S = R \setminus \{0\}$, and let $Q \in B$ be such that $K[P,Q] = K[X,Y]$. Then

$$(S^{-1}D)(P) = \begin{vmatrix} P_X & P_Y \\ P_X & P_Y \end{vmatrix} = 0 \quad \text{and} \quad (S^{-1}D)(Q) = \begin{vmatrix} P_X & P_Y \\ Q_X & Q_Y \end{vmatrix} \in K^*,$$

so $S^{-1}D$ is a triangular derivation, hence a locally nilpotent derivation, and consequently D is locally nilpotent. Observe that $\Delta_P(Q) = (S^{-1}D)(Q)$ actually belongs to $K^* \cap B = R \setminus \{0\}$, so (3) holds.

Let $b \in B$ be such that $D(B) \subseteq (b)B$. In particular, b is a common divisor of $P_X = D(Y)$ and $P_Y = D(-X)$, so $b \in B^*$. Hence D is irreducible and (1) holds.

By 2.1, $\ker D = R[H]$ for some $H \in B$. Since $D(P) = 0$, we have $P \in R[H]$ and we may write $P = f(H)$ with $f(T) \in R[T]$, T an indeterminate. Now $P_X = f'(H)H_X$ and $P_Y = f'(H)H_Y$, so $f'(H)$ is a common divisor of P_X and P_Y and consequently $f'(H) \in R^*$. Consequently $f(T) = uT + r$ with $u \in R^*$ and $r \in R$, so $\ker D = R[H] = R[P]$ and (2) holds. $\square$

Theorem 2.4. *Let R be a UFD containing $\mathbf{Q}$, let $B = R[X,Y] = R^{[2]}$ and let $K = \operatorname{qt} R$. For an R-derivation $D \neq 0$ of B, the following are equivalent:*

(1) *D is locally nilpotent;*
(2) *$D = \alpha \Delta_P$, for some $P \in B$ which is a variable of $K[X,Y]$ and satisfies $\gcd_B(P_X, P_Y) = 1$, and for some $\alpha \in R[P] \setminus \{0\}$.*

Moreover, if the above conditions are satisfied then $\ker D = R[P]$.

Proof. The fact that (2) implies (1), as well as the last assertion, follows from 2.3.

Suppose that (1) holds. By 2.1, we have $\ker D = R[P]$ for some $P \in B$. Consider $S^{-1}D : K[X,Y] \to K[X,Y]$, where $S = R \setminus \{0\}$. By part (2) of 1.1, $S^{-1}D$ is locally nilpotent and $\ker S^{-1}D = K[P]$; thus 1.2 implies that P is a variable of $K[X,Y]$ and that for some $\alpha \in K[P]$ we have

$$S^{-1}D \colon h \longmapsto \alpha \begin{vmatrix} P_X & P_Y \\ h_X & h_Y \end{vmatrix} \qquad \text{for all } h \in K[X,Y].$$

Hence $D = \alpha \Delta_P$.

We claim that $\gcd_B(P_X, P_Y) = 1$. In fact, P_X and P_Y are relatively prime in $K[X,Y]$, since P is a variable of this ring. This implies that $r \in R \setminus \{0\}$, where we define $r = \gcd_B(P_X, P_Y)$. Then, if $c \in R$ is the constant term of $P \in R[X,Y]$, r divides every coefficient of $P - c$. In other words, we have $P = rP' + c$ for some $P' \in B$, and it follows that $R[P] \subseteq R[P']$. Since $0 = D(P) = D(rP' + c) = rD(P')$, we have $P' \in \ker D = R[P]$. Hence $R[P] = R[P']$ and consequently $r \in R^*$ and $\gcd_B(P_X, P_Y) = 1$.

Next, we show that $\alpha \in R[P]$. Since $\alpha \in K[P]$ we have $\alpha = b/s$ with $b \in R[P]$, $s \in R \setminus \{0\}$ and $\gcd_B(b, s) = 1$. For each $h \in B$ we have

$$sD(h) = b\Delta_P(h) \Rightarrow s \mid b\Delta_P(h) \Rightarrow s \mid \Delta_P(h)$$

and in particular s is a common factor of $P_X = \Delta_P(Y)$ and $P_Y = \Delta_P(-X)$. By the preceding paragraph, $s \in R^*$, so $\alpha \in R[P]$ and (2) holds. □

Theorem 2.5. *For a locally nilpotent R-derivation D of $B = R[X,Y]$, the following conditions are equivalent:*

1. *$D(b) = 1$, for some $b \in B$;*
2. *D is irreducible and $B = (\ker D)^{[1]}$;*
3. *$(DB) = B$, where (DB) is the ideal of B generated by the image of D.*

The following will be needed for the proof of 2.5.

Lemma 2.6. *Let E be an integral domain containing $\mathbf{Q}$, and let $P \in E[X,Y]$ be such that $(P_X, P_Y)E[X,Y] = E[X,Y]$. Then $(\operatorname{qt} E)[P] \cap E[X,Y] = E[P]$.*

Proof. Write $L = \operatorname{qt} E$. If $L[P] \cap E[X,Y] \not\subseteq E[P]$ then we may choose $F \in L[T] \setminus E[T]$ of minimal degree such that $F(P) \in E[X,Y]$.

