COMPLETE INTERSECTION SURFACES WITH TRIVIAL MAKAR-LIMANOV INVARIANT

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Abstract. We develop a framework for studying normal rational surfaces which are connected at infinity and admit an $\mathbb{A}^1$-fibration. As an application, we obtain the following result. *Let S be an affine surface over a field of characteristic zero. If S is a complete intersection and has trivial Makar-Limanov invariant, then S is isomorphic to a hypersurface of affine 3-space with equation $XZ = P(Y)$, for some nonconstant polynomial $P(Y)$ in one variable.*

1. Introduction

Sections 6–8 of this paper define and study a set map $(K, B) \mapsto (U, \rho)$ that "constructs" all pairs (U, ρ) such that U is a normal surface which is connected at infinity and $\rho : U \to V$ is a surjective morphism whose general fiber is an affine line and whose codomain V is an affine nonsingular rational curve (it then follows that U is rational). One obtains (U, ρ) from $(\mathbb{P}^1 \times \mathbb{P}^1, p_1)$ (where $p_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the first projection) by first performing certain blowings-up, then contracting certain divisors to normal points, and finally removing certain curves; here, one can think of (K, B) as a "recipe" that dictates which blowings-up, contractions and removals to perform (for the purpose of this introduction, we don't need to know what type of objects K and B are).

The map $(K, B) \mapsto (U, \rho)$ provides a framework for investigating normal rational surfaces U which are connected at infinity and admit an $\mathbb{A}^1$-fibration, and it is one of the aims of this paper to develop that framework in a methodical way. Accordingly, sections 6–8 give several results (notably 7.4 and 7.12) that describe how the properties of the surface U are related to those of the data (K, B). As a first (and minor) reward, we obtain a 1-line proof of Rentschler's Theorem (see 7.2). The main application of the theory is a generalization of a result of Bandman and Makar-Limanov obtained in the last section of this paper; before presenting this result, we introduce some terminology. Note that more applications of the theory will be given in the forthcoming [8].

A derivation $D : R \to R$ of a ring R is said to be *locally nilpotent* if, for each $x \in R$, there exists a positive integer n such that $D^n(x) = 0$. If $\mathbf{k}$ is a field of characteristic zero then the *Makar-Limanov invariant* of a $\mathbf{k}$-algebra R, denoted $\mathrm{ML}(R)$, is the intersection of the kernels of all locally nilpotent $\mathbf{k}$-derivations of R. Thus $\mathbf{k} \subseteq \mathrm{ML}(R) \subseteq R$ and $\mathrm{ML}(R)$ is a subalgebra of R. When $\mathrm{ML}(R) = \mathbf{k}$ one says that the $\mathbf{k}$-algebra

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R has *trivial* Makar-Limanov invariant. For an affine variety X over $\mathbf{k}$ one defines $\mathrm{ML}(X) = \mathrm{ML}\big(\mathcal{O}_X(X)\big)$, where $\mathcal{O}_X(X)$ denotes the coordinate algebra of X; thus $\mathbf{k} \subseteq \mathrm{ML}(X) \subseteq \mathcal{O}_X(X)$. One says that X has *trivial* Makar-Limanov invariant if $\mathrm{ML}(X) = \mathbf{k}$. It is well known that X admits a nontrivial G_a-action if and only if the inclusion $\mathrm{ML}(X) \subseteq \mathcal{O}_X(X)$ is strict. (Remark: the sentence *"X is an affine variety over $\mathbf{k}$"* means that $X = \operatorname{Spec} R$ where R is an integral domain and a finitely generated $\mathbf{k}$-algebra. In that case we have $\mathcal{O}_X(X) \cong R$. Also note that algebraic varieties are always assumed to be irreducible and reduced. This should be remembered whenever the words *curve*, *surface*, *threefold* or *hypersurface* are encountered.)

Given a field $\mathbf{k}$ of characteristic zero, let $\mathfrak{D}(\mathbf{k})$ denote the class of $\mathbf{k}$-algebras of the form $\mathbf{k}[X,Y,Z]/(XZ - P(Y))$ for some nonconstant polynomial in one variable $P(Y) \in \mathbf{k}[Y] \setminus \mathbf{k}$. It is well known and easy to see that each member R of $\mathfrak{D}(\mathbf{k})$ is a normal domain satisfying $\mathrm{ML}(R) = \mathbf{k}$. Also consider the class of affine surfaces S over $\mathbf{k}$ satisfying $\mathcal{O}_S(S) \in \mathfrak{D}(\mathbf{k})$, and let this class of surfaces be denoted by the same symbol $\mathfrak{D}(\mathbf{k})$; in other words, a surface belongs to $\mathfrak{D}(\mathbf{k})$ if and only if it is isomorphic to a hypersurface of $\mathbb{A}^3_{\mathbf{k}}$ with equation $xz = P(y)$, with P nonconstant. By what we have already said, each member of $\mathfrak{D}(\mathbf{k})$ is in particular a normal affine surface with trivial Makar-Limanov invariant.

Bandman and Makar-Limanov gave an example in [3] of a smooth affine surface S over $\mathbb{C}$ satisfying $\mathrm{ML}(S) = \mathbb{C}$ and $S \notin \mathfrak{D}(\mathbb{C})$. In the same paper, they proved that if S is a smooth hypersurface of $\mathbb{C}^3$ satisfying $\mathrm{ML}(S) = \mathbb{C}$, then $S \in \mathfrak{D}(\mathbb{C})$. In the present paper we generalize that result by dropping the assumption on smoothness and by replacing $\mathbb{C}$ by an arbitrary field of characteristic zero. We prove the following:

Theorem 9.9. *Let R be a two-dimensional*[1] *integral domain which contains a field $\mathbf{k}$ of characteristic zero. The following conditions are equivalent.*

(a) $R \in \mathfrak{D}(\mathbf{k})$
(b) $\mathrm{ML}(R) = \mathbf{k}$ *and R is 3-generated as a $\mathbf{k}$-algebra*
(c) $\mathrm{ML}(R) = \mathbf{k}$ *and R is a complete intersection over $\mathbf{k}$.*

Here, we say that a $\mathbf{k}$-algebra R is a *complete intersection over* $\mathbf{k}$ if it is isomorphic to a quotient $\mathbf{k}[X_1, \dots, X_n]/(f_1, \dots, f_p)$ for some $n, p \in \mathbb{N}$, where $(f_1, \dots, f_p)$ is a height p prime ideal of the polynomial ring $\mathbf{k}[X_1, \dots, X_n]$. If R is a complete intersection over $\mathbf{k}$, we also call $\operatorname{Spec} R$ a complete intersection over $\mathbf{k}$. Translating the above result into geometric language gives the equivalence of (a), (b), (c) in the following:

Theorem. *Let $\mathbf{k}$ be a field of characteristic zero and S an affine surface over $\mathbf{k}$. The following conditions are equivalent.*

(a) $S \in \mathfrak{D}(\mathbf{k})$
(b) $\mathrm{ML}(S) = \mathbf{k}$ *and S is isomorphic to a hypersurface of $\mathbb{A}^3_{\mathbf{k}}$*
(c) $\mathrm{ML}(S) = \mathbf{k}$ *and S is a complete intersection over $\mathbf{k}$.*

Moreover, if we assume that $\mathbf{k}$ is algebraically closed then the above are equivalent to:

(d) $\mathrm{ML}(S) = \mathbf{k}$*, S is normal, and $S \setminus \operatorname{Sing}(S)$ has trivial canonical class.*

[1] With respect to Krull dimension.

The sentence "$S \setminus \mathrm{Sing}(S)$ has trivial canonical class" should be understood as meaning that a canonical divisor of the nonsingular surface $S \setminus \mathrm{Sing}(S)$ is linearly equivalent to zero. Equivalence of (a) and (d) follows from Theorem 9.8. (Note: H. Flenner informed us that his student Kai Ledwig recently obtained, as part of his thesis work, the equivalence of (a) and (d) in the case $\mathbf{k} = \mathbb{C}$.)

One obvious consequence of the above results is the fact that every hypersurface S of $\mathbb{A}^3_{\mathbf{k}}$ with $\mathrm{ML}(S) = \mathbf{k}$ belongs to $\mathfrak{D}(\mathbf{k})$. More generally,

Corollary. *Let X be a factorial threefold and a complete intersection over a field $\mathbf{k}$ of characteristic zero. Then every hypersurface S of X with $\mathrm{ML}(S) = \mathbf{k}$ belongs to $\mathfrak{D}(\mathbf{k})$.*

Indeed, if X is factorial and a complete intersection then every hypersurface of X is itself a complete intersection, so the claim follows from the theorem.

As a concrete application, let X be Russell's cubic, i.e., the solution-set of $x + x^2y + z^2 + t^3 = 0$ in $\mathbb{A}^4_{\mathbf{k}}$; then X satisfies the hypothesis of the Corollary, so *every hypersurface S of Russell's cubic with $\mathrm{ML}(S) = \mathbf{k}$ belongs to $\mathfrak{D}(\mathbf{k})$.*

For another example, observe that if X is a threefold satisfying $X \times \mathbb{A}^n_{\mathbf{k}} \cong \mathbb{A}^{n+3}_{\mathbf{k}}$ for some n then X satisfies the hypothesis of the Corollary, so again *every hypersurface S of X with $\mathrm{ML}(S) = \mathbf{k}$ belongs to $\mathfrak{D}(\mathbf{k})$.*

We stress that, in the last two theorems, equivalence of conditions (a–c) is valid over any field of characteristic zero. To illustrate how this can be useful, we now give a new proof of a known result. Consider the polynomial ring $B = \mathbb{C}[X, Y, Z]$ and let $0 \neq D_i : B \to B$ ($i = 1, 2$) be locally nilpotent derivations satisfying $\ker(D_1) \neq \ker(D_2)$ and $\ker(D_1) \cap \ker(D_2) \neq \mathbb{C}$. Let K be the field of fractions of $A = \ker(D_1) \cap \ker(D_2)$ and consider the K-algebra $R = K \otimes_A B$. Then $R \in \mathfrak{D}(K)$, by the main result of [5].[2] Here we just want to point out that this is a trivial consequence of theorem 9.9: it is clear that R is 3-generated as a K-algebra, and it is easy to see that $\dim R = 2$ and $\mathrm{ML}(R) = K$; so $R \in \mathfrak{D}(K)$ by 9.9.

Sections 2 (on tableaux), 3 (on surfaces) and 4–5 (on clusters) are preparatory in nature. The theory of clusters provides a convenient way of handling arbitrary sequences of blowings-up of nonsingular surfaces, and of keeping track of the combinatorial and arithmetical data associated with such sequences. Other formalisms have similar purposes (Hamburger-Noether tableaux, characteristic pairs, etc), but clusters lend themselves particularly well to the type of arguments that have to be made here, and some of the crucial steps of our reasoning would be difficult to carry out if a different formalism were used. Because clusters do not seem to be very well known by affine algebraic geometers, we found it appropriate to organize the definitions, notations and facts in an orderly and self-contained fashion, to make it easier on the reader. We do that in section 4, whereas section 5 offers what we believe to be new results in the theory of clusters.

[2]Generalizations of this can be found in [7].

Hence, the objectives of the paper go beyond merely proving Theorem 9.9. They include laying out a framework suitable for studying normal rational surfaces with $\mathbb{A}^1$-fibrations, and presenting the part of the theory of clusters which is relevant in this context.

More applications of the theory will be given in the forthcoming paper [8] by the first author: by exploiting results 7.4 and 7.12 of the present paper and developing the theory of exact tableaux, one obtains some insight into a class of surfaces which includes in particular the normal rational hypersurfaces of $\mathbb{A}^3$ which admit a nontrivial G_a-action.

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2. Tableaux

We gather here some notions which are used in sections 4, 5, 7, and 8.

2.1. **Definition.** A *tableau* is a matrix $T = \left(\begin{smallmatrix} p_1 & \cdots & p_h \\ c_1 & \cdots & c_h \end{smallmatrix}\right)$ whose entries are integers satisfying $c_i \geq p_i \geq 1$ and $\gcd(p_i, c_i) = 1$ for all $i = 1, \dots, h$. We allow $h = 0$, in which case we say that T is the *empty tableau* and write $T = \mathbb{1}$. The set of all tableaux is denoted $\mathcal{T}$. It is sometimes useful to view $\mathcal{T}$ as a monoid, the operation being concatenation:

$$\left(\begin{smallmatrix} p_1 & \cdots & p_k \\ c_1 & \cdots & c_k \end{smallmatrix}\right)\left(\begin{smallmatrix} p_{k+1} & \cdots & p_\ell \\ c_{k+1} & \cdots & c_\ell \end{smallmatrix}\right) = \left(\begin{smallmatrix} p_1 & \cdots & p_k & p_{k+1} & \cdots p_\ell \\ c_1 & \cdots & c_k & c_{k+1} & \cdots c_\ell \end{smallmatrix}\right)$$

and the identity element being the empty tableau $\mathbb{1}$.

2.2. **Definition.** Let $T = \left(\begin{smallmatrix} p_1 & \cdots & p_h \\ c_1 & \cdots & c_h \end{smallmatrix}\right)$ be a tableau.

(a) If T is empty ($h = 0$), we set $\delta(T) = 0$. If T is nonempty ($h \geq 1$), we define $\hat{c}_i = \prod_{j=i+1}^{h} c_j$ for each $i \in \{1, \dots, h\}$ (in particular, $\hat{c}_h = 1$); then we set

$$\delta(T) = \Big(\sum_{i=1}^{h} \hat{c}_i(c_i + p_i - 1)\Big) \big/ \Big(\prod_{i=1}^{h} c_i\Big),$$

which is a positive rational number.

(b) We say that T is *exact* if $\delta(T)$ is an integer.

2.3. **Lemma.**

(a) $\left\{ \left(\begin{smallmatrix} 1 \\ c \end{smallmatrix}\right) \mid c \in \mathbb{Z} \text{ and } c \geq 1 \right\}$ *is the set of all exact tableaux having* 1 *column, and is also the set of all tableaux T satisfying $\delta(T) = 1$.*

(b) $\left\{ \left(\begin{smallmatrix} 1 & 1 \\ 1 & c \end{smallmatrix}\right) \mid c \in \mathbb{Z} \text{ and } c \geq 1 \right\}$ *is the set of all exact tableaux having* 2 *columns, and is included in the set of tableaux T satisfying $\delta(T) = 2$.*

(c) *If $h > 0$ and $\left(\begin{smallmatrix} p_1 & \cdots & p_h \\ c_1 & \cdots & c_h \end{smallmatrix}\right)$ is an exact tableau then $p_h = 1$.*

Proof. Left to the reader. □

3. Preliminaries on surfaces

This section gathers definitions, notations and known facts on algebraic surfaces. All surfaces are over an algebraically closed field $\mathbf{k}$.

3.1**.** Let $f : X \to Y$ be a birational morphism of nonsingular surfaces. The *center* of f is the finite set $\operatorname{cent}(f) = \big\{ y \in Y \mid f^{-1}(y) \text{ contains more than one point} \big\}$; the *exceptional locus* of f is the set $\operatorname{exc}(f) = f^{-1}\big(\operatorname{cent}(f)\big)$.

3.2**.** By a "graph" we mean a finite undirected graph such that no edge relates a vertex to itself and at most one edge exists between any given pair of vertices. A *weighted graph* is a graph in which each vertex is assigned an integer (called its weight). If G is a weighted graph and x is either a vertex or an edge of G then one can perform the *blowing-up of* G *at* x, which is an operation which produces a new weighted graph; we assume that the reader is familiar with blowing-up of weighted graphs, and with its inverse operation the blowing-down (refer to section 1 of [6], for instance). A vertex e of a weighted graph G is said to be *contractible* if (a) e has weight (-1); (b) e has at most two neighbors; and (c) if e has two neighbors $u \neq v$ then u, v are not neighbors of each other. One can perform the blowing-down of G at e if and only if e is a contractible vertex of G. A weighted graph which doesn't have any contractible vertex is said to be *minimal*. Two weighted graphs are *equivalent* if one can be obtained from the other by a finite sequence of blowings-up and blowings-down.

3.3. **Definition.** Let $C_1, \dots, C_n$ be distinct irreducible curves on a surface W. If

(i) each C_i is a nonsingular projective curve included in $W \setminus \operatorname{Sing}(W)$

(ii) $(C_i \cdot C_j)_W \leq 1$ whenever $i \neq j$

(iii) $C_i \cap C_j \cap C_j = \varnothing$ whenever i, j, k are distinct,

one says that $D = \sum_{i=1}^n C_i$ is an *SNC-divisor* of W. We sometimes identify an SNC-divisor $D = \sum_{i=1}^n C_i$ with its support $\operatorname{supp}(D) = \bigcup_{i=1}^n C_i$. If $D = \sum_{i=1}^n C_i$ is an SNC-divisor of W then the *dual graph* of D in W, denoted $\mathcal{G}(W, D)$, is the weighted graph whose vertex-set is $\{C_1, \dots, C_n\}$, where distinct vertices C_i, C_j are joined by an edge if and only if $C_i \cap C_j \neq \varnothing$, and where the weight of the vertex C_i is $(C_i^2)_W$.

3.4**.** By a "linear chain" we mean a weighted graph of the form $\overset{x_1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{x_2}{\bullet}\!\!-\!\!\!-\ \cdots\ -\!\!\!-\!\!\overset{x_q}{\bullet}$ $(x_i \in \mathbb{Z})$. The empty graph is a linear chain. An SNC-divisor whose dual graph is a linear chain is also called a linear chain.

3.5**.** If U is a normal surface then there exists an open immersion $\mu : U \hookrightarrow W$ where W is a complete normal surface and $W \setminus U$ is the support of an SNC-divisor D of W. The equivalence class of the dual graph $\mathcal{G}(W, D)$ (with respect to equivalence of weighted graphs) is denoted $\mathcal{G}_\infty[U]$, and is uniquely determined by the isomorphism class of U.

3.6**.** Let U be a normal surface. Then there exists a *minimal SNC-resolution of singularities of* U, by which we mean a birational and proper morphism $\sigma : \hat{U} \to U$ such that

(i) $\hat{U}$ is a nonsingular surface and σ restricts to an isomorphism from $\sigma^{-1}(U_s)$ to U_s, where $U_s = U \setminus \operatorname{Sing}(U)$;

(ii) the set $\mathcal{E} = \sigma^{-1}(\operatorname{Sing} U)$ is the support of an SNC-divisor of $\hat{U}$;

(iii) no rational irreducible curve $E \subseteq \mathcal{E}$ is a contractible vertex of the dual graph of $\mathcal{E}$ in $\hat{U}$.