By the assumption, $P_X u + P_Y v = 1$ for some $u, v \in E[X,Y]$. Since $F(P) \in E[X,Y]$ implies $F'(P)P_X, F'(P)P_Y \in E[X,Y]$, we have $F'(P) = (F'(P)P_X)u + (F'(P)P_Y)v \in E[X,Y]$. By minimality of $\deg F$ we obtain $F' \in E[T]$ so, if we write $F = \sum f_i T^i$ with $f_i \in L$, we must have $f_i \in E$ for all $i > 0$ (for $\mathbf{Q} \subseteq E$). Hence $f_0 = F(P) - \sum_{i>0} f_i P^i \in E[X,Y]$, thus $f_0 \in L \cap E[X,Y] = E$ and consequently $F \in E[T]$, a contradiction. □

Proof of 2.5. If any one of the three conditions holds then clearly D is irreducible, so if we write $D = \alpha\Delta_P$ as in 2.4 then $\alpha \in R^*$. Replacing P by αP if necessary, we may arrange that $D = \Delta_P$ where $P \in B$ is a variable of $K[X,Y]$ and satisfies $\gcd_B(P_X, P_Y) = 1$. Note, also, that $\ker D = R[P]$ and that $(DB) = (P_X, P_Y)B$.

If (1) holds then (2) follows from the third part of 1.1.

If (2) holds then $R[X,Y] = R[P]^{[1]}$, so $R[X,Y] = R[P,Q]$ for some Q, and such a Q satisfies $P_XQ_Y - P_YQ_X \in R^*$; hence $(P_X, P_Y)B = B$ and (3) holds.

Assume that (3) holds, i.e., that $(P_X, P_Y)B = B$, and let $u, v \in B$ be such that $P_Xu + P_Yv = 1$. Note that, in order to show that (1) holds, it suffices to show that P is a variable of B over R (let Q be such that $B = R[P,Q]$ then, as in the preceding paragraph, we have $D(Q) \in R^*$ and (1) easily follows). The first step is:

Claim. *Given a ring homomorphism $\varphi : R \to E$ where E is a domain, let $\tilde{\varphi}$: $R[X,Y] \to E[X,Y]$ be the unique extension of φ such that $\tilde{\varphi}(X) = X$ and $\tilde{\varphi}(Y) = Y$, and let Δ^φ denote the E-derivation $\Delta_{\tilde{\varphi}(P)}$ of $E[X,Y]$. Then Δ^φ is locally nilpotent and has kernel $E[\tilde{\varphi}(P)]$. Moreover, if E is a field then $\tilde{\varphi}(P)$ is a variable of $E[X,Y]$.*

To see this, write $\overline{P} = \tilde{\varphi}(P)$ and note that $P_Xu + P_Yv = 1$ implies

$$\overline{P}_X\tilde{\varphi}(u) + \overline{P}_Y\tilde{\varphi}(v) = 1, \tag{1}$$

so $\overline{P} \notin E$ and consequently $\Delta^\varphi \neq 0$. Since the diagram

$$\begin{array}{ccc} R[X,Y] & \xrightarrow{\Delta_P} & R[X,Y] \\ \big\downarrow{\scriptstyle \tilde{\varphi}} & & \big\downarrow{\scriptstyle \tilde{\varphi}} \\ E[X,Y] & \xrightarrow{\Delta^\varphi} & E[X,Y] \end{array}$$

commutes, $(\Delta^\varphi)^n(X) = (\Delta^\varphi)^n(\tilde{\varphi}(X)) = \tilde{\varphi}({\Delta_P}^n(X)) = 0$ for $n \gg 0$, since Δ_P is locally nilpotent by 2.3; similarly, $(\Delta^\varphi)^n(Y) = 0$ for $n \gg 0$, so Δ^φ is locally nilpotent.

If E is a field then 1.2 implies that $\ker\Delta^\varphi = E[\xi]$ for some variable ξ of $E[X,Y]$ (note that E has characteristic zero since $\mathbf{Q} \subseteq R$). Thus $\overline{P} \in E[\xi]$, and by equation 1 we obtain $\overline{P} = \lambda\xi + \mu$ with $\lambda \in E^*$ and $\mu \in E$. Hence, in this case, $\overline{P}$ is a variable of $E[X,Y]$ and $\ker\Delta^\varphi = E[\,\overline{P}\,]$.

If E is not a field then let $L = \operatorname{qt} E$ and let $\psi : R \to L$ be the composition of φ with $E \hookrightarrow L$. Then the preceding paragraph implies that $\ker\Delta^\psi = L[\,\overline{P}\,]$ and, on the other hand, it is clear that Δ^ψ is an extension of Δ^φ. Hence $\ker\Delta^\varphi = \ker\Delta^\psi \cap E[X,Y] = L[\,\overline{P}\,] \cap E[X,Y]$, and this is equal to $E[\,\overline{P}\,]$ by 2.6. This proves the claim.

There are now several ways to finish the proof.

For instance, the claim implies that P is a residual variable of $R[X,Y]$; if R is noetherian then we may invoke Theorem B of [1] and conclude that P is a variable of B.

If R is not necessarely noetherian then we may apply 2.3.1 of [11] to this situation. In fact, let (S, k, K, A) of [11] correspond to our $(R \setminus \{0\}, R, R[P], B)$; then, in order to deduce $B = R[P]^{[1]}$ from the cited result, one has to verify that $R[P]$ is "S-inert" in B relative to R, and this follows from the above claim with a little bit of work.

We now give a self-contained proof. By the *length* of an element $r \in R \setminus \{0\}$, we mean the number of factors in a prime factorization of r: if $r = p_1 \cdots p_n$ where each p_i is a prime element of R, the length of r is n (the length of a unit is 0).

Let
$$\mathcal{Q} = \{Q \in B \mid K[P, Q] = K[X, Y]\}.$$
If $Q \in \mathcal{Q}$ then $\Delta_P(Q) \in B \cap K^* = R \setminus \{0\}$, so a mapping $\ell : \mathcal{Q} \to \mathbf{N}$ is defined by
$$\ell(Q) = \text{ length of } \Delta_P(Q).$$
Suppose that $Q \in \mathcal{Q}$ satisfies $\ell(Q) > 0$. Choose a prime element p of R which divides $\Delta_P(Q)$, let $E = R/pR$ and consider the canonical epimorphism $\varphi : R \to E$. Let $\tilde{\varphi}$ and Δ^φ be as in the above claim and let $\overline{P} = \tilde{\varphi}(P)$. Since p divides $\Delta_P(Q)$ we have $\Delta^\varphi(\tilde{\varphi}(Q)) = \tilde{\varphi}(\Delta_P(Q)) = 0$, so
$$\tilde{\varphi}(Q) \in \ker \Delta^\varphi = E[\,\overline{P}\,] = \tilde{\varphi}(R[P])$$
(where the first equality follows from the claim), and consequently $\tilde{\varphi}(Q) = \tilde{\varphi}(f(P))$ for some $f(T) \in R[T]$, where T is an indeterminate. Now $Q - f(P) \in \ker \tilde{\varphi} = pB$ and if we define
$$Q' = \frac{Q - f(P)}{p} \in B$$
then clearly $Q' \in \mathcal{Q}$ and
$$\Delta_P(Q') = \frac{\Delta_P(Q - f(P))}{p} = \frac{\Delta_P(Q)}{p},$$
so $\ell(Q') = \ell(Q) - 1$.

Hence there exists $Q \in \mathcal{Q}$ such that $\ell(Q) = 0$, i.e., such that $\Delta_P(Q) \in R^*$. Thus $D(\lambda Q) = 1$, where $\lambda = (DQ)^{-1} \in R^*$. $\square$

We now consider the notion of triangulability for derivations. In general, understanding which derivations are triangulable seems to be a very difficult question but, in the context of this section, the problem turns out to be relatively easy.

Definition 2.7. A derivation D of B is *triangulable over* R if there exists $X', Y' \in B$ such that $B = R[X', Y']$, $DX' \in R$ and $DY' \in R[X']$.

Since every derivation is a multiple of an irreducible one, the main problem is to understand which *irreducible* derivations are triangulable. We give a criterion for this, and then an example.

Lemma 2.8. *Let D be an irreducible, locally nilpotent R-derivation of B, and write $D = \Delta_P$ with P as in 2.4. Then the following conditions are equivalent:*

(1) *D is triangulable over R;*
(2) *there exists a variable Q of B over R such that $K[P,Q] = K[X,Y]$.*

Proof. Suppose D is triangulable and consider X', Y' such that $B = R[X',Y']$, $DX' = a \in R$ and $DY' = f(X') \in R[X']$.

If $a = 0$ then $X' \in \ker D = R[P]$, so $R[P,Y'] \supseteq R[X',Y'] = B$ and consequently $Q = Y'$ satisfies (2).

If $a \neq 0$ then let $F(T) \in R[T]$ be such that $F'(T) = f(T)$, and define $\xi = aY' - F(X')$. Then $D\xi = aDY' - F'(X')DX' = 0$, so $\xi \in \ker D = R[P]$, so $K[P,X'] \supseteq K[\xi,X'] = K[X',Y']$ and consequently $Q = X'$ satisfies condition (2).

Assume that (2) holds. Then $B \ni D(Q) = P_X Q_Y - P_Y Q_X \in K^*$, so $D(Q) \in R$.

Choose $Y' \in B$ such that $R[Y',Q] = B$. Then $K[Y',Q] = K[P,Q]$ implies that $P = \lambda Y' + f(Q)$, with $\lambda \in K^*$ and $f(T) \in K[T]$; since $R[Q,Y'] = B$ and $P \in B$, it then follows that $\lambda \in R \setminus \{0\}$ and $f(T) \in R[T]$. Now

$$0 = DP = \lambda DY' + f'(Q)DQ \implies \lambda DY' \in R[Q] \implies DY' \in R[Q],$$

so we have $B = R[Q,Y']$, $DQ \in R$ and $DY' \in R[Q]$, i.e., D is triangulable over R. $\square$

Example 2.9. If R is not a field then there exists an irreducible, locally nilpotent R-derivation $D : B \to B$ which is not triangulable over R and such that $B \neq (\ker D)^{[1]}$.

Indeed, let π be a prime element of R and let

$$P = \pi^a X + (\pi^b Y + X^c)^d \in B,$$

where a,b,c,d are positive integers, $c > 1$ and $d > 1$. Since $K[P, \pi^b Y + X^c] = K[X, \pi^b Y + X^c] = K[X,Y]$, P is a variable of $K[X,Y]$. Since P_X and P_Y are relatively prime in B, 2.4 implies that Δ_P is an irreducible, locally nilpotent R-derivation of B. Moreover, the ideal (P_X, P_Y) of B is contained in the proper ideal (π, X), so $B \neq (\ker \Delta_P)^{[1]}$ by 2.5.