Moreover, the minimal SNC-resolution of singularities of U is unique up to isomorphism. If $\sigma : \hat{U} \to U$ is the minimal SNC-resolution of singularities of U and P is a singular point of U then $\sigma^{-1}(P)$ is the support of an SNC-divisor of $\hat{U}$; the set $\sigma^{-1}(P)$ is called the *resolution locus* of P, and the dual graph of $\sigma^{-1}(P)$ in $\hat{U}$ is called the *resolution graph* of P.

3.7**.** Let X be a nonsingular projective surface, $\mathcal{E} \subset X$ a union of curves, and $\mathcal{E}_1, \dots, \mathcal{E}_r$ the connected components of $\mathcal{E}$. We say that $\mathcal{E}$ is *algebraically contractible* if there exist a normal surface $\overline{X}$ and a morphism $\pi : X \to \overline{X}$ satisfying:

- for each $i \in \{1, \dots, r\}$, $\pi(\mathcal{E}_i)$ is a point $P_i \in \overline{X}$ and $\pi^{-1}(P_i) = \mathcal{E}_i$;
- π restricts to an isomorphism from $\pi^{-1}(\overline{X} \setminus \{P_1, \dots, P_r\}) = X \setminus \mathcal{E}$ to $\overline{X} \setminus \{P_1, \dots, P_r\}$.

If π exists then it is unique, and is called the *contraction* of $\mathcal{E}$.

The following is a consequence of Artin [2] (see also Miyanishi [18], p. 53): *If $f : X \to Y$ is a birational morphism of nonsingular projective surfaces and $\mathcal{E} \subset X$ is a union of curves included in* $\operatorname{exc}(f)$*, then $\mathcal{E}$ is algebraically contractible.* In fact, one can say more: let $c : X \to \overline{X}$ be the contraction of $\mathcal{E}$ and $g : \overline{X} \dashrightarrow Y$ the birational mapping $f \circ c^{-1}$; then it is clear that g is well defined as a set map; using that $\overline{X}$ is normal, one can show that g is actually a morphism. So one obtains the following statement:

3.8**.** *Let $f : X \to Y$ be a birational morphism of nonsingular projective surfaces and $\mathcal{E} \subset X$ a union of curves included in* $\operatorname{exc}(f)$*. Then $\mathcal{E}$ is algebraically contractible and f factors as $X \xrightarrow{c} \overline{X} \xrightarrow{g} Y$, where c is the contraction of $\mathcal{E}$ and g is a proper birational morphism.*

3.9**.** Let W be a projective, nonsingular rational surface. A pencil Λ on W is called a $\mathbb{P}^1$*-ruling* if it is base-point-free and if its general member is a projective line. If Λ is a $\mathbb{P}^1$-ruling of W then by a *section* of Λ we mean an irreducible curve $H \subset W$ such that $H \cdot D = 1$ for any $D \in \Lambda$ (it then follows that $H \cong \mathbb{P}^1$).

3.10**.** Let W be a projective, nonsingular rational surface, $\rho : W \to \mathbb{P}^1$ a surjective morphism, and Λ the base-point-free pencil on W corresponding to ρ. If the general fiber of ρ is a projective line, one says that ρ is a $\mathbb{P}^1$*-fibration*. Note that ρ is a $\mathbb{P}^1$-fibration if and only if Λ is a $\mathbb{P}^1$-ruling.

3.11. **Notation.** Recall that, given $k \in \mathbb{N}$, there exists a triple $(\mathbb{F}_k, \mathbb{L}_k, \Delta_k)$ where $\mathbb{F}_k$ is a nonsingular projective rational surface, $\mathbb{L}_k$ is a base-point-free pencil on $\mathbb{F}_k$ each of whose elements is a projective line, and Δ_k is a section of $\mathbb{L}_k$ satisfying $\Delta_k^2 = -k$. Moreover, $(\mathbb{F}_k, \mathbb{L}_k, \Delta_k)$ is uniquely determined by k up to isomorphism. The surface $\mathbb{F}_k$ is called the Nagata-Hirzebruch ruled surface of degree k.

Statements 3.12 and 3.13, below, are well-known consequences of Gizatullin's results on $\mathbb{P}^1$-fibrations. Refer to [13], [17] or [18].

3.12. *Let Λ be a $\mathbb{P}^1$-ruling on a projective, nonsingular rational surface W. Then Λ has a section. Moreover, if H is a section of Λ then there exist a nonsingular projective surface $\mathbb{F}$ and a birational morphism $\pi : W \to \mathbb{F}$ satisfying:*

(a) *The exceptional locus of π is the union of the irreducible curves $C \subset W$ which are Λ-vertical*[3] *and disjoint from H.*

(b) *The linear system $\pi_*(\Lambda)$ is a base-point-free pencil on $\mathbb{F}$ each of whose elements is a projective line, and the curve $\pi(H)$ is a section of it.*

(c) $(\mathbb{F}, \pi_*(\Lambda)) = (\mathbb{F}_k, \mathbb{L}_k)$ *for some $k \in \mathbb{N}$; moreover, if $H^2 \le 0$ then $H^2 = -k$ and*
$$(\mathbb{F}, \pi_*(\Lambda), \pi(H)) = (\mathbb{F}_k, \mathbb{L}_k, \Delta_k).$$

3.13. *Let Λ be a $\mathbb{P}^1$-ruling on a projective, nonsingular rational surface W. Let H be a section of Λ, let $D \in \Lambda$ and let M be the reduced effective divisor of W satisfying* $\operatorname{supp}(M) = \operatorname{supp}(H + D)$. *Then the following hold.*

(a) *M is an SNC-divisor of W each of whose irreducible components is a $\mathbb{P}^1$.*

(b) *The dual graph $\mathcal{G}$ of M in W is a tree and the vertex H of $\mathcal{G}$ has exactly one neighbor in $\mathcal{G}$ (let v denote the unique neighbor of H in $\mathcal{G}$).*

(c) *Every vertex of $\mathcal{G}$ of weight (-1) is a contractible vertex of $\mathcal{G}$.*

(d) *If $\mathcal{G}$ has more than two vertices then every vertex of $\mathcal{G} \setminus \{H\}$ has negative weight and some vertex of $\mathcal{G} \setminus \{H, v\}$ has weight (-1).*

(e) *Let $\rho : W \to \mathbb{P}^1$ be the $\mathbb{P}^1$-fibration corresponding to Λ, let C be an irreducible component of D satisfying $(C^2)_W = -1$, and let $\sigma : W \to W'$ be the contraction of C. Then the rational map $\rho \circ \sigma^{-1} : W' \dashrightarrow \mathbb{P}^1$ is a morphism and a $\mathbb{P}^1$-fibration.*

(f) *If $\mathcal{G}$ has two vertices then the weight of v is 0.*

4. Preliminaries on clusters

This section recalls the notion of cluster, which provides convenient terminology and notations for dealing with arbitrary finite sequences of blowings-up of nonsingular surfaces. Refer to Chapter 1 of [1] for general background.

Throughout this section, we fix a nonsingular algebraic surface S over an algebraically closed field of arbitrary characteristic. By a "point over S," we mean either a point of S or a point infinitely near a point of S. Consider the partially ordered set $(S^*, \le)$, where S^* is the set of points over S and $P < Q$ means that Q is infinitely near P. The partial order $\le$ is called the *natural order*, and the symbol "$\le$" will always stand for that order. The minimal elements of $(S^*, \le)$ are called "proper points" of S and correspond bijectively to the closed points of S. We leave it to the reader to convince himself that $(S^*, \le)$ can be rigorously defined, and that this can be done in such a way that the claims contained in this section are true.

[3] A curve $C \subset W$ is said to be Λ-*vertical* if it is included in the support of an element of Λ.

4.1. **Definition.** A *cluster on* S is a (possibly empty) finite subset K of S^* with the property that, for any $P, Q \in S^*$, the conditions $P \le Q$ and $Q \in K$ imply $P \in K$. A cluster is always regarded as being partially ordered by the natural order. Note that if K is a cluster on S then each minimal element of K is a proper point of S. If K is a cluster on S then a *subcluster* of K is any subset of K which is itself a cluster on S.

4.2. Given $P \in S^*$, define $K_P = \big\{ x \in S^* \mid x \le P \big\}$. Then K_P is a nonempty cluster on S, and is totally ordered by natural order.

4.3. Given a cluster K on S, one defines the *blowing-up of* S *along* K, denoted

$$\pi_K : S_K \to S,$$

as follows. Choose a total order $\preceq$ on K which extends the natural order (which means that $P \le Q \Rightarrow P \preceq Q$), write $K = \{P_1, \dots, P_n\}$ where $P_1 \prec \cdots \prec P_n$, and consider $S_n \xrightarrow{\pi_n} \cdots \xrightarrow{\pi_1} S_0 = S$ where $S_i \xrightarrow{\pi_i} S_{i-1}$ is the blowing-up of S_{i-1} at the proper point P_i of S_{i-1}. Then the morphism $\pi_1 \circ \cdots \circ \pi_n : S_n \to S_0$ is the blowing-up $\pi_K : S_K \to S$ of S along K.

Actually, the blowing-up $\pi_K : S_K \to S$ is only defined up to equivalence (given nonsingular surfaces Y_1, Y_2 and proper birational morphisms $f_i : Y_i \to S$ $(i = 1, 2)$, declare that f_1, f_2 are *equivalent* if there exists an isomorphism of varieties $\theta : Y_1 \to Y_2$ such that $f_2 \circ \theta = f_1$).

4.4. If K is a cluster on S and K' a subcluster of K then $K \setminus K'$ is a cluster on $S_{K'}$ and we have the commutative diagram:

$$\begin{tikzcd}
 & S_{K'} \arrow[dr, "\pi_{K'}"] & \\
S_K \arrow[ur, "\pi_{K\setminus K'}"] \arrow[rr, "\pi_K"'] & & S
\end{tikzcd}$$

4.5. We write $\mathrm{Div}(S)$ for the group of Weil divisors of S, and $\mathrm{Cl}(S)$ for the divisor class group of S.

4.6. Let K be a cluster on S and let $\pi_K : S_K \to S$ be the blowing-up of S along K. Given $P \in K$, one can define the corresponding exceptional curve E_P as follows. Choose a total order $\preceq$ on K which extends the natural order, write $K = \{P_1, \dots, P_n\}$ where $P_1 \prec \cdots \prec P_n$, and consider the factorization

$$S_K = S_n \xrightarrow{\pi_n} \cdots \xrightarrow{\pi_1} S_0 = S \tag{1}$$

of $\pi_K : S_K \to S$, where $S_i \xrightarrow{\pi_i} S_{i-1}$ is the blowing-up of S_{i-1} at the proper point P_i of S_{i-1}. Then there is a unique i such that $P = P_i$ and we set $E_P = \pi_i^{-1}(P) \subset S_i$. The strict transform (resp. total transform) of E_P on S_K is denoted $\widetilde{E}_P^K \subset S_K$ (resp. $\overline{E}_P^K \in \mathrm{Div}(S_K)$); observe that $\widetilde{E}_P^K$ and $\overline{E}_P^K$ are independent of the choice of $\preceq$. Given $D \in \mathrm{Div}(S)$, we write $\widetilde{D}^K, \overline{D}^K \in \mathrm{Div}(S_K)$ for the strict transform and total transform of D, respectively.

4.7. Given $P \in S^*$ and $D \in \mathrm{Div}(S)$, consider the cluster $K_{(P)} = \{ x \in K \mid x < P \}$ on S, the corresponding blowing-up $\pi_{K_{(P)}} : S_{K_{(P)}} \to S$ and the strict transform $\widetilde{D}^{K_{(P)}} \in \mathrm{Div}\left(S_{K_{(P)}}\right)$ of D. As P is a proper point of $S_{K_{(P)}}$, it makes sense to consider the multiplicity of P on $\widetilde{D}^{K_{(P)}}$; we denote this integer by $e_P(D)$ and call it the *multiplicity of P on D*. So each point $P \in S^*$ has a multiplicity on $D \in \mathrm{Div}(S)$.

4.8. **Lemma.** *Let K be a cluster on S and consider $\pi_K : S_K \to S$.*
(a) *If $D \in \mathrm{Div}(S)$ then $\overline{D}^K = \widetilde{D}^K + \sum_{P \in K} e_P(D) \overline{E}_P^K$.*
(b) *If $\kappa \in \mathrm{Div}(S)$ is a canonical divisor of S then $\overline{\kappa}^K + \sum_{P \in K} \overline{E}_P^K$ is a canonical divisor of S_K.*

Proof. See 1.1.18 and 1.1.26(7) of [1]. □

4.9. Given a cluster K on S and an irreducible curve $G \subset S$, we define

$$K_G = \{ P \in K \mid e_P(G) > 0 \},$$

which is a subcluster of K. In short, K_G is the set of points $P \in K$ which lie on a strict transform of G.

4.10. **Definition.** Let K be a cluster on S. Choose a total order $\preceq$ which extends the natural order, and write $K = \{P_1, \dots, P_n\}$ where $P_1 \prec \cdots \prec P_n$. The pair $(K, \preceq)$ determines the $n \times n$ matrix $\mathbf{Q}(K, \preceq) = \mathbf{Q}(K) = \mathbf{Q}$ defined as follows. For each $j \in \{1, \dots, n\}$, define $a_{1j}, \dots, a_{nj} \in \mathbb{N}$ by $\overline{E}_{P_j}^K = \sum_{i=1}^n a_{ij} \widetilde{E}_{P_i}^K$ (see 4.6 for the notations $\widetilde{E}_{P_i}^K$ and $\overline{E}_{P_j}^K$). Then let $\mathbf{Q}$ be the $n \times n$ matrix[4] whose jth column is $\begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix}$.

We write $\mathbf{Q}_i(K)$ for the i-th row of $\mathbf{Q}(K)$. For the last row of $\mathbf{Q}(K)$ we may write $\mathbf{Q}_n(K)$ (if $\mathbf{Q}(K)$ is $n \times n$) or $\mathbf{Q}_*(K)$ (if we prefer not to mention the "n").

Given a subset A of K, we write $\mathbf{Q}_A$ for the $(n - |A|) \times n$ submatrix of $\mathbf{Q}$ obtained by deleting the ith row of $\mathbf{Q}$ for each $P_i \in A$.

The following observation is trivial, but useful:

4.11. **Lemma.** *Let K be any cluster on S, choose a total order $\preceq$ extending the natural order $\leq$, write the elements of K as $P_1 \prec \cdots \prec P_n$ and consider the $n \times n$ matrix $\mathbf{Q} = \mathbf{Q}(K, \preceq) = (a_{ij})$ defined in 4.10. Then for any $i, j \in \{1, \dots, n\}$,*
(a) *a_{ij} is a nonnegative integer*
(b) *$a_{ii} = 1$*
(c) *if $a_{ij} \neq 0$, then $P_j \leq P_i$ (in particular $j \leq i$, so $\mathbf{Q}$ is lower triangular).*

Proof. These claims follow immediately from $\overline{E}_{P_j}^K = a_{1j} \widetilde{E}_{P_1}^K + \cdots + a_{nj} \widetilde{E}_{P_n}^K$, which is the definition of $\mathbf{Q}$. □

[4] In the terminology of [1], $\mathbf{Q}$ is the inverse of the proximity matrix of the cluster K.

Concentric clusters and tableaux

Recall that if K is a cluster on S and $\leq$ is the natural order, then $(K, \leq)$ is a partially ordered set.

4.12. **Definition.** A cluster K on S is said to be *concentric* if $(K, \leq)$ is totally ordered.

Consider a sequence $S_n \xrightarrow{\pi_n} \cdots \xrightarrow{\pi_1} S_0 = S$, where $S_i \xrightarrow{\pi_i} S_{i-1}$ is the blowing-up of S_{i-1} at a proper point P_i of S_{i-1}. Then $K = \{P_1, \dots, P_n\}$ is a cluster on S, $\pi_1 \circ \cdots \circ \pi_n$ is the blowing-up of S along K, and the condition

$$\pi_i(P_{i+1}) = P_i \text{ for all } i = 1, \dots, n-1 \tag{2}$$

is equivalent to K being concentric. Sequences $S_n \xrightarrow{\pi_n} \cdots \xrightarrow{\pi_1} S_0 = S$ satisfying (2) determine combinatorial and arithmetical objects which have been studied extensively by algebraic geometers. In 4.13, we explain how a concentric cluster determines a tableau (refer to 2.1 for the definition of tableau).

4.13. **Definition.** Consider a triple (S, K, C) of the following type:

(∗) S is a nonsingular projective surface, K is a concentric cluster on S, $C \subset S$ is a nonsingular irreducible curve and, if $K \neq \varnothing$, C passes through the unique minimal element of K.

Then (S, K, C) determines a tableau $T(S, K, C) \in \mathfrak{T}$ which we now proceed to define.

Write $K = \{P_1, \dots, P_n\}$ with $P_1 < \cdots < P_n$ (where $<$ is the natural order) and factor $\pi_K : S_K \to S$ as

$$S = S_0 \xleftarrow{\pi_1} S_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} S_n = S_K$$

where π_i is the blowing-up of S_{i-1} at P_i; also let $E_i = \pi_i^{-1}(P_i) \subset S_i$. For each i such that $1 \leq i \leq n$, note that the point $P_i \in S_{i-1}$ belongs to either 1 or 2 irreducible components of the closed subset $(\pi_1 \circ \cdots \circ \pi_{i-1})^{-1}(C)$ of S_{i-1}; if P_i belongs to 1 component (resp. 2 components), we say that π_i is *sprouting* (resp. *subdivisional*) *with respect to* (S, C). Clearly, if $K \neq \varnothing$ then π_1 is sprouting with respect to (S, C). Let $h(S, K, C)$ denote the number of blowings-up among $\pi_1, \dots, \pi_n$ which are sprouting with respect to (S, C). We now define the tableau $T(S, K, C)$.