To see that Δ_P is not triangulable over R, let us suppose the contrary. Then, by 2.8, there exists a variable Q of B over R which satisfies $K[P,Q] = K[X,Y]$. Thus $K[P,Q] = K[P, \pi^b Y + X^c]$ and consequently $Q = \lambda(\pi^b Y + X^c) + g(P)$ for some $\lambda \in K^*$ and $g(T) \in K[T]$. Multiplying by a suitable element of K^* yields

$$a_0 Q = a_1(\pi^b Y + X^c) + G(P), \tag{2}$$

where a_0, $a_1 \in R \setminus \{0\}$, $G(T) = \sum_{i=0}^{d} r_i T^i \in R[T]$ ($r_i \in R$) and

$$\gcd_R(a_0, a_1, r_0, \dots, r_d) = 1. \tag{3}$$

Let $E = R/\pi R$, let $\varphi : R \to E$ be the canonical epimorphism and define $\tilde{\varphi} \colon R[X,Y] \to E[X,Y]$ as in the proof of 2.5. Also, let $G^{(\varphi)} = \sum \varphi(r_i) T^i \in E[T]$.

If $\pi \mid a_0$ then equation 2 yields $0 = \varphi(a_1)X^c + G^{(\varphi)}(X^{cd})$, and this implies that $\varphi(a_1) = 0$ and $G^{(\varphi)} = 0$. In view of equation 3, this is impossible.

If $\pi \nmid a_0$ then equation 2 gives

$$\varphi(a_0)\tilde{\varphi}(Q) = \varphi(a_1)X^c + G^{(\varphi)}(X^{cd}), \quad \text{where } \varphi(a_0) \in E \setminus \{0\}.$$

Since Q is a variable of B over R, it follows that $\tilde{\varphi}(Q)$ is a variable of $E[X,Y]$ over E, and hence that $\varphi(a_1)X^c + G^{(\varphi)}(X^{cd})$ is a variable of $(\operatorname{qt} E)[X,Y]$. This is absurd, so we must conclude that Δ_P is not triangulable over R.

3. Derivations of rank two

Throughout this section, $\mathbf{k}$ is a field of characteristic zero and $R_n = \mathbf{k}^{[n]}$.

Using the results of section 2, we immediately obtain a description of all locally nilpotent derivations of $\mathbf{k}^{[n]}$ of rank at most two.

Definition 3.1. Given a coordinate system $\gamma = (X_1, \dots, X_n)$ of R_n and an element $P \in R_n = \mathbf{k}[X_1, \dots, X_n]$, define a derivation $\Delta_P^\gamma : R_n \to R_n$ by

$$\Delta_P^\gamma = -P_{X_n} \frac{\partial}{\partial X_{n-1}} + P_{X_{n-1}} \frac{\partial}{\partial X_n} \ .$$

Remark. A convenient notation is $\gamma = (X_1, \dots, X_{n-2}, Y, Z)$ and $\Delta_P^\gamma = -P_Z \, \partial/\partial Y + P_Y \, \partial/\partial Z$.

Corollary 3.2. *For a* $\mathbf{k}$*-derivation* $D \neq 0$ *of* $R_n = \mathbf{k}^{[n]}$*, the following are equivalent:*

1. D *is locally nilpotent and* $\operatorname{rank} D \le 2$*;*
2. $D = \alpha \, \Delta_P^\gamma$ *for some* γ*,* P *and* α *satisfying*
   - $\gamma = (X_1, \dots, X_{n-2}, Y, Z)$ *is a coordinate system of* R_n*,*
   - $P \in R_n$ *is a variable of* $\mathbf{k}(X_1, \dots, X_{n-2})[Y, Z]$ *satisfying* $\gcd_{R_n}(P_Y, P_Z) = 1$*,*
   - α *is a nonzero element of* $\mathbf{k}[X_1, \dots, X_{n-2}, P]$*.*

Moreover, if the above two conditions are satisfied then the following hold:

3. $\ker D = \mathbf{k}[X_1, \dots, X_{n-2}, P]$*;*
4. Δ_P^γ *is irreducible;*
5. $\Delta_P^\gamma(R_n)$ *contains a nonzero element of* $\mathbf{k}[X_1, \dots, X_{n-2}]$*.*

Corollary 3.3. *For a locally nilpotent derivation* D *of* $R_n = \mathbf{k}^{[n]}$ *of rank at most two, the following conditions are equivalent:*

1. $D(f) = 1$*, for some* $f \in R_n$*;*
2. $(D\,R_n) = R_n$*, where* $(D\,R_n)$ *is the ideal of* R_n *generated by the image of* D*.*

Corollary 3.4. *Let* $D \neq 0$ *be a locally nilpotent derivation of* $R_n = \mathbf{k}^{[n]}$ *of rank at most two, and write* $D = \alpha \, \Delta_P^\gamma$ *where* γ*,* α *and* P *are as in 3.2. Then the following are equivalent:*

1. $\operatorname{rank} D = 1$*;*

(2) $R_n = (\ker D)^{[1]}$;
(3) $(P_Y, P_Z)R_n = R_n$.

Proofs. Each one of the above results deals with R-derivations of $R_n = R[Y, Z]$, where $R = \mathbf{k}[X_1, \dots, X_{n-2}]$ and where $(X_1, \dots, X_{n-2}, Y, Z)$ is a suitably chosen coordinate system of R_n. Thus 3.2 follows from 2.3 and 2.4, and 3.3 is a consequence of 2.5. For 3.4, note that $\ker D = \ker \Delta_P^\gamma$ and apply 2.5 to Δ_P^γ. □

4. A non-triangulable derivation of $\mathbf{k}[X, Y, Z]$ fixing a line

Throughout this section, $\mathbf{k}$ is a field of characteristic zero and $R_n = \mathbf{k}^{[n]}$. We do not assume that $\mathbf{k}$ is algebraically closed, except in 4.7.