4.13.1. If $h(S, K, C) = 0$ (i.e., $K = \varnothing$), we set $T(S, K, C) = \mathbb{1}$ (the empty tableau).

4.13.2. Assume that $h(S, K, C) = 1$, i.e., $K \neq \varnothing$ and π_1 is the only sprouting blowing-up among $\pi_1, \dots, \pi_n$. Then $\pi_K^{-1}(C)$ is the support of an SNC-divisor of S_n whose dual graph is a linear chain as follows:

$$\underset{\widetilde{C}^K}{\bullet} \text{---} \underbrace{\overset{a_1}{\bullet} \text{---} \cdots \text{---} \overset{a_s}{\bullet}}_{p} \text{---} \underset{E_n}{\overset{-1}{\bullet}} \text{---} \underbrace{\overset{a_{s+1}}{\bullet} \text{---} \cdots \text{---} \overset{a_{n-1}}{\bullet}}_{c} \qquad (a_i \in \mathbb{Z}) \tag{3}$$

where $\widetilde{C}^K \subset S_n$ denotes the strict transform of C. Let p and c be the determinants[5] of the subtrees indicated by the braces in diagram (3) and define $T(S, K, C) = \binom{p}{c}$. It

[5]For the definition of the determinant of a weighted graph, see for instance 3.15 of [9]; note that the determinant of the empty weighted graph is equal to 1.

is well known that p, c are integers satisfying $1 \le p \le c$ and $\gcd(p,c) = 1$; thus $\binom{p}{c}$ is indeed a tableau.

4.13.3. More generally, assume that $h(S,K,C) \ge 1$ and let $j_1 < \cdots < j_h$ (where $h = h(S,K,C)$) be the elements of

$$\big\{\, j \mid 1 \le j \le n \text{ and } \pi_j \text{ is sprouting with respect to } (S,C) \,\big\}.$$

Note that $j_1 = 1$ and also define $j_{h+1} = n+1$ and $E_0 = C$. For each $\nu \in \{1, \dots, h\}$, let $K_\nu = \big\{\, P_i \mid j_\nu \le i < j_{\nu+1} \,\big\}$; then

$$(S_{j_\nu - 1}, K_\nu, E_{j_\nu - 1}) \text{ satisfies condition } (*) \text{ of 4.13 and } h(S_{j_\nu - 1}, K_\nu, E_{j_\nu - 1}) = 1,$$

so it makes sense to define $\binom{p_\nu}{c_\nu} = T(S_{j_\nu - 1}, K_\nu, E_{j_\nu - 1})$ as in 4.13.2. This defines $\binom{p_\nu}{c_\nu}$ for $\nu = 1, \dots, h$. We may therefore define

$$T(S,K,C) = \left(\begin{smallmatrix} p_1 & \cdots & p_h \\ c_1 & \cdots & c_h \end{smallmatrix} \right).$$

4.14. **Remarks.** Suppose that (S,K,C) satisfies condition $(*)$ of 4.13 and let the notation (P_i, π_i, E_i, etc.) be as in 4.13.

(a) The number of columns of the tableau $T(S,K,C)$ is equal to the number $h(S,K,C)$ of blowings-up among $\pi_1, \dots, \pi_n$ which are sprouting with respect to (S,C). In particular, $T(S,K,C) = \mathbb{1}$ if and only if $K = \varnothing$; and $T(S,K,C)$ has 1 column if and only if $K \neq \varnothing$ and π_1 is the only sprouting blowing-up.

(b) $K = K_C$ if and only if $T(S,K,C) = \binom{1}{c}$ for some $c \ge 1$ (indeed, $K = K_C$ is equivalent to "$h(S,K,C) = 1$ and E_n is a neighbor of $\widetilde{C}^K$ in diagram (3)"; cf. 4.9 for the definition of K_C). Also, K is a singleton if and only if $T(S,K,C) = \binom{1}{1}$.

(c) If $T(S,K,C)$ has at least two columns and j is any element of $\{2, \dots, n\}$ such that π_j is sprouting then

(S, K', C) and (S_{j-1}, K'', E_{j-1}) satisfy condition $(*)$ of 4.13 and there holds $T(S,K,C) = T(S,K',C)\, T(S_{j-1}, K'', E_{j-1})$,

where we define $K' = \{P_1, \dots, P_{j-1}\}$ and $K'' = \{P_j, \dots, P_n\}$.

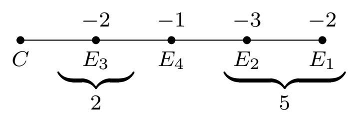
Given the importance of 4.13, we give:

4.15. **Example.** We use the following notations: $S_i \xrightarrow{\pi_i} S_{i-1}$ is the blowing-up of S_{i-1} at $P_i \in S_{i-1}$, $E_i = \pi_i^{-1}(P_i) \subset S_i$, and if $\Gamma \subset S_i$ is a curve and $j > i$ then the strict transform of Γ on S_j is denoted by the same symbol Γ. We consider a sequence of blowings-up $S_5 \xrightarrow{\pi_5} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = S$ satisfying the following conditions.

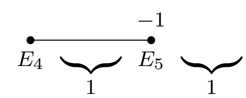
Let $C \subset S$ be a nonsingular curve, let $P_1 \in S_0 = S$ be any point of C, let $P_2 \in S_1$ be the point $E_1 \cap C$, let $P_3 \in S_2$ be the point $E_2 \cap C$, let $P_4 \in S_3$ be the point $E_2 \cap E_3$, and let $P_5 \in S_4$ be a point of E_4 which does not belong to $E_2 \cup E_3$.

This gives $S_5 \xrightarrow{\pi_5} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = S$, and $K = \{P_1, \dots, P_5\}$ is a concentric cluster on S. Clearly, (S,K,C) satisfies condition $(*)$ of 4.13. Also suppose that $C' \subset S$ is a nonsingular curve such that $C' \neq C$ and $(C' \cdot C)_{P_1} = 2$; then (S,K,C') also satisfies condition $(*)$ of 4.13. We compute the tableaux $T(S,K,C)$ and $T(S,K,C')$.

(a) (S, K, C). Since π_1 and π_5 are the blowings-up which are sprouting with respect to (S, C), $T(S, K, C)$ has two columns: $T(S,K,C) = \left(\begin{smallmatrix} p_1 & p_2 \\ c_1 & c_2 \end{smallmatrix}\right)$. Using the clusters $K_1 = \{P_1, P_2, P_3, P_4\}$ on S and $K_2 = \{P_5\}$ on S_4, we find that $\binom{p_1}{c_1} = T(S, K_1, C)$ and $\binom{p_2}{c_2} = T(S_4, K_2, E_4)$. The dual graphs of $(\pi_1 \circ \cdots \circ \pi_4)^{-1}(C)$ and $\pi_5^{-1}(E_4)$ are



and



where, in the first (resp. the second) graph, the weight of C (resp. of E_4) is not indicated, as it is irrelevant; so $T(S, K_1, C) = \binom{2}{5}$ and $T(S_4, K_2, E_4) = \binom{1}{1}$, and consequently $T(S, K, C) = \left(\begin{smallmatrix} 2 & 1 \\ 5 & 1 \end{smallmatrix}\right)$.

(b) (S, K, C'). As π_1, π_3, π_5 are the blowings-up which are sprouting with respect to (S, C'), we have 3 columns: $T(S, K, C') = \left(\begin{smallmatrix} p_1 & p_2 & p_3 \\ c_1 & c_2 & c_3 \end{smallmatrix}\right)$. Let $K_1 = \{P_1, P_2\}$, $K_2 = \{P_3, P_4\}$, and $K_3 = \{P_5\}$, then $T(S, K_1, C) = \binom{p_1}{c_1}$, $T(S_2, K_2, E_2) = \binom{p_2}{c_2}$, and $T(S_4, K_3, E_4) = \binom{p_3}{c_3}$. The reader may verify that $T(S, K, C') = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \end{smallmatrix}\right)$.

4.16. **Notation.** Let K be a cluster on S and $\preceq$ a total order on K extending the natural order. Write $K = \{P_1, \dots, P_n\}$ where $P_1 \prec \cdots \prec P_n$. Given a subset A of K, define the $n \times 1$ matrix

$$\mathbf{1}_A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \text{where} \quad a_i = \begin{cases} 1 & \text{if } P_i \in A, \\ 0 & \text{if } P_i \notin A. \end{cases}$$

Given (S, K, C) satisfying condition $(*)$ of 4.13, we will need (in section 7) to compute the products $\mathbf{Q}_*(K)\mathbf{1}_K$ and $\mathbf{Q}_*(K)\mathbf{1}_{K_C}$ ($\mathbf{Q}_*(K)$ is defined in 4.10, K_C in 4.9, $\mathbf{1}_K$ and $\mathbf{1}_{K_C}$ in 4.16). Result 4.18 answers this question, but first we need the following:

4.17. **Notation.** Let p, c be integers satisfying $1 \le p \le c$ and $\gcd(p, c) = 1$ (or equivalently, $\binom{p}{c} \in \mathfrak{T}$). Consider the Euclidean algorithm of $(x_0, x_1) = (c, p)$:

$$\begin{aligned} x_0 &= q_1 x_1 + x_2 \\ &\ \ \vdots \\ x_{s-2} &= q_{s-1} x_{s-1} + x_s \\ x_{s-1} &= q_s x_s \end{aligned}$$

where all x_i and q_i are positive integers and $x_1 > \cdots > x_s = 1$. Then we define:

$$X(p, c) = \big(\ \underbrace{x_1 \ \dots \ x_1}_{q_1} \ \ \underbrace{x_2 \ \dots \ x_2}_{q_2} \ \ \cdots \ \ \underbrace{x_s \ \dots \ x_s}_{q_s}\ \big),$$

which is a $1 \times t$ matrix with $t = q_1 + \cdots + q_s$.
Remark. It is easily verified that $\sum_{i=1}^{s} q_i x_i = c + p - 1$ (this will be used later).

4.18. **Proposition.** *Suppose that* (S, K, C) *satisfies condition* $(*)$ *of* 4.13 *and write*

$$T(S, K, C) = \left(\begin{smallmatrix} p_1 & \cdots & p_h \\ c_1 & \cdots & c_h \end{smallmatrix}\right).$$

Suppose that $K \neq \varnothing$ (or equivalently, $h \geq 1$). For each $j \in \{1, \dots, h\}$, define $\hat{c}_j = \prod_{i=j+1}^{h} c_i$ (so in particular $\hat{c}_{h-1} = c_h$ and $\hat{c}_h = 1$).

(a) $\mathbf{Q}_*(K) = \big(\hat{c}_1 X(p_1, c_1) \quad \cdots \quad \hat{c}_h X(p_h, c_h)\big)$

(b) *If we write $K = \{P_1, \dots, P_n\}$ where $P_1 < \cdots < P_n$, then $K_C = \{P_1, \dots, P_m\}$ where $m = \lceil c_1/p_1 \rceil$.*[6]

(c) $\mathbf{Q}_*(K)\mathbf{1}_K = \sum_{i=1}^{h} \hat{c}_i(c_i + p_i - 1)$ *and* $\mathbf{Q}_*(K)\mathbf{1}_{K_C} = \prod_{i=1}^{h} c_i$.

The rest of the section is devoted to the proof of 4.18.

4.18.1. **Lemma.** *Suppose that (S, K, C) satisfies condition $(*)$ of 4.13 and that $T(S, K, C) = \binom{p}{c}$ with $p \neq 1$. Define q and r by*

$$c = qp + r \quad (q, r \in \mathbb{N}, \quad 0 < r < p).$$

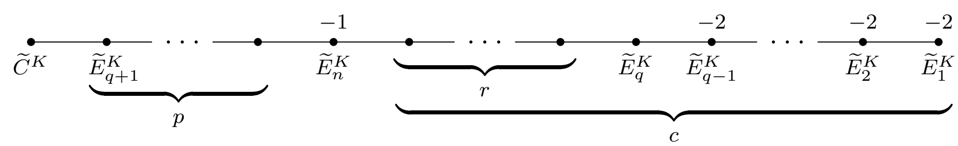
With notations ($S = S_0 \xleftarrow{\pi_1} S_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} S_n$, P_i, E_i) as in 4.13, the following hold:

(a) $K_C = \{P_1, \dots, P_{q+1}\}$

(b) *Let $K' = \{P_{q+1}, P_{q+2}, \dots, P_n\}$; then K' is a cluster on S_q, (S_q, K', E_q) satisfies condition $(*)$ of 4.13 and $T(S_q, K', E_q) = \binom{r}{p}$.*

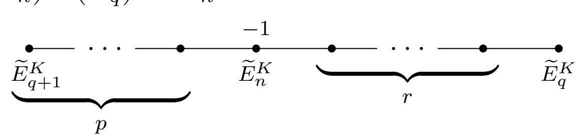
Proof. One can check that the dual graph of $(\pi_1 \circ \cdots \circ \pi_n)^{-1}(C)$ in S_n is as follows:

(4)



where, as usual, the numbers under the braces are determinants. In particular, $\widetilde{E}^K_{q+1}$ meets $\widetilde{C}^K$ in $S_n = S_K$, so P_{q+1} is the greatest element of K_C and (a) is true.

Let $K' = \{P_{q+1}, P_{q+2}, \dots, P_n\}$; then it is clear that K' is a cluster on S_q and that (S_q, K', E_q) satisfies condition $(*)$ of 4.13. Moreover, (4) shows that the dual graph of $(\pi_{q+1} \circ \cdots \circ \pi_n)^{-1}(E_q)$ in S_n is



and this picture immediately implies that $T(S_q, K', E_q) = \binom{r}{p}$. So we are done. $\square$

4.18.2. **Lemma.** *Suppose that (S, K, C) satisfies condition $(*)$ of 4.13 and that*

$$T(S, K, C) = \begin{pmatrix} p \\ c \end{pmatrix}.$$

(a) *If we write $K = \{P_1, \dots, P_n\}$ where $P_1 < \cdots < P_n$, then $K_C = \{P_1, \dots, P_m\}$ where $m = \lceil c/p \rceil$.*

(b) *The last row of $\mathbf{Q}(K)$ is $\mathbf{Q}_*(K) = X(p, c)$.*

[6] For $x \in \mathbb{R}$, let $\lceil x \rceil = \min\big([x, \infty) \cap \mathbb{Z}\big)$.

Proof. By 4.18.1, assertion (a) is true whenever $p \neq 1$; it is a simple matter to verify that it continues to be true when $p = 1$. Assertion (b) is proved by induction on the number s of equations in the Euclidean algorithm (we let the notation be as in 4.17). If $s = 1$ then $p = 1$, in which case the claim is easy to prove. Assume that $s > 1$, i.e., that $p \neq 1$. Write $K' = \{P_{q_1+1}, P_{q_1+2}, \dots, P_n\}$. By 4.18.1, (S_{q_1}, K', E_{q_1}) satisfies condition $(*)$ of 4.13 and $T(S_{q_1}, K', E_{q_1}) = \binom{x_2}{x_1}$. The number of equations in the Euclidean algorithm of (x_1, x_2) is precisely $s-1$, so by the inductive hypothesis the Lemma is true for the triple (S_{q_1}, K', E_{q_1}). This gives:

(a′) $K'_{E_{q_1}} = \{P_{q_1+1}, \dots, P_{q_1+m}\}$, where $m = \lceil x_1/x_2 \rceil$

(b′) $\mathbf{Q}_*(K') = \big(\underbrace{x_2 \dots x_2}_{q_2} \ \ \dots \ \ \underbrace{x_s \dots x_s}_{q_s} \big)$.

(Remark: until the end of the proof, we use the definition of m given in (a′), not the one given in the statement of the lemma.) It follows that

$$\mathbf{Q}_*(K) = \big(\ a_1 \ \dots \ a_{q_1} \ \ \underbrace{x_2 \dots x_2}_{q_2} \ \ \dots \ \ \underbrace{x_s \dots x_s}_{q_s} \big)$$

where, for each $j \in \{1, \dots, q_1\}$, a_j is the coefficient of E_n in the divisor $\overline{E}_j^K \in \operatorname{Div}(S_n)$. To complete the proof, there only remains to show that $a_j = x_1$ for all $j = 1, \dots, q_1$.