If α is an algebraic action of G_a on $\mathbf{A}^n$, and if D is the corresponding locally nilpotent derivation of R_n, then it is well-known that the set of fixed points of α is the vanishing set of the ideal (DR_n) of R_n (i.e., the ideal generated by $D(R_n)$); and it is equally well-known that α is a triangulable action if and only if D is triangulable, where:

Definition 4.1. Let D be a $\mathbf{k}$-derivation of R_n. We say that D is *triangulable* if there exists a coordinate system $(X_1, \dots, X_n)$ of R_n such that $D(X_1) \in \mathbf{k}$ and $D(X_i) \in \mathbf{k}[X_1, \dots, X_{i-1}]$ for $2 \le i \le n$.

Note that all triangulable derivations are locally nilpotent. As to fixed points, the following terminology is useful:

Definition 4.2. Let B be an integral domain of characteristic zero and let $D : B \to B$ be a locally nilpotent derivation. Then $\operatorname{Fix}(D)$ denotes the closed subset $V(DB)$ of $\operatorname{Spec} B$, and the closed points of $\operatorname{Fix}(D)$ are called the *fixed points* of D.

We are about to give an example (see 4.3) of a non-triangulable locally nilpotent derivation D of R_3 such that $\operatorname{Fix}(D)$ is a line. Although several examples of non-triangulable locally nilpotent derivations of R_n are known, it seems that 4.3 is the first one which is also irreducible. As far as we know, all examples before this one were obtained by multiplying a triangulable derivation by a cleverly chosen element of its kernel. So 4.3 gives a negative answer to the question whether every locally nilpotent derivation of R_3 is a multiple of a triangulable derivation.

Another interesting feature of 4.3 is its set of fixed points. The usual strategy for proving that a derivation D is not triangulable is to show that $\operatorname{Fix}(D)$ is not the set of fixed points of a triangulable derivation; this has in fact raised the question whether $\operatorname{Fix}(D)$ is the only obstruction for triangulability (see [12] p. 169 or [3] p. 675). However, if D is the derivation of R_3 given in 4.3 then it is easy to find a triangulable derivation T such that $\mathrm{Fix}\,(T) = \mathrm{Fix}\,(D)$ as sets; further, if one defines the *scheme of fixed points* of D to be $\mathit{Fix}(D) = \mathrm{Spec}\, R_3/(DR_3)$ then $\mathit{Fix}(D) \cong \mathit{Fix}(T)$ as schemes.[1]

Example 4.3. There exists a rank two locally nilpotent derivation D of $R_3 = \mathbf{k}[X, Y, Z]$ with the following properties:

(1) D is irreducible;
(2) D is non-triangulable;
(3) the set $\mathrm{Fix}\,(D)$ is a line.

Indeed, let $D = \Delta_P^\gamma$ where $\gamma = (X, Y, Z)$ and $P = XY + (XZ + Y^2)^2$. If we write $K = \mathbf{k}(X)$ then $K[P, XZ + Y^2] = K[Y, Z]$ so P is a variable of $K[Y, Z]$. Since

$$P_Y = X + 4Y(XZ + Y^2) \quad \text{and} \quad P_Z = 2X(XZ + Y^2)$$

are relatively prime in R_3, 3.2 implies that D is an irreducible locally nilpotent derivation of rank one or two. We easily see that $\mathrm{rank}\, D = 2$, for instance by using 3.4 and $(P_Y, P_Z)R_3 \neq R_3$. One easily checks that the ideal $(DR_3) = (P_Y, P_Z)R_3$ of R_3 satisfies $(X^2, Y^5) \subset (DR_3) \subset (X, Y)$, so $\mathrm{Fix}\,(D)$ is the Z-axis in $\mathbf{A}^3$. It remains to prove that D is not triangulable, but this requires some preliminary work. In fact, the main purpose of this section is to prove that D is not triangulable.

Remark. The above example is 2.9 with (X, Y, Z) in place of (π, X, Y) and with $a = b = 1$ and $c = d = 2$. So, by 2.9, it follows that $D : \mathbf{k}[X, Y, Z] \to \mathbf{k}[X, Y, Z]$ is not triangulable *over* $\mathbf{k}[X]$. We don't know if non-triangulability over $\mathbf{k}[X]$ implies non-triangulability.[2]

We begin with two lemmas on polynomial rings in one variable. We denote the nilradical of a ring A by $\mathrm{nil}\,(A)$ and the reduced ring $A/\mathrm{nil}\,(A)$ by A_{red}. If I is an ideal of A then we denote its radical by $\sqrt{I}$.

Lemma 4.4. *Consider the polynomial ring $A[T] = A^{[1]}$, where A is a noetherian ring. If $a \in A^*$ and $n \in \mathrm{nil}\,(A[T])$ then $A[aT + n] = A[T]$.*

Proof. Write $A_0 = A[aT + n]$ and $I = \mathrm{nil}\,(A_0)$, and note that $\mathrm{nil}\,(A) \subseteq I \subseteq \mathrm{nil}\,(A[T])$ implies that $\mathrm{nil}\,(A[T]) = I\,A[T]$. Now $A_0 + I\,A[T]$ is a subring of $A[T]$ which contains A and also $a^{-1}(aT + n) - a^{-1}n = T$, so

$$A[T] = A_0 + I\,A[T].$$

By induction, it follows that $A[T] = A_0 + I^d A[T]$ for all $d \geq 1$. Since A_0 is noetherian we have $I^d = 0$ for $d \gg 0$, so $A[T] = A_0$. $\square$

[1] It is interesting to ask whether there exists a triangulable T such that $\mathit{Fix}(D)$ and $\mathit{Fix}(T)$ are isomorphic as subschemes of $\mathbf{A}^3$, but we have not tried to answer this.