For each $j \in \{1, \dots, q_1\}$, let $\widetilde{E}_j \subset S_{q_1}$ be the strict transform of E_j and set

$$D_j = \widetilde{E}_j + \widetilde{E}_{j+1} + \dots + \widetilde{E}_{q_1} \in \operatorname{Div}(S_{q_1}).$$

Then D_j is the total transform of E_j in S_{q_1} and consequently $\overline{E}_j^K = \overline{D}_j^{K'}$. Thus

$$\overline{E}_j^K = \overline{D}_j^{K'} = \widetilde{D}_j^{K'} + \sum_{P \in K'} e_P(D_j) \overline{E}_P^{K'}$$

by 4.8. Now $e_P(D_j) = e_P(E_{q_1})$ for all $P \in K'$, and by (b′) we have

$$e_P(E_{q_1}) = \begin{cases} 1, & \text{if } P \in \{P_{q_1+1}, \dots, P_{q_1+m}\}, \\ 0, & \text{if } P \in K' \setminus \{P_{q_1+1}, \dots, P_{q_1+m}\}. \end{cases}$$

So $\overline{E}_j^K = \widetilde{D}_j^{K'} + \sum_{i=1}^m \overline{E}_{q_1+i}^{K'}$. The coefficient of E_n in this divisor is

$$a_j = \text{sum of the first } m \text{ entries in the row } \mathbf{Q}_*(K') = \big(\underbrace{x_2 \dots x_2}_{q_2} \ \ \dots \ \ \underbrace{x_s \dots x_s}_{q_s} \big).$$

Taking into account that

$$m = \lceil x_1/x_2 \rceil = \begin{cases} q_2 + 1, & \text{if } x_2 \neq 1, \\ q_2, & \text{if } x_2 = 1, \end{cases}$$

we obtain $a_j = x_1$ in all cases, which completes the proof. $\square$

4.18.3. **Lemma.** *Suppose that (S, K, C) satisfies condition $(*)$ of 4.13 and let the notation ($\pi_i : S_i \to S_{i-1}$, P_i, etc) be as in 4.13. Suppose that $T(S, K, C)$ has at least* *two columns, let $j > 1$ be the greatest element of $\{1, \dots, n\}$ such that π_j is sprouting with respect to (S, C), and consider the clusters $K' = \{P_1, \dots, P_{j-1}\}$ on S and $K'' = \{P_j, \dots, P_n\}$ on S_{j-1}. Define $\binom{p}{c} = T(S_{j-1}, K'', E_{j-1})$ and note that this is the rightmost column of $T(S, K, C)$. Then*

$$\mathbf{Q}_*(K) = \begin{pmatrix} c\,\mathbf{Q}_*(K') & \mathbf{Q}_*(K'') \end{pmatrix}.$$

That is, if $\mathbf{Q}_*(K') = \begin{pmatrix} a_1 & \cdots & a_{j-1} \end{pmatrix}$ *and* $\mathbf{Q}_*(K'') = \begin{pmatrix} a_j & \cdots & a_n \end{pmatrix}$*, then*

$$\mathbf{Q}_*(K) = \begin{pmatrix} ca_1 & \cdots & ca_{j-1} & a_j & \cdots & a_n \end{pmatrix}.$$

Proof. It is clear that $\mathbf{Q}_*(K) = \begin{pmatrix} \alpha_1 & \cdots & \alpha_{j-1} & a_j & \cdots & a_n \end{pmatrix}$ where, for each $\nu \in \{1, \dots, j-1\}$, α_ν is the coefficient of E_n in the divisor $\overline{E}_\nu^K \in \operatorname{Div}(S_n)$. The integer ν being fixed, let $D = \overline{E}_\nu^{K'} \in \operatorname{Div}(S_{j-1})$ and note that for each $P \in K''$ we have $e_P(D) = a_\nu e_P(E_{j-1})$. So the coefficient α_ν of E_n in

$$\overline{E}_\nu^K = \overline{D}^{K''} = \widetilde{D}^{K''} + \sum_{P \in K''} e_P(D)\overline{E}_P^{K''} = \widetilde{D}^{K''} + a_\nu \sum_{P \in K''} e_P(E_{j-1})\overline{E}_P^{K''}$$

is $\alpha_\nu = a_\nu y$, where y denotes the unique entry of the following 1×1 matrix:

$$\mathbf{Q}_*(K'') \cdot \begin{pmatrix} e_{P_j}(E_{j-1}) \\ \vdots \\ e_{P_n}(E_{j-1}) \end{pmatrix}.$$

Now y is the sum of the first m entries of $\mathbf{Q}_*(K'')$, where $m = |K''_{E_{j-1}}|$. Recall that $T(S_{j-1}, K'', E_{j-1}) = \binom{p}{c}$ and let q_i, x_i be the natural numbers determined by the Euclidean algorithm of $(x_0, x_1) = (c, p)$ (notation as in 4.17). Then 4.18.2 implies that $m = \lceil c/p \rceil$ and that

$$\mathbf{Q}_*(K'') = \begin{pmatrix} \underbrace{x_1 \dots x_1}_{q_1} & \underbrace{x_2 \dots x_2}_{q_2} & \cdots & \underbrace{x_s \dots x_s}_{q_s} \end{pmatrix},$$

so $y = x_0 = c$ and $\alpha_\nu = a_\nu c$. This proves the Lemma. □

Proof of 4.18. We prove (a) and (b) by induction on h. By 4.18.2, the result is true when $h = 1$. Assume that $h > 1$. As in 4.13.3, let $j > 1$ be the greatest element of $\{1, \dots, n\}$ such that π_j is sprouting and consider the clusters $K' = \{P_1, \dots, P_{j-1}\}$ and $K'' = \{P_j, \dots, P_n\}$. Then the definition of $T(S, K, C)$ gives

$$T(S, K', C) = \begin{pmatrix} p_1 & \cdots & p_{h-1} \\ c_1 & \cdots & c_{h-1} \end{pmatrix} \quad \text{and} \quad T(S_{j-1}, K'', E_{j-1}) = \begin{pmatrix} p_h \\ c_h \end{pmatrix}.$$

Now 4.18.3 implies that $\mathbf{Q}_*(K) = (c_h \mathbf{Q}_*(K') \ \ \mathbf{Q}_*(K''))$ and the inductive hypothesis gives

$$\mathbf{Q}_*(K'') = X(p_h, c_h) = \hat{c}_h X(p_h, c_h)$$
$$\mathbf{Q}_*(K') = \big(\hat{c}'_1 X(p_1, c_1) \ \ \cdots \ \ \hat{c}'_{h-1} X(p_{h-1}, c_{h-1})\big)$$

with $\hat{c}'_j = \prod_{i=j+1}^{h-1} c_i = \hat{c}_j / c_h$. This proves assertion (a).

As π_j is sprouting we have $P_j \notin \widetilde{C}^{K'}$, so $K_C \subseteq K'$ and hence $K_C = K'_C$. By the inductive hypothesis, $K'_C = \{P_1, \dots, P_m\}$ where $m = \lceil c_1/p_1 \rceil$, so (b) is proved.

It was remarked at the end of 4.17 that the sum of all entries in $X(p, c)$ is $c + p - 1$; this together with assertion (a) gives the first part of assertion (c). To prove the last claim, let $m = \lceil c_1/p_1 \rceil$; then, by assertion (b), the product $\mathbf{Q}_*(K)\mathbf{1}_{K_C}$ is the sum of the first m entries of $\mathbf{Q}_*(K)$, which is equal to the sum of the first m entries of $\hat{c}_1 X(p_1, c_1)$. We leave it to the reader to verify that this is equal to $\prod_{i=1}^{h} c_i$. □

5. Further properties of clusters

The aim of this section is to prove 5.3 and 5.5, which appear to be new results in the theory of clusters. The first one is very general. The second one has been designed for a specific use, but it turns out that the situation to which it applies is still fairly general. The two results are of interest for their own sake.

Result 5.3 is needed for proving 5.5, and 5.5 is used in the proof of 8.3.

Throughout, we fix a nonsingular projective surface S over an algebraically closed field, and we consider clusters on S.

5.1. **Definition.** Let K be a cluster on S and (Q, G) a pair such that $G \subset S$ is a nonsingular curve and Q is a minimal element of K satisfying $Q \in G$. Let $\preceq$ be a total order on K extending the natural order $\leq$, and write $K = \{P_1, \dots, P_n\}$ where $P_1 \prec \cdots \prec P_n$. We say that $(K, \preceq)$ is (Q, G)-*exhaustive* if, for each $i \in \{1, \dots, n\}$ satisfying $P_i \geq Q$, the conditions (a) and (b) below are satisfied.

We introduce the notation which is needed for stating these conditions. Consider the subcluster $K_i = \{P_1, \dots, P_i\}$ of K (where i is such that $P_i \geq Q$) and factor π_K as $S_K \xrightarrow{\pi_{K \setminus K_i}} S_{K_i} \xrightarrow{\pi_{K_i}} S$. Let $\mathcal{G}_i$ be the dual graph of $\pi_{K_i}^{-1}(G)$ in S_{K_i}; note that $\mathcal{G}_i$ is a tree and that $\widetilde{E}_{P_i}^{K_i}$ and $\widetilde{G}^{K_i}$ are distinct vertices of it; let

$$\overset{\widetilde{E}_{P_i}^{K_i}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{C_1}{\bullet}\!\!-\!\!\!-\ \cdots\ -\!\!\!-\!\!\overset{C_s}{\bullet} \quad \text{(where } C_s = \widetilde{G}^{K_i} \text{ and } s \geq 1) \tag{5}$$

be the unique simple path in $\mathcal{G}_i$ from $\widetilde{E}_{P_i}^{K_i}$ to $\widetilde{G}^{K_i}$. Then the conditions that are required to hold are the following:

(a) If $s \geq 2$ then $C_{s-1} \cap C_s \cap \operatorname{cent}(\pi_{K \setminus K_i}) = \varnothing$;
(b) if $s > 2$ then $(C_2 \cup \cdots \cup C_{s-1}) \cap \operatorname{cent}(\pi_{K \setminus K_i}) = \varnothing$.

5.2. **Remark.** Let K be a cluster on S, $G \subset S$ a nonsingular curve, Q a minimal element of K satisfying $Q \in G$ and $\preceq$ a total order on K extending the natural order. Also consider the subcluster $K' = \{ x \in K \mid x \geq Q \}$ of K and the restriction $\preceq'$ of $\preceq$ to K'. Then $(K, \preceq)$ is (Q, G)-exhaustive if and only if $(K', \preceq')$ is (Q, G)-exhaustive.

5.3. **Lemma.** *Let K be a cluster on S, let $Q_1, \dots, Q_r$ be distinct minimal elements of K and $G_1, \dots, G_r$ nonsingular curves on S such that $Q_i \in G_i$ for all i (where $G_1, \dots, G_r$ are not necessarily distinct). Then there exists a total order $\preceq$ on K which extends the natural order and such that $(K, \preceq)$ is simultaneously (Q_i, G_i)-exhaustive for all*

$i = 1, \dots, r$. Moreover, given an arbitrary total order $\preceq_0$ on the set $\{Q_1, \dots, Q_r\}$, we can choose $\preceq$ so that its restriction to $\{Q_1, \dots, Q_r\}$ be $\preceq_0$.

Proof. We first prove the following special case:

(6) *Let K be a cluster on S which has a unique minimal element Q, and let $G \subset S$ be a nonsingular curve such that $Q \in G$. Then there exists a total order $\preceq$ on K which extends the natural order and such that $(K, \preceq)$ is (Q, G)-exhaustive.*

Let K' be any subcluster of K such that $K' \neq K$. Factor π_K as

$$S_K \xrightarrow{\pi_{K \setminus K'}} S_{K'} \xrightarrow{\pi_{K'}} S$$

and let D be the unique SNC-divisor of $S_{K'}$ such that $\pi_{K'}^{-1}(G) = \operatorname{supp}(D)$. Note that the dual graph $\mathcal{G} = \mathcal{G}(S_{K'}, D)$ of D in $S_{K'}$ is a tree and consider the vertex $\widetilde{G}^{K'}$ of $\mathcal{G}$. If D_i is an irreducible component of D then let $d(D_i)$ be the distance, in the tree $\mathcal{G}$, between the vertices $\widetilde{G}^{K'}$ and D_i (i.e., the length of the unique simple path in $\mathcal{G}$ going from $\widetilde{G}^{K'}$ to D_i.) If $x \in D$ then let $D_1, \dots, D_n$ be the distinct irreducible components of D such that $x \in D_i$ (so $n = 1$ or 2) and set

$$f(x) = \big(\min \{ d(D_i) \mid 1 \le i \le n \},\ \max \{ d(D_i) \mid 1 \le i \le n \} \big).$$

This defines a set map $f : D \to \mathbb{N}^2$. Let $\mathbb{N}^2$ be ordered by the lexicographic order and define a strict partial order $\lhd$ on the set D by stipulating that

$$\text{for any } x, y \in D, \quad x \lhd y \iff f(x) <_{\text{lex}} f(y).$$

Note that $\operatorname{cent}(\pi_{K \setminus K'})$ is a nonempty finite set of points of D; by a *satellite* of K', we mean a minimal element of $\big(\operatorname{cent}(\pi_{K \setminus K'}), \lhd \big)$.[7] We stress that if K' is any subcluster of K such that $K' \neq K$ then there exists at least one satellite P of K', and for any such P, $K' \cup \{P\}$ is a subcluster of K. It follows that there exists at least one sequence

$$\varnothing \subset \{P_1\} \subset \{P_1, P_2\} \subset \cdots \subset \{P_1, \dots, P_n\} = K$$

of subclusters of K such that, for each $i \in \{1, \dots, n\}$, P_i is a satellite of $\{P_1, \dots, P_{i-1}\}$ (in particular $P_1 = Q$ is the unique satellite of $\varnothing$). Define a total order $\preceq$ on K by $P_1 \prec \cdots \prec P_n$ and note that $\preceq$ extends the natural order.

We claim that $(K, \preceq)$ is (Q, G)-exhaustive. To see this, fix $i \in \{1, \dots, n\}$ and let us verify that conditions (a) and (b) of 5.1 are satisfied for $P_i \in K$. Consider the factorization

$$S_K = S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_1} S_0 = S$$

of $\pi_K : S_K \to S$, where $\pi_j : S_j \to S_{j-1}$ is the blowing-up of S_{j-1} at P_j. Then

$$\begin{tikzcd} S_n \arrow[r, "\pi_{K \setminus K_i}"'] \arrow[rr, bend left, "\pi_{K \setminus K_{i-1}}"] & S_i \arrow[r, "\pi_i"'] \arrow[rr, bend left, "\pi_{K_i}"] & S_{i-1} \arrow[r, "\pi_{K_{i-1}}"'] & S_0 \end{tikzcd}$$

[7] In other words, the elements $P \in \operatorname{cent}(\pi_{K \setminus K'})$ satisfying $f(P) = \min \{ f(x) \mid x \in \operatorname{cent}(\pi_{K \setminus K'}) \}$ are the satellites of K'.

where $K_{i-1} = \{P_1, \dots, P_{i-1}\}$, $K_i = \{P_1, \dots, P_{i-1}, P_i\}$ and P_i is a satellite of K_{i-1}. This means that P_i is a minimal element of $(M, \lhd)$, where we define $M = \operatorname{cent}(\pi_{K \setminus K_{i-1}})$.

Let the notation be as in (5). There is nothing to prove if $s = 1$, so assume that $s \geq 2$; note that $\pi_i(C_1), \dots, \pi_i(C_s)$ are distinct curves on S_{i-1}. If some point $x \in C_2 \cup \dots \cup C_s$ belongs to $\operatorname{cent}(\pi_{K \setminus K_i})$ then $y = \pi_i(x)$ must lie on $\pi_i(C_2) \cup \dots \cup \pi_i(C_s)$ and must be an element of M; as $P_i \in \pi_i(C_1) \setminus \big(\pi_i(C_2) \cup \dots \cup \pi_i(C_s)\big)$, the definition of the relation $\lhd$ on M implies that $y \lhd P_i$, which contradicts the fact that P_i is a satellite of K_{i-1}. So, such a point x does not exist and consequently $(C_2 \cup \dots \cup C_s) \cap \operatorname{cent}(\pi_{K \setminus K_i}) = \varnothing$. It follows that (a) and (b) hold for P_i. So $(K, \preceq)$ is (Q, G)-exhaustive and (6) is proved.

Now suppose that K, $Q_1, \dots, Q_r$ and $G_1, \dots, G_r$ satisfy the hypothesis of 5.3. For each $i \in \{1, \dots, r\}$, let $K_i = \big\{ x \in K \mid x \geq Q_i \big\}$; also define $K_{r+1} = K \setminus (\bigcup_{i=1}^{r} K_i)$. Then $K = \bigcup_{i=1}^{r+1} K_i$ where $K_1, \dots, K_{r+1}$ are pairwise disjoint subclusters of K and, for each $i \leq r$, K_i has exactly one minimal element Q_i. It follows from (6) that, for each $i \in \{1, \dots, r\}$, there exists a total order $\preceq_i$ on K_i which extends the natural order and such that $(K_i, \preceq_i)$ is (Q_i, G_i)-exhaustive; let also $\preceq_{r+1}$ be any total order on K_{r+1} which extends the natural order, and let $\preceq_0$ be an arbitrary total order on the set $K_0 = \{Q_1, \dots, Q_r\}$. Choose a total order $\preceq$ on K such that, for each $i \in \{0, 1, \dots, r+1\}$, the restriction of $\preceq$ to K_i is $\preceq_i$. Then $\preceq$ extends the natural order on K, because if $x \in K_i$, $y \in K_j$ and $1 \leq i < j$, then x, y are not comparable by natural order. By 5.2, $(K, \preceq)$ is (Q_i, G_i)-exhaustive for each $i \in \{1, \dots, r\}$. $\square$

Recall from the introduction of section 4 that S^* is the set of "points over S."

5.4. **Definition.** Any nonsingular curve $G \subset S$ determines a map $T_G : S^* \to \mathfrak{T}$ as follows. Given $P \in S^*$ we consider the set $K_P = \big\{ x \in S^* \mid x \leq P \big\}$, which is a nonempty concentric cluster over S, and the unique minimal element Q of K_P. If $Q \in G$ then (S, K_P, G) satisfies condition $(*)$ of 4.13, so the tableau $T(S, K_P, G) \in \mathfrak{T}$ is defined; we define $T_G(P) = T(S, K_P, G)$ in this case. If $Q \notin G$ then set $T_G(P) = \mathbb{1}$ (the empty tableau). Note that $T_G(P) \neq \mathbb{1}$ if and only if $Q \in G$.

5.5. **Proposition.** *Let (K, G, P_1, Z) be such that K is a nonempty cluster on S, $G \subset S$ is a nonsingular curve, P_1 is a minimal element of K such that $P_1 \in G$, and $Z \subset S_K$ is a (possibly empty) finite union of curves satisfying:*

(i) *Z is a proper subset of $\pi_K^{-1}(P_1)$*
(ii) *the dual graph of $\mathcal{L} = \widetilde{G}^K \cup Z$ in S_K is a linear chain*
(iii) *$\widetilde{E}_{P_1}^K$ is either included in Z or disjoint from Z*
(iv) *P_1 is not a maximal element of K_G*
(v) *each irreducible component C of Z satisfies $(C^2)_{S_K} \leq -2$.*

Then there exists $P \in K$ satisfying $P \geq P_1$, $\widetilde{E}_P^K \cap \mathcal{L} \neq \varnothing$ and $\widetilde{E}_P^K \not\subseteq \mathcal{L}$, and such that the tableau $T_G(P)$ is one of the following:

(a) $T_G(P) = \binom{p}{c}$ *for some p, c such that $1 \leq p < c$*
(b) $T_G(P) = \left(\begin{smallmatrix} p & 1 \\ c & N \end{smallmatrix}\right)$ *for some p, c, N such that $1 \leq p < c$ and $N \geq 1$.*

Proof. First consider the case where $Z = \varnothing$. Let P be a maximal element of $\big\{ x \in K_G \mid x \geq P_1 \big\}$. Then $P \geq P_1$ and P is a maximal element of K_G; consequently $\widetilde{E}_P^K \cap \mathcal{L} \neq \varnothing$, $\widetilde{E}_P^K \nsubseteq \mathcal{L}$ and $T_G(P) = T(S, K_P, G) = \binom{1}{c}$ for some $c \geq 1$. If $c = 1$ then $P = P_1$, which contradicts (iv), so in fact $c > 1$ and $T_G(P)$ is of the form displayed in (a).

From now-on, assume that $Z \neq \varnothing$. As Z is a nonempty proper subset of $\pi_K^{-1}(P_1)$, the set

$$A = \big\{ P \in K \mid \widetilde{E}_P^K \cap Z \neq \varnothing \text{ and } \widetilde{E}_P^K \nsubseteq Z \big\} \tag{7}$$

is nonempty. By 5.3, we may choose a total order $\preceq$ on K which extends the natural order $\leq$, such that $(K, \preceq)$ is (P_1, G)-exhaustive, and such that P_1 is the least element of $(K, \preceq)$; write $K = \{P_1, \dots, P_n\}$, $P_1 \prec \dots \prec P_n$. Let P be the least element of $(A, \preceq)$. Then P satisfies $P \geq P_1$, $\widetilde{E}_P^K \cap \mathcal{L} \neq \varnothing$ and $\widetilde{E}_P^K \nsubseteq \mathcal{L}$.

We claim that $T_G(P)$ is as required by the Proposition, i.e., satisfies (a) or (b).

This claim is clear if $P \in K_G$. Indeed, we then have $T_G(P) = \binom{1}{c}$ for some $c \geq 1$, and if $c = 1$ then $P = P_1$, so $P_1 \in A$, which contradicts (iii); so in fact $c > 1$ and $T_G(P)$ is of the form displayed in (a). So from now-on we may assume that

$$P \notin K_G. \tag{8}$$

Let i be such that $P = P_i$ and define

$$K_P = \big\{ Q \in K \mid Q \leq P \big\} \quad \text{and} \quad K_i = \{P_1, \dots, P_i\} = \big\{ Q \in K \mid Q \preceq P \big\}.$$

Then $K_P \subseteq K_i$ are subclusters of K, so π_K factors as in the following diagram

$$\begin{tikzcd} S_K \arrow[d, "\pi_{K\setminus K_i}"'] \arrow[ddrr, "\pi_K"] & & \\ S_{K_i} \arrow[dr, "\pi_{K_i\setminus K_P}"'] \arrow[drr, "\pi_{K_i}"] & & \\ & S_{K_P} \arrow[r, "\pi_{K_P}"'] & S \end{tikzcd} \qquad \begin{aligned} \mathcal{G} &= \mathcal{G}\big(S_K,\, \pi_K^{-1}(G)\big) \\ \mathcal{G}_i &= \mathcal{G}\big(S_{K_i},\, \pi_{K_i}^{-1}(G)\big) \\ \mathcal{G}_P &= \mathcal{G}\big(S_{K_P},\, \pi_{K_P}^{-1}(G)\big) \end{aligned} \tag{9}$$

(note that $K_i \setminus K_P$ is a cluster on S_{K_P} and $K \setminus K_i$ is a cluster on S_{K_i}) where we also define the dual graphs $\mathcal{G}$, $\mathcal{G}_i$ and $\mathcal{G}_P$ (which are in fact trees). Let

$$\gamma_i = \overset{\widetilde{E}_P^{K_i}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{C_1}{\bullet}\!\!-\!\!\!- \;\cdots\; -\!\!\!-\!\!\overset{C_s}{\bullet} \quad (\text{where } C_s = \widetilde{G}^{K_i} \text{ and } s \geq 1) \tag{10}$$

be the unique simple path in $\mathcal{G}_i$ going from $\widetilde{E}_P^{K_i}$ to $\widetilde{G}^{K_i}$. Observe that if $s = 1$ then $\widetilde{E}_P^{K_i}$ meets $\widetilde{G}^{K_i}$ in S_{K_i}, which contradicts (8); so:

$$s \geq 2. \tag{11}$$

We claim that if $s > 2$ then the following hold:

$$\text{For each } j \in \{2, \dots, s-1\},\ (C_j^2)_{S_{K_i}} \leq -2; \tag{12}$$

$$\text{for each } j \in \{2, \dots, s-1\},\ C_j \text{ is not a branch point of } \mathcal{G}_i. \tag{13}$$

Indeed, consider any $j \in \{2, \dots, s-1\}$. Observe that the dual graph of $\mathcal{L}$ in S_K is connected and is a subgraph of the tree $\mathcal{G}$; as $\widetilde{G}^K \subseteq \mathcal{L}$ and $\widetilde{E}_P^K \cap \mathcal{L} \neq \varnothing$, it follows that the simple path γ in $\mathcal{G}$ going from $\widetilde{E}_P^K$ to $\widetilde{G}^K$ is included in $\mathcal{L}$, except for its initial vertex $\widetilde{E}_P^K$. As C_{j-1}, C_j, C_{j+1} are vertices of γ_i, it follows that $\widetilde{C}_{j-1}^K, \widetilde{C}_j^K, \widetilde{C}_{j+1}^K$ are vertices of γ and consequently:

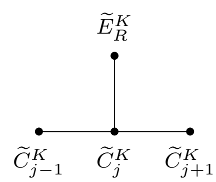
$$\widetilde{C}_{j-1}^K \cup \widetilde{C}_j^K \cup \widetilde{C}_{j+1}^K \subseteq \mathcal{L} \quad \text{and} \quad \widetilde{C}_j^K \subseteq Z.$$

Because $(K, \preceq)$ is (P_1, G)-exhaustive and $P \geq P_1$, the center of $\pi_{K \setminus K_i} : S_K \to S_{K_i}$ is disjoint from $C_2 \cup \dots \cup C_{s-1}$ (this is 5.1(b)); so:

$$\operatorname{cent}(\pi_{K \setminus K_i}) \cap C_j = \varnothing.$$

In particular, $(C_j^2)_{K_i}$ is equal to the self-intersection number of $\widetilde{C}_j^K$ in S_K, which is at most (-2) by (v) (because we noted that $\widetilde{C}_j^K \subseteq Z$). So (12) is proved.

We prove (13) by contradiction: suppose that $j \in \{2, \dots, s-1\}$ is such that C_j is a branch point of $\mathcal{G}_i$. Then C_j has a neighbor $\widetilde{E}_R^{K_i}$ (in $\mathcal{G}_i$) which does not belong to γ_i; note that $R \in K_i$ and $R \neq P = P_i$, so $R \prec P$. We observed in the above paragraph that $\operatorname{cent}(\pi_{K \setminus K_i}) \cap C_j = \varnothing$; as C_j meets each one of $\widetilde{E}_R^{K_i}$, C_{j-1} and C_{j+1} in S_{K_i}, it follows that $\widetilde{C}_j^K$ still meets each one of $\widetilde{E}_R^K$, $\widetilde{C}_{j-1}^K$ and $\widetilde{C}_{j+1}^K$ in S_K. So we have the following subgraph of $\mathcal{G}$:



Since $\widetilde{C}_{j-1}^K \cup \widetilde{C}_j^K \cup \widetilde{C}_{j+1}^K \subseteq \mathcal{L}$ and $\mathcal{L}$ does not have branch points, $\widetilde{E}_R^K \nsubseteq \mathcal{L}$ and hence $\widetilde{E}_R^K \nsubseteq Z$; on the other hand, we have $\widetilde{E}_R^K \cap Z \neq \varnothing$, because $\widetilde{E}_R^K \cap \widetilde{C}_j^K \neq \varnothing$; so $R \in A$ (cf. (7)). We already observed that $R \prec P$, so this contradicts the fact that P is the least element of $(A, \preceq)$. This proves (13).

Now consider the unique simple path γ_P in $\mathcal{G}_P$ going from $\widetilde{E}_P^{K_P}$ to $\widetilde{G}^{K_P}$:

$$\gamma_P = \overset{\widetilde{E}_P^{K_P}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{D_1}{\bullet}\!\!-\!\!\!-\ \cdots\ \longrightarrow\!\!\overset{D_r}{\bullet} \quad \text{(where } D_r = \widetilde{G}^{K_P} \text{ and } r \geq 1\text{).} \tag{14}$$

Then

$$r \geq 2, \tag{15}$$

for otherwise we would have a contradiction with (8). We claim that if $r > 2$ then:

(16) For each $j \in \{2, \dots, r-1\}$, D_j is not a branch point of $\mathcal{G}_P$.

Indeed, suppose that $j \in \{2, \dots, r-1\}$ is such that D_j is a branch point of $\mathcal{G}_P$. Then the strict transform of D_j via $\pi_{K_i \setminus K_P} : S_{K_i} \to S_{K_P}$ is a branch point of $\mathcal{G}_i$ and is equal to C_{j_1} for some $j_1 \in \{2, \dots, s-1\}$; this contradicts (13), so (16) is proved. It follows from (16) that $T_G(P)$ is one of the following tableaux:[8]

(a′) $T_G(P) = \binom{1}{1}^\nu \binom{p}{c}$ for some ν, p, c such that $\nu \geq 0$ and $1 \leq p < c$
(b′) $T_G(P) = \binom{1}{1}^\nu \left(\begin{smallmatrix} p & 1 \\ c & N \end{smallmatrix}\right)$ for some ν, p, c, N such that $\nu \geq 0$, $1 \leq p < c$ and $N \geq 1$
(c′) $T_G(P) = \binom{1}{1}^\nu$ for some $\nu \geq 1$.

Indeed, if $T_G(P)$ is not one of them then $T_G(P) = \binom{1}{1}^\nu \binom{p}{c} T'$ where $\nu \geq 0$, $\binom{p}{c} \neq \binom{1}{1}$, $T' \in \mathfrak{T}$, and where T' is neither $\mathbb{1}$ nor of the form $\binom{1}{N}$. Recall that $T_G(P) = T(S, K_P, G)$ where K_P is concentric. Write $K_P = \{Q_1, \dots, Q_m\}$ where $P_1 = Q_1 < Q_2 < \dots < Q_m = P$, and factor $\pi_{K_P} : S_{K_P} \to S$ as

$$S_{K_P} = S_m \xrightarrow{\pi_m} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = S, \tag{17}$$

where $S_j \xrightarrow{\pi_j} S_{j-1}$ is the blowing-up of S_{j-1} at $Q_j \in S_{j-1}$. Refer to 4.13 and 4.14 for the following argument. As $T(S, K_P, G) = \binom{1}{1}^\nu \binom{p}{c} T'$, the blowings-up $\pi_1, \dots, \pi_{\nu+1}$ are sprouting, and at least one of $\pi_{\nu+2}, \dots, \pi_m$ is sprouting; let ℓ be the least element of $\{\nu+2, \dots, m\}$ such that π_ℓ is sprouting. Then π_{K_P} factors as

$$S_m \xrightarrow{\pi_{K''}} S_{\ell-1} \xrightarrow{\pi_{K'}} S_0,$$

where $K' = \{Q_1, \dots, Q_{\ell-1}\}$ and $K'' = \{Q_\ell, \dots, Q_m\}$, and we have $T(S, K', G) = \binom{1}{1}^\nu \binom{p}{c}$ and $T(S_{\ell-1}, K'', E_{\ell-1}) = T'$ (where E_j denotes the curve $\pi_j^{-1}(Q_j) \subset S_j$).

For $j \in \{0, \dots, m\}$, let $\mathcal{H}_j$ be the dual graph of $(\pi_1 \circ \dots \circ \pi_j)^{-1}(G)$ in S_j. As $\binom{p}{c} \neq \binom{1}{1}$, $E_{\ell-1}$ has two neighbors in $\mathcal{H}_{\ell-1}$ and consequently its strict transform in S_ℓ is a branch point of $\mathcal{H}_\ell$ (because π_ℓ is sprouting); so $\widetilde{E}_{\ell-1}^{K''}$ is a branch point of $\mathcal{H}_m$. As $T' \neq \mathbb{1}$, it follows that $K'' \neq \varnothing$, so E_m and $\widetilde{E}_{\ell-1}^{K''}$ are distinct vertices of $\mathcal{H}_m$. If these two vertices are neighbors in $\mathcal{H}_m$ then $T' = \binom{1}{N}$ for some $N \geq 1$, which contradicts our assumption; so E_m and $\widetilde{E}_{\ell-1}^{K''}$ are not neighbors. Also, our choice of ℓ implies that $\ell > 1$ and hence that $\widetilde{E}_{\ell-1}^{K''}$ is not the strict transform of G.

We have shown that $\mathcal{G}_P = \mathcal{H}_m$ has a branch point which is distinct from $\widetilde{G}^{K_P}$, distinct from $E_P = E_m$, and which is not a neighbor of E_P. As γ_P passes through every branch point of $\mathcal{G}_P$, it follows that this branch point is one of $D_2, \dots, D_{r-1}$, which contradicts (16).

This proves that $T_G(P)$ is one of the tableaux described in statements (a′–c′). To complete the proof of the Proposition, we have to show that the first column of $T_G(P)$ is not $\binom{1}{1}$. The following trivial fact will be used below:

(18) Let $\mathcal{H}'$ be a weighted tree obtained from a weighted tree $\mathcal{H}$ by a finite sequence of blowings-up. Suppose that $\overset{u}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{v}{\bullet}$ is a subgraph of $\mathcal{H}$ and that $\overset{u}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{w_1}{\bullet}\!\!-\!\!- \;\dots\; -\!\!-\!\!\overset{w_p}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{v}{\bullet}$ is a subgraph of $\mathcal{H}'$. If no w_j is a branch point of $\mathcal{H}'$ or has weight (-1) in $\mathcal{H}'$, then $p = 0$, i.e., u, v are neighbors in $\mathcal{H}'$.

[8] The notation uses the fact that the set $\mathfrak{T}$ of tableaux is a monoid; for instance $\binom{1}{1}^\nu$ is the tableau $\left(\begin{smallmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{smallmatrix}\right)$ where there are ν columns.

Let us first prove:

(19) The point $D_{r-1} \cap D_r$ of S_{K_P} does not belong to $\operatorname{cent}(\pi_{K \setminus K_P})$.

Refer to (14) and (9) for the notation, and recall that $r \geq 2$ by (15) (so that D_{r-1} and E_P are distinct vertices in γ_P). As D_{r-1} is a vertex of γ_P distinct from E_P and $\widetilde{G}^{K_P}$, it follows that $\widetilde{D}^{K_i}_{r-1}$ is a vertex of γ_i distinct from $\widetilde{E}^{K_i}_P$ and $\widetilde{G}^{K_i}$; thus $\widetilde{D}^{K_i}_{r-1} = C_j$ for some $j \in \{1, \dots, s-1\}$. Now $\mathcal{G}_i$ is obtained from $\mathcal{G}_P$ by a sequence of blowings-up, $\overset{D_{r-1}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\widetilde{G}^{K_P}}{\bullet}$ is a subgraph of $\mathcal{G}_P$ and $\overset{\widetilde{D}^{K_i}_{r-1}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{C_{j+1}}{\bullet} \;\cdots\; \overset{C_{s-1}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\widetilde{G}^{K_i}}{\bullet}$ is a subgraph of $\mathcal{G}_i$; by (13), (12) and (18), it follows that $\widetilde{D}^{K_i}_{r-1}$ and $\widetilde{G}^{K_i}$ are neighbors in $\mathcal{G}_i$.

Consider the factorization $S_K \xrightarrow{\pi_{K \setminus K_i}} S_{K_i} \xrightarrow{\pi_{K_i \setminus K_P}} S_{K_P}$ of $\pi_{K \setminus K_P}$ and let $Q \in S_{K_P}$ be the point $D_{r-1} \cap D_r$. We showed that $\widetilde{D}^{K_i}_{r-1}$ and $\widetilde{G}^{K_i}$ meet in S_{K_i}, so $Q \notin \operatorname{cent}(\pi_{K_i \setminus K_P})$ and $\pi^{-1}_{K_i \setminus K_P}(Q)$ is the point $C_{s-1} \cap C_s$ in S_{K_i}; now this point does not belong to $\operatorname{cent}(\pi_{K \setminus K_i})$ because $(K, \preceq)$ is (P_1, G)-exhaustive, $P_i \geq P_1$ and $s \geq 2$. Consequently $\pi^{-1}_{K \setminus K_P}(Q)$ is a single point, which proves (19).

Consider the concentric subcluster $X = \{\, x \in K_G \mid x \geq P_1 \,\}$ of K. We claim:

(20) $$X \subseteq K_P.$$

Indeed, suppose that $X \nsubseteq K_P$ and consider the least element Q of $X \setminus K_P$. Then Q is a minimal element of $K \setminus K_P$ and hence a proper point of S_{K_P} and an element of $\operatorname{cent}(\pi_{K \setminus K_P})$. As $Q \in X$, Q must be the point $D_{r-1} \cap D_r$. This contradicts (19), so (20) is true.

Let us use again the notation $K_P = \{Q_1, \dots, Q_m\}$, $P_1 = Q_1 < \cdots < Q_m = P$, and factor π_{K_P} as in (17). By (iv), X contains at least two elements; so (20) implies that $Q_1, Q_2 \in X$, which implies that π_2 (see (17)) is a subdivisional blowing-up. Hence, the first column of $T(S, K_P, G)$ is not $\binom{1}{1}$. This completes the proof. $\square$

6. Construction of certain normal surfaces

The aim of this section is to define a set map $\mathcal{P} \to \overline{\mathcal{C}}$ and study some of its properties. We define $\overline{\mathcal{C}}$ in 6.1, $\mathcal{P}$ in 6.3, and the map $\mathcal{P} \to \overline{\mathcal{C}}$ in 6.5. The map $\mathcal{P} \to \overline{\mathcal{C}}$ will serve as a framework for studying a certain class of surfaces.

All varieties are over an algebraically closed field $\mathbf{k}$ of characteristic zero.

6.1. **Notation.** Let $\mathcal{C}$ be the set of pairs (U, ρ) where U is a normal surface which is connected at infinity, $\rho : U \to V$ is a surjective morphism whose general fiber is an affine line, and V is a curve isomorphic to an open subset W of $\mathbb{P}^1$ such that $W \neq \varnothing$ and $W \neq \mathbb{P}^1$. Note that the fact that ρ exists implies that U is also rational. Elements (U, ρ) and (U', ρ') of $\mathcal{C}$ (where $\rho : U \to V$ and $\rho' : U' \to V'$) are said to be *equivalent* if there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\cong"] \arrow[d, "\rho"'] & U' \arrow[d, "\rho'"] \\ V \arrow[r, "\cong"] & V' \end{tikzcd}$$

where the horizontal arrows are isomorphisms of varieties. The set of equivalence classes is denoted $\overline{\mathcal{C}}$ and the equivalence class of $(U,\rho) \in \mathcal{C}$ is denoted $[U,\rho] \in \overline{\mathcal{C}}$.

6.2. **Notations.** Until the end of section 6, let $S, \mathbb{L}, \infty, F, \Delta$ be the following objects:

- $S = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{L} = \big\{ \{x\} \times \mathbb{P}^1 \mid x \in \mathbb{P}^1 \big\}$ (so $\mathbb{L}$ is a pencil on S);
- choose a point of $\mathbb{P}^1$ and call it "∞";
- $F = \{\infty\} \times \mathbb{P}^1$ (so $F \in \mathbb{L}$);
- $\Delta = \mathbb{P}^1 \times \{\infty\}$ (so Δ is a section of $\mathbb{L}$ and $(\Delta^2)_S = 0$).