[2] It was recently discovered that non-triangulability over k[X] does imply non-triangulability (see Theorem 3.4 of [2]). Hence, the fact that D is not triangulable is also a consequence of [2].

Lemma 4.5. *Let k be a field and let B be a noetherian k-algebra satisfying $B = k[z] + \mathrm{nil}\,(B)$ for some $z \in B$. If there exists a subalgebra A of B such that $B = A^{[1]}$ then $B = A[z]$ and, consequently, A is k-isomorphic to B/zB.*

Proof. Write $N = \mathrm{nil}\,(B)$ and let $\cong$ mean k-isomorphism. Observe that $B_{\mathrm{red}} = (A^{[1]})_{\mathrm{red}} \cong (A_{\mathrm{red}})^{[1]}$ and that $B_{\mathrm{red}} = (k[z] + N)/N \cong k[z]/(N \cap k[z]) = k[\bar{z}]$, where $\bar{z}$ is the image of z in $k[z]/(N \cap k[z])$. Thus $\bar{z}$ must be transcendental over k, since $(A_{\mathrm{red}})^{[1]}$ contains a transcendental element. It follows that $(A_{\mathrm{red}})^{[1]} \cong k^{[1]}$, and hence that $A_{\mathrm{red}} \cong k$. So $A = k + \mathrm{nil}\,(A)$ and, consequently, if $t \in B$ satisfies $B = A[t]$ then $B = k[t] + N$. By the assumption,

$$k[t] + N = k[z] + N,$$

so there exist $f, g \in k[T]$ (where T is an indeterminate) and $m, n \in N$ such that $z = f(t) + n$ and $t = g(z) + m$. Then $t = g(f(t) + n) + m = g(f(t)) + n'$ for some $n' \in N$; since n' actually belongs to $N \cap k[t] = 0$, we have $t = g(f(t))$ and consequently $f(t) = at + b$, for some $a \in k^*$ and $b \in k$. Now

$$B = A[t] = A[at + b] = A[at + b + n],$$

where the last equality follows from 4.4 and the fact that A is noetherian (a homomorphic image of the noetherian ring $B = A^{[1]}$). Since $z = at + b + n$, we conclude that $B = A[z]$. $\square$

Next, we point out that the rank of a locally nilpotent derivation D is related to the minimum dimension of a component of $\mathrm{Fix}\,(D)$:

Lemma 4.6. *Let $D \neq 0$ be a locally nilpotent derivation of R_n such that $\mathrm{Fix}\,(D) \neq \emptyset$. Write $\mathrm{Fix}\,(D) = F_{d_1} \cup \cdots \cup F_{d_s}$, where $0 \leq d_1 < \cdots < d_s \leq n-1$ and $F_{d_i} \neq \emptyset$ is closed and of pure dimension d_i ($i = 1, \ldots, s$). Then:*

(1) $\mathrm{rank}\, D \geq n - d_1$;
(2) $d_s < n - 1$ *if and only if D is irreducible.*

Remark. In the event D is triangulable, it is known that $\mathrm{Fix}\,(D) \cong Z \times \mathbf{A}^1$ for some affine variety Z (see [9]). If D is both triangulable and reducible, then $F_{n-1} \cong Z' \times \mathbf{A}^{\mathrm{rank}\, D}$ for some affine variety Z' (see [7]).

Proof. Note first that (2) is simply the definition of irreducibility in geometric terms.

For (1), let I denote the ideal (DR_n) of R_n. If $r = \mathrm{rank}\, D$ then there exists a coordinate system $(X_1, \ldots, X_n)$ of R_n such that $DX_i = 0$ for all $i > r$, so $I = (DX_1, \ldots, DX_r)$ and consequently $\mathrm{ht}\, I \leq r$. But I defines the closed set $\mathrm{Fix}\,(D)$, whence $\mathrm{ht}\, I = n - d_1$. $\square$

We will also need to understand some of the properties of triangulable derivations fixing a line. (By a *line* in $\mathbf{A}^3$ we mean the zero-set of an ideal (X, Y), where X, Y belong to some coordinate system (X, Y, Z) of R_3.) For the sake of simplicity, $\mathbf{k}$ is assumed to be algebraically closed in the following:

Proposition 4.7. *Assume that* $\mathbf{k}$ *is algebraically closed. For a* $\mathbf{k}$*-derivation* T *on* $R_3 = \mathbf{k}^{[3]}$*, the following two conditions are equivalent:*

(1) T *is triangulable and* $\operatorname{Fix}(T)$ *is a single line in* $\mathbf{A}^3$*;*
(2) *For some coordinate system* (X, Y, Z) *of* R_3*,* T *satisfies*
$$TX = 0, \qquad TY = X^a f(X), \qquad TZ = Y^b + Xg(X, Y),$$
where a *and* b *are positive integers,* $f(X) \in \mathbf{k}[X]$ *satisfies* $f(0) \neq 0$ *and* $g(X, Y) \in \mathbf{k}[X, Y]$ *is such that* $Y^b + \alpha g(\alpha, Y) \in \mathbf{k}^*$ *for each root* $\alpha \in \mathbf{k}$ *of* $f(X)$.