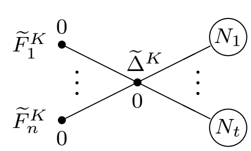
6.3. **Definition.** Let $\mathcal{P}$ be the set of pairs (K, B) satisfying:

- K is a cluster on S all of whose minimal elements are points of $S \setminus (F \cup \Delta)$;
- B is the support of a divisor on S_K (where $\pi_K : S_K \to S$ denotes the blowing-up of S along K) and satisfies $\widetilde{F}^K \cup \widetilde{\Delta}^K \subseteq B \subseteq \pi_K^{-1}(F \cup \Delta \cup G_1 \cup \cdots \cup G_s)$, for some finite subset $\{G_1, \dots, G_s\}$ of $\mathbb{L}$;
- each irreducible component C of B satisfies $(C^2)_{S_K} \neq -1$.

For each $(K,B) \in \mathcal{P}$ we write $B = B_\infty \cup \mathcal{E}$, where B_∞ is the connected component of B which contains $\widetilde{F}^K \cup \widetilde{\Delta}^K$ and $\mathcal{E}$ is the union of the other connected components of B. Observe that $\mathcal{E} \subseteq \operatorname{exc}(\pi_K)$ and that each irreducible component E of $\mathcal{E}$ satisfies $(E^2)_{S_K} \leq -2$.

6.4. **Remark.** Let $(K,B) \in \mathcal{P}$ and let $F_1, \dots, F_n, C_1, \dots, C_t$ denote the distinct elements of $\big\{ G \in \mathbb{L} \mid \widetilde{G}^K \subseteq B \big\}$, where $F_i \cap \min K = \varnothing$ and $C_i \cap \min K \neq \varnothing$. Then $n \geq 1$ (because $F \in \{F_1, \dots, F_n\}$), $t \geq 0$, and the dual graph $\mathcal{G}(S_K, B_\infty)$ of B_∞ in S_K has the form

(21)



$(n \geq 1,\ t \geq 0)$

where $N_1, \dots, N_t$ are branches in which every vertex has weight ≤ -2 and $\widetilde{C}_i^K$ is the vertex of N_i which is adjacent to $\widetilde{\Delta}^K$.

6.5. **Definition.** We proceed to define a set map from $\mathcal{P}$ to $\overline{\mathcal{C}}$. Let $(K,B) \in \mathcal{P}$, and write $B = B_\infty \cup \mathcal{E}$ as in 6.3. As $\mathcal{E} \subseteq \operatorname{exc}(\pi_K)$, paragraph 3.8 implies that $\mathcal{E}$ is algebraically contractible in the sense of 3.7 and that $\pi_K : S_K \to S$ factors as $S_K \xrightarrow{\overline{\sigma}} \overline{S_K} \xrightarrow{\overline{\pi}} S$, where $\overline{\sigma}$ is the contraction of $\mathcal{E}$ and $\overline{\pi}$ is a proper birational morphism. Note that $\overline{\sigma}(\mathcal{E})$ is exactly the singular locus of the normal complete surface $\overline{S_K}$ (indeed, each irreducible component E of $\mathcal{E}$ satisfies $(E^2)_{S_K} \leq -2$; so, for each connected component

$\mathcal{E}_i$ of $\mathcal{E}$, the point $\overline{\sigma}(\mathcal{E}_i)$ must be singular). Let $p_1 : S = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the first projection and note that $\mathbb{L}$ (cf. 6.2) is the set of fibers of p_1. We have:

$$\begin{tikzcd} S_K \arrow[rr, "\pi_K"] \arrow[dr, "\overline{\sigma}"] & & S \arrow[dr, "p_1"] & \\ & \overline{S_K} \arrow[ur, "\overline{\pi}"'] & & \mathbb{P}^1 \end{tikzcd}$$

Consider the set $\Gamma_{(K,B)}$ of pairs (U, ρ) satisfying:

6.5.1. *U is a surface, $\rho : U \to V$ is a surjective morphism, and there exist open immersions $U \hookrightarrow \overline{S_K}$ and $V \hookrightarrow \mathbb{P}^1$ such that*

(1) *the diagram* $\begin{tikzcd} \overline{S_K} \arrow[r, "\overline{\pi}"] & S \arrow[r, "p_1"] & \mathbb{P}^1 \\ U \arrow[u, hook] \arrow[rr, "\rho"'] & & V \arrow[u, hook] \end{tikzcd}$ *is commutative;*

(2) *the image of $U \hookrightarrow \overline{S_K}$ is equal to the complement of $\overline{\sigma}(B_\infty)$ in $\overline{S_K}$;*

(3) *the image of $V \hookrightarrow \mathbb{P}^1$ does not contain the point ∞ of $\mathbb{P}^1$.*

We claim that $\Gamma_{(K,B)} \in \overline{\mathcal{C}}$. To see this, let us first check that $\Gamma_{(K,B)} \neq \varnothing$. Indeed, let $U \subset \overline{S_K}$ be the complement of $\overline{\sigma}(B_\infty)$ in $\overline{S_K}$, and let $V \subset \mathbb{P}^1$ be the image of U via $p_1 \circ \overline{\pi}$. Then restricting $p_1 \circ \overline{\pi}$ gives a surjective morphism $\rho : U \to V$ which makes diagram 6.5.1(1) commute. Since the inverse image of $\infty \in \mathbb{P}^1$ by $p_1 \circ \overline{\pi}$ is $\overline{\pi}^{-1}(F) = \overline{\sigma}(\widetilde{F}^K)$, which is included in $\overline{\sigma}(B_\infty)$ and hence disjoint from U, we have $\infty \notin V$. So $(U, \rho) \in \Gamma_{(K,B)}$.

We also note that if (U, ρ) is any element of $\Gamma_{(K,B)}$ then U is normal and connected at infinity (because $\overline{S_K}$ is normal and $\overline{\sigma}(B_\infty)$ is connected), and the general fiber of ρ is an affine line (because if P is a general point of $\mathbb{P}^1$ then $(p_1 \circ \overline{\pi})^{-1}(P)$ is a projective line in $\overline{S_K}$ which meets $\overline{\sigma}(B_\infty)$ in one point). This shows that $\Gamma_{(K,B)} \subseteq \mathcal{C}$. It is clear that $\Gamma_{(K,B)}$ is an equivalence class, i.e., an element of $\overline{\mathcal{C}}$, and that $(K, B) \mapsto \Gamma_{(K,B)}$ defines a set map from $\mathcal{P}$ to $\overline{\mathcal{C}}$.

Let the notation $(U_{(K,B)}, \rho_{(K,B)})$ stand for an arbitrary element of $\Gamma_{(K,B)}$. So the set map that we have just defined is

$$\mathcal{P} \to \overline{\mathcal{C}}, \qquad (K, B) \mapsto [U_{(K,B)}, \rho_{(K,B)}].$$

If it is convenient, we may choose $U_{(K,B)}$ to be the complement of $\overline{\sigma}(B_\infty)$ in $\overline{S_K}$.

6.6. **Lemma.** *Let $(K, B) \in \mathcal{P}$ and let the notation be as in* 6.5.

(a) *There is an isomorphism of surfaces $U_{(K,B)} \setminus \operatorname{Sing}(U_{(K,B)}) \cong S_K \setminus B$.*

(b) *The dual graph $\mathcal{G}(S_K, B_\infty)$ is a minimal element of $\mathcal{G}_\infty[U_{(K,B)}]$* (cf. 3.5).

(c) *Consider the morphism $\hat{\sigma} : S_K \setminus B_\infty \to \overline{S_K} \setminus \overline{\sigma}(B_\infty) \cong U_{(K,B)}$ obtained by restricting the morphism $\overline{\sigma}$ of* 6.5. *Then $\hat{\sigma}$ is the minimal SNC-resolution of singularities of $U_{(K,B)}$* (cf. 3.6) *and $\hat{\sigma}^{-1}(\operatorname{Sing} U_{(K,B)}) = \mathcal{E}$. So the connected components of $\mathcal{E}$ are the resolution loci of the singular points of $U_{(K,B)}$.*

Proof. Refer to 6.5 and take $U = U_{(K,B)}$ to be the complement of $\overline{\sigma}(B_\infty)$ in $\overline{S_K}$. Then $\overline{\sigma}$ restricts to an isomorphism from $S_K \setminus B$ to $U \setminus \operatorname{Sing}(U)$, so (a) is proved.

As $\overline{\sigma}$ also restricts to an isomorphism from a neighborhood of B_∞ to a neighborhood of $\overline{\sigma}(B_\infty)$, we see that $\overline{\sigma}(B_\infty)$ is a connected SNC-divisor of $\overline{S_K}$ whose dual graph can be identified with $\mathcal{G}(S_K, B_\infty)$. As $\overline{\sigma}(B_\infty)$ is the complement of U, $\mathcal{G}(S_K, B_\infty) \in \mathcal{G}_\infty[U]$. From (21), we see that no vertex of $\mathcal{G}(S_K, B_\infty)$ has weight (-1), so $\mathcal{G}(S_K, B_\infty)$ is a minimal element of $\mathcal{G}_\infty[U]$. So (b) is proved.

It is clear from 6.5 that $\overline{\sigma} : S_K \to \overline{S_K}$ is the minimal SNC-resolution of singularities of $\overline{S_K}$. Assertion (c) follows from this. $\square$

6.7. **Proposition.** *The set map $\mathcal{P} \to \overline{\mathcal{C}}$ defined in 6.5 is surjective.*

Proof. Let $(U, \rho) \in \mathcal{C}$. Here, $\rho : U \to V$ is a surjective morphism with general fiber $\mathbb{A}^1$, and V is isomorphic to an open subset W of $\mathbb{P}^1$ such that $W \neq \varnothing$ and $W \neq \mathbb{P}^1$.

Let $\sigma : \hat{U} \to U$ be a minimal SNC-resolution of singularities of U (cf. 3.6). Let $\hat{\rho} : \hat{U} \to V$ be the composite $\hat{U} \xrightarrow{\sigma} U \xrightarrow{\rho} V$ and note that $\hat{\rho}$ is a surjective morphism whose general fiber is $\mathbb{A}^1$. There exists a commutative diagram

$$\begin{tikzcd} \hat{U} \arrow[r, hook, "i"'] \arrow[d, "\hat{\rho}"'] & \overline{U} \arrow[d, "\overline{\rho}"] \\ V \arrow[r, hook, "j"] & \mathbb{P}^1 \end{tikzcd} \tag{22}$$

where the "$\hookrightarrow$" are open immersions, $\overline{U}$ is a nonsingular projective surface, $\overline{U} \setminus \hat{U}$ is the support of an SNC-divisor of $\overline{U}$ and $\overline{\rho}$ is a morphism. Here, when choosing the open immersion $j : V \hookrightarrow \mathbb{P}^1$, we make sure that $\infty \in \mathbb{P}^1 \setminus V$ (the point ∞ of $\mathbb{P}^1$ was fixed at the beginning of the section). Let Λ be the base-point-free pencil on $\overline{U}$ which corresponds to $\overline{\rho}$. As the general fiber of $\hat{\rho}$ is $\mathbb{A}^1$ and $\operatorname{char} \mathbf{k} = 0$, it follows that the general fiber of $\overline{\rho}$ is a $\mathbb{P}^1$ which meets $\overline{U} \setminus \hat{U}$ in one point. Consequently, exactly one irreducible component H of $\overline{U} \setminus \hat{U}$ is Λ-horizontal,[9] and $\overline{\rho}$ restricts to an isomorphism from H to $\mathbb{P}^1$. We summarize this as:

(23) Λ is a $\mathbb{P}^1$-ruling on $\overline{U}$, exactly one irreducible component H of $\overline{U} \setminus \hat{U}$ is Λ-horizontal, and H is a section of Λ.

So $\overline{U} \setminus \hat{U}$ is a tree of projective lines (by 3.13 and the fact that U is connected at infinity). By parts (a), (c) and (e) of 3.13, if C is a vertical component of $\overline{U} \setminus \hat{U}$ such that $(C^2)_{\overline{U}} = -1$ then C meets at most two other irreducible components of $\overline{U} \setminus \hat{U}$, and the contraction of C yields a new diagram (22) in which $\overline{U} \setminus \hat{U}$ has one less irreducible component. Consequently, any diagram (22) which minimizes the number of irreducible components of $\overline{U} \setminus \hat{U}$ satisfies the additional condition:

(24) No Λ-vertical component C of $\overline{U} \setminus \hat{U}$ satisfies $(C^2)_{\overline{U}} = -1$.

[9]A curve $C \subset \overline{U}$ is said to be Λ-*vertical* if it is included in the support of an element of Λ. If C is not Λ-vertical, we say that it is Λ-*horizontal.*

We choose such a diagram.

Since in (22) we arranged that $\infty \notin V$, $\overline{\rho}^{-1}(\infty)$ is entirely contained in $\overline{U} \setminus \hat{U}$. By (24), no irreducible component of $\overline{\rho}^{-1}(\infty)$ has self-intersection (-1); it follows (e.g. from part (d) of 3.13) that $\overline{\rho}^{-1}(\infty)$ is an irreducible curve; let us use the notation $F_\infty = \overline{\rho}^{-1}(\infty)$, then $F_\infty \in \Lambda$, so $(F_\infty)^2 = 0$. Moreover, the vertex F_∞ of the dual graph $\mathcal{G}(\overline{U}, \overline{U} \setminus \hat{U})$ has a unique neighbor in this graph, namely, H. Thus, by blowing-up $\overline{U}$ at a point Q of the curve F_∞ and then shrinking[10] the strict transform of F_∞, we may replace the diagram (22) by another one in which the self-intersection number of H has either increased by 1 (if $Q \in F_\infty \setminus H$) or decreased by 1 (if Q is the point $F_\infty \cap H$); moreover, this operation does not change the number of irreducible components of $\overline{U} \setminus \hat{U}$, so the new diagram still satisfies (24). It follows that we may choose a diagram (22) which satisfies (24) and in which we have $(H^2)_{\overline{U}} = 0$. We fix such a diagram until the end of the proof. Note:

(25) Each irreducible component C of $\overline{U} \setminus \hat{U}$ satisfies $(C^2)_{\overline{U}} \neq -1$.

Indeed, if C is an irreducible component of $\overline{U} \setminus \hat{U}$ then either $C = H$, in which case $(C^2)_{\overline{U}} = 0$, or C is Λ-vertical, in which case $(C^2)_{\overline{U}} \neq -1$ by (24). So (25) is true.

In view of (23) and of the fact that $(H^2)_{\overline{U}} = 0$, we may consider a birational morphism $\pi' : \overline{U} \to \mathbb{F}_0 = S$ as in 3.12, with $\mathrm{exc}(\pi')$ equal to the union of all Λ-vertical curves in $\overline{U}$ disjoint from H. Composing π', if necessary, with an automorphism of S, we arrange $\pi'_*(\Lambda) = \mathbb{L}$ and $\pi'(H) = \Delta$ (recall that the notations $S, \mathbb{L}, \Delta, F$ were fixed at the beginning of section 6). We claim that there exists an automorphism θ of $S = \mathbb{P}^1 \times \mathbb{P}^1$ such that the morphism $\pi = \theta \circ \pi' : \overline{U} \to S$ still satisfies $\pi_*(\Lambda) = \mathbb{L}$ and $\pi(H) = \Delta$, and moreover makes the diagram

(26)
$$\begin{tikzcd} \overline{U} \arrow[r, "\pi"] \arrow[d, "\overline{\rho}"'] & S \arrow[dl, "p_1"] \\ \mathbb{P}^1 & \end{tikzcd}$$

commute. Indeed, the condition $\pi'_*(\Lambda) = \mathbb{L}$ implies that the two morphisms $\overline{U} \overset{p_1 \circ \pi'}{\underset{\overline{\rho}}{\rightrightarrows}} \mathbb{P}^1$ determine the same pencil on $\overline{U}$ (namely, Λ), and hence that they differ by an automorphism θ_1 of $\mathbb{P}^1$ ($\theta_1 \circ p_1 \circ \pi' = \overline{\rho}$). Let $\theta_2 = \mathrm{id} : \mathbb{P}^1 \to \mathbb{P}^1$, then $\theta = (\theta_1, \theta_2)$ has the desired property. We have $\pi(F_\infty) = F$ by commutativity of (26), and $\mathrm{exc}(\pi)$ is of course equal to $\mathrm{exc}(\pi')$, i.e., is the union of all Λ-vertical curves in $\overline{U}$ disjoint from H.

Let K be the cluster on S such that π is the blowing-up of S along K, i.e., $\overline{U} \xrightarrow{\pi} S$ is the same as $S_K \xrightarrow{\pi_K} S$. Thus $\widetilde{F}^K = F_\infty$ and $\widetilde{\Delta}^K = H$.

Recall from 3.6 that the set $\mathcal{E} = \sigma^{-1}(\mathrm{Sing}\, U)$ is the support of an SNC-divisor of $\hat{U}$; in particular, it is a union of complete curves; so $\mathcal{E}$ is closed in $\overline{U}$ and is therefore the support of an SNC-divisor of $\overline{U}$. As $(\overline{U} \setminus \hat{U}) \cap \mathcal{E} = \varnothing$, the set $B = (\overline{U} \setminus \hat{U}) \cup \mathcal{E}$ is the

[10] By 3.13(e), shrinking the strict transform of F_∞ does yield a new diagram (22).

support of an SNC-divisor of $\overline{U}$. We claim that $(K,B) \in \mathcal{P}$. To see this, we have to verify the following conditions:

(i) all minimal elements of K are points of $S \setminus (F \cup \Delta)$;

(ii) $\widetilde{F}^K \cup \widetilde{\Delta}^K \subseteq B \subseteq \pi_K^{-1}(F \cup \Delta \cup G_1 \cup \cdots \cup G_s)$, for some finite subset $\{G_1, \dots, G_s\}$ of $\mathbb{L}$;

(iii) each irreducible component C of B satisfies $(C^2)_{S_K} \neq -1$.

The set of minimal elements of K is precisely the center of π. Since $\operatorname{exc}(\pi)$ is disjoint from $F_\infty \cup H$, it follows that the center of π is disjoint from $F \cup \Delta$. So (i) is clear.

The inclusion $\widetilde{F}^K \cup \widetilde{\Delta}^K \subseteq B$ is clear. To prove (ii), we have to show that if C is an irreducible component of B then $\pi(C) = \Delta$ or there exists $G \in \mathbb{L}$ such that $\pi(C) \subseteq G$. Note that this is clear if $C = H$ or $C \subseteq \operatorname{exc}(\pi)$. If $C \subseteq \mathcal{E}$ then C is a Λ-vertical curve in $\overline{U}$ disjoint from H (since it is disjoint from $\overline{U} \setminus \hat{U}$), so $C \subseteq \operatorname{exc}(\pi)$ and we are done in that case. So we may assume that $C \subseteq \overline{U} \setminus \hat{U}$, $C \neq H$, and $C \nsubseteq \operatorname{exc}(\pi)$; then C is Λ-vertical and $\pi(C)$ is a curve, so $\pi(C) = G$ for some $G \in \mathbb{L}$, and we are done proving (ii).

In the last paragraph we noted that $\mathcal{E} \subseteq \operatorname{exc}(\pi)$. It follows that each irreducible component C of $\mathcal{E}$ satisfies $(C^2)_{\overline{U}} \leq -1$, and that if $(C^2)_{\overline{U}} = -1$ then C is a contractible vertex of the dual graph of $\mathcal{E}$ in $\overline{U}$; such a vertex cannot exist by part (iii) of 3.6, so in fact we have $(C^2)_{\overline{U}} \leq -2$ for every irreducible component C of $\mathcal{E}$. So, to prove (iii), we may assume that C is an irreducible component of $\overline{U} \setminus \hat{U}$; then $(C^2)_{\overline{U}} \neq -1$ follows from (25). So (iii) is proved, and consequently $(K,B) \in \mathcal{P}$.

There remains to show that the elements (U, ρ) and $(U_{(K,B)}, \rho_{(K,B)})$ of $\mathcal{C}$ are equivalent. To see this, we follow the definition of $(U_{(K,B)}, \rho_{(K,B)})$ given in 6.5.

First note that the fact that U is connected at infinity implies that $\overline{U} \setminus \hat{U}$ is a connected component of B; so $B_\infty = \overline{U} \setminus \hat{U}$ and the "$\mathcal{E}$" of the present argument is equal to the "$\mathcal{E}$" defined in 6.3. Let $\overline{\sigma} : S_K \to \overline{S_K}$ be the contraction of $\mathcal{E}$, then (see 6.5) $\pi = \pi_K : S_K \to S$ factors as $S_K \xrightarrow{\overline{\sigma}} \overline{S_K} \xrightarrow{\overline{\pi}} S$, for some $\overline{\pi}$. Since σ and $\overline{\sigma}$ are the contractions of $\mathcal{E}$ in $\hat{U}$ and $\overline{U}$ respectively, and since $\hat{U}$ is an open subset of $\overline{U}$, there exists an open immersion $U \hookrightarrow \overline{S_K}$ which makes (I) a commutative square, in the following diagram:

$$
\begin{tikzcd}
\overline{U} = S_K \arrow[r, "\overline{\sigma}"] & \overline{S_K} \arrow[r, "\overline{\pi}"] & S \arrow[r, "p_1"] & \mathbb{P}^1 \\
\hat{U} \arrow[u, hook, "i"] \arrow[r, "\sigma"'] \arrow[ur, phantom, "\mathrm{(I)}"] & U \arrow[u, hook] \arrow[rr, "\rho"] \arrow[urr, phantom, "\mathrm{(II)}"] & & V \arrow[u, hook, "j"']
\end{tikzcd} \tag{27}
$$

Note that the image of $U \hookrightarrow \overline{S_K}$ is the complement of $\overline{\sigma}(B_\infty)$, and that the image of $V \hookrightarrow \mathbb{P}^1$ does not contain the point ∞; so, in order to prove that $[U, \rho] = [U_{(K,B)}, \rho_{(K,B)}]$, it suffices to verify that diagram (II) commutes, in (27) (compare (II) with 6.5.1(1)). Clearly, commutativity of (II) is a consequence of the following assertions:

(iv) σ is an epimorphism;

(v) the square (I) is commutative;

(vi) the "external square" (I,II) is commutative, i.e., $p_1 \circ \overline{\pi} \circ \overline{\sigma} \circ i = j \circ \rho \circ \sigma$.

In fact, only (vi) needs to be explained: commutativity of (26) gives $p_1 \circ \overline{\pi} \circ \overline{\sigma} = \overline{\rho}$, and $\rho \circ \sigma = \hat{\rho}$ by definition of $\hat{\rho}$; so (vi) is simply the fact that (22) is a commutative diagram.

Since (iv–vi) are true, (II) is commutative and hence $[U_{(K,B)}, \rho_{(K,B)}] = [U, \rho]$. $\square$

7. Properties of (K, B) and of $U_{(K,B)}$

Throughout this section, varieties are over an algebraically closed field $\mathbf{k}$ of characteristic zero, and $S, \mathbb{L}, \Delta, F$ are as in 6.2.

The purpose of section 6 is to define the map $\mathcal{P} \to \overline{\mathcal{C}}$, $(K, B) \mapsto [U_{(K,B)}, \rho_{(K,B)}]$, and to show that it is surjective (see 6.7). In the present section, our aim is to study how the properties of the surface $U_{(K,B)}$ are related to those of the data (K, B). We consider the following properties of $U_{(K,B)}$:

- $U_{(K,B)} \cong \mathbb{A}^2$ (in 7.1);
- $U_{(K,B)}$ is affine (in 7.4);
- $U_{(K,B)} \setminus \operatorname{Sing} U_{(K,B)}$ has trivial canonical class (in 7.11).

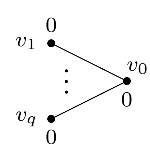
7.1. **Lemma.** *Let $(K, B) \in \mathcal{P}$.*

(a) *$K = \varnothing$ if and only if $U_{(K,B)} \cong V \times \mathbb{A}^1$ for some open subset V of $\mathbb{P}^1$ such that $V \neq \mathbb{P}^1$. Moreover, if these conditions hold then B has $|\mathbb{P}^1 \setminus V| + 1$ irreducible components and $\Delta \cup F \subseteq B$.*

(b) *$U_{(K,B)} \cong \mathbb{A}^2$ if and only if $(K, B) = (\varnothing, \Delta \cup F)$.*

Proof. (a) If $K = \varnothing$ then Definition 6.3 implies that $B = \Delta \cup F_1 \cup \dots \cup F_n$ (for some distinct $F_1, \dots, F_n \in \mathbb{L}$, where $F \in \{F_1, \dots, F_n\}$) and that $U_{(K,B)} = S \setminus B$. So $U_{(K,B)} \cong V \times \mathbb{A}^1$ where V is $\mathbb{P}^1$ minus n points, and B has $|\mathbb{P}^1 \setminus V| + 1$ components.

Conversely, suppose that $U_{(K,B)} \cong V \times \mathbb{A}^1$ where V is $\mathbb{P}^1$ minus q points, $q \geq 1$. Then

(28) 

is an element of $\mathcal{G}_\infty[U_{(K,B)}]$, and $\mathcal{G}(S_K, B_\infty)$ (which is pictured in (21)) is also an element of $\mathcal{G}_\infty[U_{(K,B)}]$, by 6.6. So (21) and (28) are equivalent weighted graphs, and consequently $t = 0$ in (21). This means that $B_\infty = \widetilde{\Delta}^K \cup \widetilde{F}_1^K \cup \dots \cup \widetilde{F}_n^K$ for some $F_1, \dots, F_n \in \mathbb{L}$, where $F_i \cap \min K = \varnothing$ for all i; consequently, $\operatorname{exc}(\pi_K) \subset S_K \setminus B_\infty$. The fact that $U_{(K,B)}$ is nonsingular implies that $\mathcal{E} = \varnothing$; so $B = B_\infty$ and $S_K \setminus B_\infty = U_{(K,B)}$, so $S_K \setminus B_\infty$ is affine and hence cannot contain a complete curve. Since $\operatorname{exc}(\pi_K) \subset S_K \setminus B_\infty$, it follows that $\operatorname{exc}(\pi_K) = \varnothing$, so $K = \varnothing$.

(b) If $(K, B) \in \mathcal{P}$ satisfies $U_{(K,B)} \cong \mathbb{A}^2$, then (a) implies that $K = \varnothing$, that B has 2 irreducible components, and that $\Delta \cup F \subseteq B$; so $B = \Delta \cup F$. The converse is trivial. $\square$

7.2. **Remark.** By 7.1, exactly one element (K,B) of $\mathcal{P}$ satisfies $U_{(K,B)} \cong \mathbb{A}^2$. So 6.7 implies:

> *up to equivalence* 6.1, $\mathbb{A}^2$ *admits exactly one surjective morphism* $\mathbb{A}^2 \to \mathbb{A}^1$ *with general fiber* $\mathbb{A}^1$.

This is "Rentschler's Theorem" [19]. (To recover Rentschler's formulation, one uses the well-known correspondence — cf. for instance 2.3 of [10] — between $\mathbb{A}^1$-fibrations and kernels of nonzero locally nilpotent derivations.)

7.3. **Notations.** (1) Given a cluster K on S, define

$$\mathbb{L}_{(K)} = \big\{\, G \in \mathbb{L} \mid G \text{ contains some minimal element of } K \,\big\}.$$

(2) Given $(K,B) \in \mathcal{P}$, define $K(B) = \big\{\, P \in K \mid \widetilde{E}_P^K \subseteq B \,\big\}$.

Regarding affineness of $U_{(K,B)}$, we have the following fact.

7.4. **Lemma.** *Let $(K,B) \in \mathcal{P}$. Then $U_{(K,B)}$ is affine if and only if*

$$(*) \qquad \text{for all } P \in K \setminus K(B), \quad \widetilde{E}_P^K \cap B_\infty \neq \varnothing.$$

Moreover, if $U_{(K,B)}$ is affine then

(a) *the surface $\overline{S}_K$ defined in* 6.5 *is projective;*

(b) *$\widetilde{G}^K \subseteq B$ for all $G \in \mathbb{L}_{(K)}$.*

Proof. Let the notation be as in 6.5 and take $U = U_{(K,B)}$ to be the complement of $\overline{\sigma}(B_\infty)$ in $\overline{S}_K$.

Consider $P \in K \setminus K(B)$. Then $\widetilde{E}_P^K \nsubseteq \mathcal{E}$, so $\overline{\sigma}(\widetilde{E}_P^K)$ is a complete curve in $\overline{S}_K$. Since $\overline{\sigma}$ restricts to an isomorphism from a neighborhood of B_∞ to a neighborhood of $\overline{\sigma}(B_\infty)$,

$$\overline{\sigma}(\widetilde{E}_P^K) \subset U \iff \widetilde{E}_P^K \cap B_\infty = \varnothing.$$

So it is clear that if U is affine then $(*)$ holds.

Conversely, suppose that $(*)$ holds. Let us first show that

$$(29) \qquad \text{any irreducible curve } \bar{C} \text{ in } \overline{S}_K \text{ meets } \overline{\sigma}(B_\infty).$$

Let C be the unique irreducible curve in S_K such that $\overline{\sigma}(C) = \bar{C}$; observe that $C \nsubseteq \mathcal{E}$ because $\overline{\sigma}(C)$ is one-dimensional. There are two cases. (i) If π_K maps C to a curve then $\pi_K(C) \cap (\Delta \cup F) \neq \varnothing$, so $C \cap (\widetilde{\Delta}^K \cup \widetilde{F}^K) \neq \varnothing$, so $\bar{C} \cap \overline{\sigma}(B_\infty) \neq \varnothing$. (ii) If π_K maps C to a point then $C = \widetilde{E}_P^K$ for some $P \in K$; either $P \in K(B)$, in which case $C \subseteq B$, so $C \subseteq B_\infty$, so $\bar{C} \cap \overline{\sigma}(B_\infty) \neq \varnothing$, or $P \in K \setminus K(B)$, in which case $\bar{C} \cap \overline{\sigma}(B_\infty) \neq \varnothing$ by $(*)$. This proves (29).

Let us say that a Weil divisor $D \in \operatorname{Div}(\overline{S}_K)$ is *positive* if D is effective, $D \neq 0$, $\operatorname{supp}(D)$ is included in the nonsingular locus of $\overline{S}_K$, and each irreducible component C of D satisfies $(C \cdot D) > 0$ in $\overline{S}_K$. Observe that if D is positive and C is an irreducible curve included in the nonsingular locus of $\overline{S}_K$ and satisfying $C \cap \operatorname{supp}(D) \neq \varnothing$ and $C \nsubseteq \operatorname{supp}(D)$, then $nD + C$ is positive for $n > 0$ large enough.

Consider the curves $\bar{F}, \bar{\Delta} \subset \overline{S}_K$ defined by $\bar{\Delta} = \overline{\sigma}(\widetilde{\Delta}^K)$ and $\bar{F} = \overline{\sigma}(\widetilde{F}^K)$. Then the divisor $\bar{\Delta} + \bar{F}$ is positive. In view of the observation made in the preceding paragraph and of the fact that $\overline{\sigma}(B_\infty)$ is connected, it follows that there exists a positive divisor D satisfying $\operatorname{supp}(D) = \overline{\sigma}(B_\infty)$. The fact that D is positive together with (29) imply:

$$(C \cdot D) > 0 \text{ in } \overline{S}_K, \text{ for all irreducible curves } C \subset \overline{S}_K.$$

So D is ample by Nakai's criterion. It follows that $U = \overline{S}_K \setminus \operatorname{supp}(D)$ is affine and that $\overline{S}_K$ is projective. So $(*)$ implies that $U_{(K,B)}$ is affine and that (a) is true.

Finally, suppose that $U_{(K,B)}$ is affine (or equivalently, that $(*)$ holds) and consider $G \in \mathbb{L}_{(K)}$. Choose a minimal element Q of K such that $Q \in G$, and a maximal element P of K such that $P \geq Q$. Then $P \in K \setminus K(B)$, so $\widetilde{E}_P^K \cap B_\infty \neq \varnothing$ by $(*)$, and this implies that $\widetilde{G}^K \subseteq B_\infty$. So if $U_{(K,B)}$ is affine then (b) holds. □

7.5. **Notation.** $\mathcal{P}_0 = \big\{ (K, B) \in \mathcal{P} \mid \text{the surface } S_K \setminus B \text{ has trivial canonical class} \big\}$.

Our next objective is to describe the set $\mathcal{P}_0$, and this is achieved in 7.11. We are interested in $\mathcal{P}_0$ because of:

7.6. **Lemma.** *For any $(K, B) \in \mathcal{P}$, there holds*

$$(K, B) \in \mathcal{P}_0 \iff U_{(K,B)} \setminus \operatorname{Sing} U_{(K,B)} \textit{ has trivial canonical class.}$$

Proof. Follows from 6.6. □

If T is a subset of a group $\mathbb{G}$, we write $\langle T \rangle$ for the subgroup of $\mathbb{G}$ generated by T.

7.7. **Lemma.** *Let $(K, B) \in \mathcal{P}$ and let $\preceq$ be a total order on K extending the natural order. Then $(K, B) \in \mathcal{P}_0$ if and only if*

$$\mathbf{Q}_{K(B)} \mathbf{1}_K \in \big\langle \big\{ \mathbf{Q}_{K(B)} \mathbf{1}_{K_G} \mid G \in \Gamma \big\} \big\rangle, \tag{30}$$

where:

- $\Gamma = \big\{ G \in \mathbb{L} \mid \widetilde{G}^K \subseteq B$ *and some minimal element of K lies on G* $\big\}$
- $K(B) = \big\{ P \in K \mid \widetilde{E}_P^K \subseteq B \big\}$
- K_G *is defined in* 4.9*;* $\mathbf{1}_K$ *and* $\mathbf{1}_{K_G}$ *are defined in* 4.16
- $\mathbf{Q} = \mathbf{Q}(K, \preceq)$ *and* $\mathbf{Q}_{K(B)}$ *are defined in* 4.10.

Proof. Let $(K, B) \in \mathcal{P}$ and let $\pi_K : S_K \to S$ be the blowing-up of S along K. Then $(K, B) \in \mathcal{P}_0$ if and only if $S_K \setminus B$ has trivial canonical class, if and only if

$$\kappa_{S_K} \in \mathbb{B}, \tag{31}$$

where $\kappa_{S_K} \in \operatorname{Cl}(S_K)$ denotes the canonical class of S_K and $\mathbb{B}$ denotes the subgroup of $\operatorname{Cl}(S_K)$ generated by the irreducible components of B. Note that[11]

$$\mathbb{B} = \big\langle \{\widetilde{F}^K, \widetilde{\Delta}^K\} \cup \big\{ \widetilde{G}^K \mid G \in \Gamma_1 \big\} \cup \big\{ \widetilde{E}_P^K \mid P \in K(B) \big\} \big\rangle,$$

[11]We use the same notation for a divisor $D \in \operatorname{Div}(S_K)$ and for its linear equivalence class $D \in \operatorname{Cl}(S_K)$.

where we define $\Gamma_1 = \big\{ G \in \mathbb{L} \mid \widetilde{G}^K \subseteq B \big\}$. We have $\Gamma \subseteq \Gamma_1$, and if $G \in \Gamma_1 \setminus \Gamma$ then $\widetilde{G}^K$ is linearly equivalent to $\widetilde{F}^K$; so the above equality simplifies to

$$\mathbb{B} = \big\langle \{\widetilde{F}^K, \widetilde{\Delta}^K\} \cup \big\{ \widetilde{G}^K \mid G \in \Gamma \big\} \cup \big\{ \widetilde{E}_P^K \mid P \in K(B) \big\} \big\rangle. \tag{32}$$

Let $G \in \Gamma$. As G is linearly equivalent to F, the total transform (cf. 4.8)

$$\overline{G}^K = \widetilde{G}^K + \textstyle\sum_{P \in K} e_P(G) \overline{E}_P^K = \widetilde{G}^K + \sum_{P \in K_G} \overline{E}_P^K$$

of G is linearly equivalent to $\overline{F}^K = \widetilde{F}^K$, so we have

$$\widetilde{G}^K = \widetilde{F}^K - \textstyle\sum_{P \in K_G} \overline{E}_P^K \quad \text{(equality in } \mathrm{Cl}(S_K)).$$

In view of (32), this gives

$$\mathbb{B} = \big\langle \{\widetilde{F}^K, \widetilde{\Delta}^K\} \cup \big\{ \textstyle\sum_{P \in K_G} \overline{E}_P^K \mid G \in \Gamma \big\} \cup \big\{ \widetilde{E}_P^K \mid P \in K(B) \big\} \big\rangle. \tag{33}$$

Recall that the divisor class group $\mathrm{Cl}(S)$ is a free $\mathbb{Z}$-module and that $\{F, \Delta\}$ is a basis of it; also, $\mathrm{Cl}(S_K)$ is a free $\mathbb{Z}$-module with basis $\{\widetilde{F}^K, \widetilde{\Delta}^K\} \cup \big\{ \widetilde{E}_P^K \mid P \in K \big\}$. Any $D \in \mathrm{Div}(S)$ is linearly equivalent to $aF + b\Delta$ for some $a, b \in \mathbb{Z}$. It follows that, for any $D \in \mathrm{Cl}(S)$, the total transform $\overline{D}^K$ belongs to the subgroup of $\mathrm{Cl}(S_K)$ generated by $\overline{F}^K = \widetilde{F}^K$ and $\overline{\Delta}^K = \widetilde{\Delta}^K$, and so belongs to $\mathbb{B}$. In particular, if $\kappa_S \in \mathrm{Cl}(S)$ is the canonical class of S, then $\overline{\kappa}_S^K \in \mathbb{B}$. As $\kappa_{S_K} = \overline{\kappa}_S^K + \sum_{P \in K} \overline{E}_P^K$ (cf. 4.8), condition (31) is equivalent to $\sum_{P \in K} \overline{E}_P^K \in \mathbb{B}$; as $\mathrm{Cl}(S_K) = \big\langle \widetilde{F}^K, \widetilde{\Delta}^K \big\rangle \oplus \big\langle \big\{ \widetilde{E}_P^K \mid P \in K \big\} \big\rangle$, it follows that (31) is equivalent to

$$\textstyle\sum_{P \in K} \overline{E}_P^K \in \big\langle \big\{ \sum_{P \in K_G} \overline{E}_P^K \mid G \in \Gamma \big\} \cup \big\{ \widetilde{E}_P^K \mid P \in K(B) \big\} \big\rangle, \tag{34}$$

where each $\overline{E}_P^K$ and each $\widetilde{E}_P^K$ is to be interpreted as an element of $\mathrm{Cl}(S_K)$.