Moreover, if the above conditions are satisfied then the following hold:

(3) T *is irreducible,* $\operatorname{rank} T = 2$ *and*
$$\ker T = \mathbf{k}[X, X^a f(X)Z - \tfrac{1}{b+1} Y^{b+1} - XG(X, Y)],$$
where G *is any polynomial in* $\mathbf{k}[X, Y]$ *such that* $G_Y = g$.
(4) *Let* $B = R_3/(TR_3)$*. Then there exists a subalgebra* A *of* B *such that* $B = A^{[1]}$*, and* A *is unique up to* $\mathbf{k}$*-isomorphism. Moreover,* $\dim_{\mathbf{k}} A = ab$.
(5) *Let* $\pi : \operatorname{Spec} R_3 \to \operatorname{Spec}(\ker T)$ *be the morphism determined by the inclusion* $\ker T \hookrightarrow R_3$ *and, given a closed point* ξ *of* $\operatorname{Spec}(\ker T)$*, let* $n(\xi)$ *denote the number of irreducible components of the fiber* $\pi^{-1}(\xi)$*. Then* $n(\xi)$ *is either* 1 *or* $b+1$*, and both values are realized.*

Remark. The integers a and b and the function $\xi \mapsto n(\xi)$ defined in the above statement are independent of the choice of (X, Y, Z). Indeed, this is clear for n since it is defined in geometric terms. Since $b + 1$ is the maximum value of $n(\xi)$, and since $ab = \dim_{\mathbf{k}} A$, we see that a and b are completely determined by T.

Proof. Suppose that (1) holds. Since T is triangulable, there exists a coordinate system (X, Y, Z) of R_3 such that $TX \in \mathbf{k}$, $TY \in \mathbf{k}[X]$ and $TZ \in \mathbf{k}[X, Y]$. Then $\operatorname{Fix}(T) \neq \emptyset$ implies that the ideal $(TR_3) = (TX, TY, TZ)$ is proper, so $TX = 0$. Hence,
$$TX = 0, \qquad TY \in \mathbf{k}[X], \qquad TZ \in \mathbf{k}[X, Y].$$
Since $TY, TZ \in \mathbf{k}[X, Y]$, and since $\operatorname{Fix}(T)$ is the zero-set in $\mathbf{A}^3$ of the ideal (TY, TZ), $\operatorname{Fix}(T) = S \times \mathbf{A}^1$ where S is the zero-set of TY and TZ in $\mathbf{A}^2 = \operatorname{Spec} \mathbf{k}[X, Y]$. Since $\operatorname{Fix}(T)$ is an irreducible curve by assumption, S is a single point. Replacing if

necessary (X,Y,Z) by $(X-x_0,Y-y_0,Z)$ for suitable $x_0,y_0\in\mathbf{k}$, we obtain that $S=\{(0,0)\}$, and consequently

$$\sqrt{(TY,TZ)}=(X,Y) \text{ in the ring } \mathbf{k}[X,Y].$$

It immediately follows that $TY=X^af(X)$ and $TZ=\lambda Y^b+Xg(X,Y)$, where $\lambda\in\mathbf{k}^*$, a and b are positive integers, $f(X)\in\mathbf{k}[X]$ satisfies $f(0)\neq 0$, and $g(X,Y)\in\mathbf{k}[X,Y]$ is such that $\lambda Y^b+\alpha g(\alpha,Y)\in\mathbf{k}^*$ for every root α of $f(X)$. Replacing Y by $\lambda^{-1/b}Y$ gives the desired expression for TY and TZ, thus (2) holds.

The proof that (2) implies (1) is a straightforward verification which we omit. From now-on, assume that (1) and (2) hold.

(3) Let $P=-X^af(X)Z+\frac{1}{b+1}Y^{b+1}+XG(X,Y)$ where $G_Y=g$. Then P is a variable of $\mathbf{k}(X)[Y,Z]$ and

$$P_Y=Y^b+Xg(X,Y), \qquad P_Z=-X^af(X)$$

are relatively prime in $\mathbf{k}[X,Y,Z]$. Observe that $T=-P_Z\frac{\partial}{\partial Y}+P_Y\frac{\partial}{\partial Z}$, i.e., $T=\Delta_P^\gamma$ with $\gamma=(X,Y,Z)$. Thus 3.2 implies that $\ker T=\mathbf{k}[X,P]$, T is irreducible and $\operatorname{rank} T$ is 1 or 2. In the notation of 4.6, $d_1=1$ for T, so $\operatorname{rank} T\geq n-d_1=2$. (Alternately, $\operatorname{rank} T\neq 1$ follows from 3.4 and $(P_Y,P_Z)R_3\neq R_3$, and irreducibility of T follows from 4.6 and $d_s=1$.)

(4) We have $(TR_3)=(X^af(X),Y^b+Xg(X,Y))$ so

$$B=\mathbf{k}[X,Y,Z]/(X^af,Y^b+Xg)=A_0{}^{[1]},$$

with $A_0=\mathbf{k}[X,Y]/(X^af,Y^b+Xg)$. Now $\dim_{\mathbf{k}} A_0$ is the total intersection number (at finite distance) of the two affine plane curves $X^af=0$ and $Y^b+Xg=0$. By the conditions stated in (2), these two curves meet only at the origin so $\dim_{\mathbf{k}} A_0$ is their local intersection number at that point:

$$\dim_{\mathbf{k}} A_0=\dim_{\mathbf{k}}\ \mathbf{k}[[X,Y]]/(X^af,Y^b+Xg)=ab.$$

Observe that $B=\mathbf{k}[x,y,z]$ where x,y,z are the images in B of X,Y,Z respectively, and that x and y are nilpotent (because the conditions of (2) imply that $\sqrt{(TR_3)}=(X,Y)$). Thus $B=\mathbf{k}[z]+\operatorname{nil}(B)$, and 4.5 implies that any subalgebra A of B such that $B=A^{[1]}$ is $\mathbf{k}$-isomorphic to B/zB. Hence A is unique, up to $\mathbf{k}$-isomorphism.

(5) Since $\ker T=\mathbf{k}[X,P]$, the morphism π may be identified with $\pi\colon\mathbf{A}^3\to\mathbf{A}^2$, $(x,y,z)\mapsto(x,P(x,y,z))$. Let $\xi=(\alpha,\beta)$ be a closed point of $\mathbf{A}^2$.

If $\alpha=0$ then $\pi^{-1}(\xi)=\{(0,y,z)\in\mathbf{A}^3\mid\frac{1}{b+1}Y^{b+1}=\beta\}$; this has $b+1$ components if $\beta\neq 0$, and 1 component if $\beta=0$.

If α is a root of f then the fiber is the set of (α,y,z) such that $\frac{1}{b+1}Y^{b+1}+\alpha G(\alpha,Y)=\beta$. Now the derivative of the left hand side with respect to Y is in $\mathbf{k}^*$, by the conditions stated in (2). Thus the equation is of the form $c_1Y+c_2=\beta$, with $c_1,c_2\in\mathbf{k}$ and $c_1\neq 0$, and the fiber is irreducible.

If α is neither 0 nor a root of f then the fiber is the set of (α, y, z) such that $cz + h(y) = \beta$, where $c = -\alpha^a f(\alpha) \in \mathbf{k}^*$ and $h(Y) = \frac{1}{b+1}Y^{b+1} + \alpha G(\alpha, Y)$. So the fiber is irreducible in this case. $\square$

We can now finish the proof of 4.3.

4.8. *The derivation D defined in* 4.3 *is not triangulable.*

Proof. We may assume that $\mathbf{k}$ is algebraically closed. Indeed, let $\overline{\mathbf{k}}$ be the algebraic closure of $\mathbf{k}$ and extend D to $\overline{D} = \Delta_P^\gamma\colon \overline{\mathbf{k}}[X,Y,Z] \to \overline{\mathbf{k}}[X,Y,Z]$. Then it suffices to prove that $\overline{D}$ is not triangulable. So, from now on, we assume that $\mathbf{k} = \overline{\mathbf{k}}$.

Let $B = \mathbf{k}[X,Y,Z]/I$, where $I = (DR_3) = (X + 4Y(XZ+Y^2),\, 2X(XZ+Y^2))$, and write $x = X + I$, $y = Y + I$, $z = Z + I \in B$. Then we have X^2, $Y^5 \in I$, so $x, y \in \operatorname{nil}(B)$ and

$$B = \mathbf{k}[z] + \operatorname{nil}(B).$$

If A is any subalgebra of B satisfying $B = A^{[1]}$ then 4.5 implies that $A \cong B/zB$. Now

$$\begin{aligned} B/zB &\cong \mathbf{k}[X,Y,Z]/(X + 4Y(XZ+Y^2),\, 2X(XZ+Y^2),\, Z) \\ &\cong \mathbf{k}[X,Y]/(X + 4Y^3,\, XY^2) \cong \mathbf{k}[Y]/(Y^5), \end{aligned}$$

so we may record the following for later use:

(4) If A is any subalgebra of B such that $B = A^{[1]}$ then $\dim_{\mathbf{k}} A = 5$.

On the other hand, let $\pi : \operatorname{Spec} R_3 \to \operatorname{Spec}(\ker D)$ be the morphism determined by the inclusion $\ker D \hookrightarrow R_3$. Then we may identify π with $\pi : \mathbf{A}^3 \to \mathbf{A}^2$, $(x,y,z) \mapsto (x, xy + (xz+y^2)^2)$, and we easily obtain

$$(5) \qquad \pi^{-1}(\alpha, \beta) \cong \begin{cases} \text{a union of 4 lines,} & \text{if } \alpha = 0 \text{ and } \beta \neq 0; \\ \text{a line,} & \text{otherwise.} \end{cases}$$

Suppose that D is triangulable, and let a, b and $n(\xi)$ be defined as in[3] 4.7. The cited result and (5) imply that $b + 1 = 4$, and the same result and (4) imply that $ab = 5$. This is clearly absurd, so D is not triangulable. $\square$

Remark. More generally, consider $P = X^sY + (X^tZ + Y^u)^v$ where $s, t \geq 1$ and $u, v \geq 2$ are integers. Let $D = \Delta_P^\gamma$, where $\gamma = (X,Y,Z)$. Then it is easy to show that D is a rank 2 locally nilpotent derivation of $\mathbf{k}[X,Y,Z]$, D is irreducible, and $\operatorname{Fix}(D)$ is a line. Imitating the proof of 4.8, one sees that if $uv - 1 \nmid s(u-1)$ then D is not triangulable.

[3] See also the remark following 4.7.

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Department of Mathematics, University of Ottawa, Ottawa, Canada K1N 6N5
E-mail address: daniel@zenon.mathstat.uottawa.ca

Department of Mathematics, University of Southern Indiana, Evansville, IN 47712, USA
E-mail address: freudenb.ucs@smtp.usi.edu