Choose a total order $\preceq$ of K which extends the natural order and write $K = \{P_1, \dots, P_n\}$ such that $P_1 \prec \cdots \prec P_n$. Note that (34) takes place in the subgroup of $\mathrm{Cl}(S_K)$ generated by $\{\widetilde{E}_{P_1}^K, \dots, \widetilde{E}_{P_n}^K\}$ (which is a free abelian group with basis $\{\widetilde{E}_{P_1}^K, \dots, \widetilde{E}_{P_n}^K\}$). Using coordinates with respect to the basis $\{\widetilde{E}_{P_1}^K, \dots, \widetilde{E}_{P_n}^K\}$, we see that statement (34) is equivalent to $\mathbf{Q}\mathbf{1}_K$ belonging to the subgroup of $\mathbb{Z}^n$ generated by all $\mathbf{Q}\mathbf{1}_{K_G}$ such that $G \in \Gamma$ and all $\mathbf{1}_{\{P\}}$ such that $P \in K(B)$; and this is equivalent to $\mathbf{Q}_{K(B)}\mathbf{1}_K$ being a linear combination (over $\mathbb{Z}$) of the columns $\mathbf{Q}_{K(B)}\mathbf{1}_{K_G}$ such that $G \in \Gamma$. So we are done. □

Recall the meaning of S^* from the introduction of section 4.

7.8. **Notation.** Given $P \in S^*$, let G_P denote the unique element of $\mathbb{L}$ which passes through the least element of the cluster $K_P = \big\{ x \in S^* \mid x \leq P \big\}$.

7.9. **Definition.** Each point $P \in S^*$ (where $S = \mathbb{F}_0$ as before) determines a tableau $T(P)$ as follows. Let G_P be as in 7.8. Then (S, K_P, G_P) satisfies condition $(*)$ of 4.13, so a tableau $T(S, K_P, G_P)$ is defined. We set $T(P) = T(S, K_P, G_P)$, and we note that $T(P) \neq \mathbb{1}$. Note that $T(P) = T_{G_P}(P)$ where $T_{G_P}(P)$ is defined in 5.4.

7.10. **Remark.** Let $P \in S^*$. Then P is a proper point of S if and only if $T(P) = \binom{1}{1}$. More generally, the condition "$T(P) = \binom{1}{c}$ for some $c \geq 1$" is equivalent to $P \in K_{G_P}$. (These claims follow from 4.14(b)).

7.11. **Proposition.** *Let $(K, B) \in \mathcal{P}$. Then $(K, B) \in \mathcal{P}_0$ if and only if the following conditions are satisfied:*

(i) *For each $P \in K \setminus K(B)$, $T(P)$ is an exact tableau;*
(ii) *for any $P, Q \in K \setminus K(B)$, if $G_P = G_Q$ then $\delta T(P) = \delta T(Q)$;*
(iii) *$\widetilde{G}^K \subseteq B$, for all $G \in \mathbb{L}_{(K)}$.*

Proof. For any $P \in S^*$, define the integers $a(P), b(P)$ by

$$a(P) = \sum_{i=1}^{h} \hat{c}_i(c_i + p_i - 1) \quad \text{and} \quad b(P) = \prod_{i=1}^{h} c_i,$$

where the notation is defined by $T(P) = \left(\begin{smallmatrix} p_1 & \cdots & p_h \\ c_1 & \cdots & c_h \end{smallmatrix}\right)$. Then $a(P) > 0$, $b(P) > 0$, $\delta T(P) = a(P)/b(P)$, and $T(P)$ is exact if and only if $a(P)/b(P)$ is an integer.

Let $(K, B) \in \mathcal{P}$. Define a map $q : K \times K \to \mathbb{N}$ by stipulating that $\overline{E}_Q^K = \sum_{P \in K} q(P, Q)\widetilde{E}_P^K$ for all choices of $(P, Q) \in K \times K$. Note that if $q(P, Q) \neq 0$ then Q belongs to the cluster $K_P = \left\{ x \in S^* \mid x \leq P \right\}$. Choose a total order $\preceq$ extending the natural order $\leq$, write the elements of K as $P_1 \prec \cdots \prec P_n$ and consider $\mathbf{Q} = \mathbf{Q}(K, \preceq)$.

Let $P \in K$ and $G \in \mathbb{L}$. Then $P = P_i$ for some $i \in \{1, \dots, n\}$, and the i-th row $\mathbf{Q}_i$ of $\mathbf{Q}(K, \preceq)$ satisfies

$$\mathbf{Q}_i \mathbf{1}_K = a(P) \quad \text{and} \quad \mathbf{Q}_i \mathbf{1}_{K_G} = b(P)\delta_{G_P}^G, \qquad \text{where} \quad \delta_{G_P}^G = \begin{cases} 1 & \text{if } G = G_P \\ 0 & \text{else.} \end{cases} \tag{35}$$

Indeed,

$$\mathbf{Q}_i \mathbf{1}_K = \sum_{Q \in K} q(P, Q) = \sum_{Q \in K_P} q(P, Q) = \mathbf{Q}_*(K_P)\mathbf{1}_{K_P} = a(P),$$

the last equality by part (c) of 4.18, and

$$\mathbf{Q}_i \mathbf{1}_{K_G} = \sum_{Q \in K_G} q(P, Q) = \sum_{Q \in K_P \cap K_G} q(P, Q).$$

If $G \neq G_P$ then $K_P \cap K_G = \varnothing$ and $\mathbf{Q}_i \mathbf{1}_{K_G} = 0$. If $G = G_P$ then $\sum_{Q \in K_P \cap K_G} q(P, Q) = \mathbf{Q}_*(K_P)\mathbf{1}_{(K_P)_G} = b(P)$, again by part (c) of 4.18. Thus (35) is correct.

Suppose that $(K, B) \in \mathcal{P}_0$. Then, by 7.7, there exists a family $\left(m_G\right)_{G \in \Gamma}$ of integers such that

$$\mathbf{Q}_{K(B)} \mathbf{1}_K = \sum_{G \in \Gamma} m_G \mathbf{Q}_{K(B)} \mathbf{1}_{K_G}. \tag{36}$$

Let $P \in K \setminus K(B)$; then $P = P_i$ for some i, and the row $\mathbf{Q}_i$ is present in $\mathbf{Q}_{K(B)}$. Considering that row in (36) and using (35) gives

$$a(P) = \sum_{G \in \Gamma} m_G b(P) \delta_{G_P}^G = b(P) \sum_{G \in \Gamma} m_G \delta_{G_P}^G,$$

so $T(P)$ is exact. As $a(P) \neq 0$, it follows that $\sum_{G\in\Gamma} m_G\delta^G_{G_P} \neq 0$, so $G_P \in \Gamma$ and

$$\delta T(P) = \sum_{G\in\Gamma} m_G\delta^G_{G_P} = m_{G_P}.$$

So (i) and (ii) hold, and moreover $G_P \in \Gamma$ (so $\widetilde{G}^K_P \subseteq B$) for each $P \in K \setminus K(B)$. This last condition implies that (iii) holds. Indeed, consider $G \in \mathbb{L}_{(K)}$. Then there exists $P \in K$ satisfying $G_P = G$, and we may choose this P to be a maximal element of K. Then $(\widetilde{E}^K_P)^2 = -1$ in S_K, so the fact that $(K, B) \in \mathcal{P}$ implies that $\widetilde{E}^K_P \nsubseteq B$, i.e., $P \in K \setminus K(B)$. Then $\widetilde{G}^K = \widetilde{G}^K_P \subseteq B$, so (iii) holds.

The converse is left to the reader. □

Let us reformulate 7.11 as follows:

7.12. **Corollary.** *Consider a normal surface U which is connected at infinity and which admits a dominant morphism $U \to \mathbb{A}^1$ whose general fiber is an affine line. Then there exists $(K, B) \in \mathcal{P}$ such that $U \cong U_{(K,B)}$. Moreover, given any such (K, B), the condition*

$$U \setminus \operatorname{Sing} U \text{ has trivial canonical class}$$

is satisfied if and only if the following conditions hold:

(i) *For each $P \in K \setminus K(B)$, $T(P)$ is an exact tableau;*
(ii) *for any $P, Q \in K \setminus K(B)$, if $G_P = G_Q$ then $\delta T(P) = \delta T(Q)$;*
(iii) *$\widetilde{G}^K \subseteq B$, for all $G \in \mathbb{L}_{(K)}$.*

Proof. Let $f : U \to \mathbb{A}^1$ be the morphism given in the assumption, let $V = f(U)$, and let $\rho : U \to V$ be f regarded as a morphism from U to V. Then $(U, \rho) \in \mathcal{C}$, so 6.7 implies that there exists $(K, B) \in \mathcal{P}$ such that $[U_{(K,B)}, \rho_{(K,B)}] = [U, \rho]$. Then $U \cong U_{(K,B)}$.

Consider any $(K, B) \in \mathcal{P}$ such that $U \cong U_{(K,B)}$. By 7.6, the condition "$U \setminus \operatorname{Sing} U$ has trivial canonical class" is equivalent to $(K, B) \in \mathcal{P}_0$, which is equivalent to (i–iii) by 7.11. □

8. Pairs (K, B) satisfying $U_{(K,B)} \in \mathfrak{D}(\mathbf{k})$

We continue to assume that varieties are over an algebraically closed field $\mathbf{k}$ of characteristic zero. See the introduction for the definition of the class $\mathfrak{D}(\mathbf{k})$ of surfaces. The aim of this section is to prove result 8.4.

8.1. **Lemma.** *Let $(K, B) \in \mathcal{P}$ and suppose that either $(K, B) = (\varnothing, \Delta \cup F)$ or the following conditions hold:*

- *$\mathbb{L}_{(K)}$ is a singleton $\{G\}$*
- *$K = K_G$*
- *$B = \widetilde{F}^K \cup \widetilde{\Delta}^K \cup \widetilde{G}^K \cup \big(\bigcup_{P\in K'} \widetilde{E}^K_P\big)$, where*

$$K' = \big\{\, x \in K \mid x \text{ is not a maximal element of } K \,\big\}.$$

Then $U_{(K,B)} \in \mathfrak{D}(\mathbf{k})$.

One can give a direct proof of the above fact, but that is somewhat tedious. Instead, we deduce the result from [4, 5.4.5]. Alternatively, if $\mathbf{k} = \mathbb{C}$ then we could derive it from [12, 4.10] and [14, 3.10].

Proof of 8.1. Let the notation be as in 6.5 and take $U = U_{(K,B)}$ to be the complement of $\overline{\sigma}(B_\infty)$ in $\overline{S}_K$. If $(K,B) = (\varnothing, \Delta \cup F)$ then $U_{(K,B)} = \mathbb{A}^2$ by 7.1, so $U_{(K,B)} \in \mathfrak{D}(\mathbf{k})$. Assume that $(K,B) \neq (\varnothing, \Delta \cup F)$. Then the assumption on (K,B) implies that condition $(*)$ of 7.4 is satisfied, so $U_{(K,B)}$ is affine and $\overline{S}_K$ is projective (where $\overline{S}_K$ is defined in 6.5). Moreover, $U_{(K,B)}$ is the complement of $\overline{\sigma}(B_\infty)$ in $\overline{S}_K$, $\overline{\sigma}(B_\infty)$ is the support of an SNC-divisor D of $\overline{S}_K$, each irreducible component of D is a rational curve and the dual graph of D in $\overline{S}_K$ is $\overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{-n}{\bullet}$ for some $n \geq 2$. Now Theorem 5.4.5 of [4] implies in particular the following statement:

> *Let X be a normal projective rational surface and D an SNC-divisor of X all of whose irreducible components are rational curves. Suppose that $X \setminus \operatorname{supp}(D)$ is affine and that the dual graph of D in X is $\overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{-1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{-n}{\bullet}$ where $n \geq 2$. Then $X \setminus \operatorname{supp}(D) \in \mathfrak{D}(\mathbf{k})$.*

Moreover, it is clear that this assertion remains true if, in its statement, we replace the graph $\overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{-1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{-n}{\bullet}$ by $\overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{-n}{\bullet}$. So we obtain $U_{(K,B)} \in \mathfrak{D}(\mathbf{k})$. $\square$

8.2. **Lemma.** *Suppose that $(K,B) \in \mathcal{P}_0$ is such that $\mathbb{L}_{(K)}$ is a singleton $\{G\}$ and $K_G \nsubseteq K(B)$. Then $K = K_G$.*

Proof. Consider the set of tableaux $\Sigma = \big\{\, T(P) \mid P \in K \setminus K(B) \,\big\}$. Since $K_G \nsubseteq K(B)$, some $T \in \Sigma$ satisfies $\delta(T) = 1$ (pick $P_0 \in K_G \setminus K(B)$; by 7.10, $T(P_0) = \binom{1}{c}$ for some c, so 2.3 implies that $\delta T(P_0) = 1$).

Note that $G_P = G$ for all $P \in K$. So, in view of 7.11, we have $\delta(T) = \delta(T')$ for all $T, T' \in \Sigma$. By the first paragraph, $\delta(T) = 1$ for all $T \in \Sigma$.

Let P be a maximal element of K. Then $(\widetilde{E}_P^K)^2 = -1$ in S_K, so $\widetilde{E}_P^K \nsubseteq B$ (because $(K,B) \in \mathcal{P}$), so $P \in K \setminus K(B)$ and consequently $T(P) \in \Sigma$. By the preceding paragraph, $\delta T(P) = 1$; by 2.3(a), it follows that $T(P) = \binom{1}{c}$ for some c; then 7.10 implies that $P \in K_G$. Hence, all maximal elements of K belong to K_G. As K_G is a subcluster of K, it follows that $K = K_G$. $\square$

8.3. **Lemma.** *Suppose that $(K,B) \in \mathcal{P}_0$ is such that $K \neq \varnothing$ and such that the dual graph of B_∞ is a linear chain. Then $\mathbb{L}_{(K)}$ is a singleton $\{G\}$, the set $\big\{\, H \in \mathbb{L} \mid \widetilde{H}^K \subseteq B \,\big\}$ is equal to $\{F, G\}$, and $K = K_G$.*

Proof. As in 6.4, let $F_1, \dots, F_n, C_1, \dots, C_t$ denote the distinct elements of $\big\{\, H \in \mathbb{L} \mid \widetilde{H}^K \subseteq B \,\big\}$, where $F_i \cap \min K = \varnothing$ and $C_i \cap \min K \neq \varnothing$. Then $\widetilde{\Delta}^K$ has $n + t$ neighbors in the dual graph (21) of B_∞; consequently, $n + t \leq 2$. We have $\mathbb{L}_{(K)} = \{C_1, \dots, C_t\}$ by 7.11, so $t \geq 1$, because $K \neq \varnothing$. We have $n \geq 1$, because $F \in \{F_1, \dots, F_n\}$. Consequently, $n = 1 = t$. We change the notation and write $\mathbb{L}_{(K)} = \{G\}$; then $\big\{\, H \in \mathbb{L} \mid \widetilde{H}^K \subseteq B \,\big\} = \{F, G\}$.

There remains to show that $K = K_G$. In view of 8.2, we may assume throughout:

$$K_G \subseteq K(B). \tag{37}$$

We show that (37) leads to a contradiction, and this will complete the proof. First note that $K_G \neq \varnothing$ (because $G \in \mathbb{L}_{(K)}$) and consequently K_G has a maximal element. Let P be any maximal element of K_G; then $P \in K(B)$ by (37), so $\widetilde{E}_P^K$ is an irreducible component of B; moreover, $\widetilde{E}_P^K \cap \widetilde{G}^K \neq \varnothing$ because P is a maximal element of K_G. As $\widetilde{G}^K$ is not a branch point of the dual graph of B_∞, it follows that K_G has only one (hence exactly one) maximal element. As K_G is a cluster, it must then be totally ordered by the natural order. In particular, K_G has a unique minimal element, so K has a unique minimal element; let $P_1 \in K$ be that element.

Note that for each $P \in K$, the tableaux $T_G(P)$ (cf. 5.4) and $T(P)$ (cf. 7.9) are in fact equal: $T_G(P) = T(S, K_P, G) = T(P)$ where $K_P = \{\, x \in K \mid x \leq P \,\}$. We claim:

(38) *There exists $P \in K \setminus K(B)$ such that the tableau $T(P)$ is one of the following:*
- (a) $T(P) = \binom{p}{c}$ *for some p, c such that $1 \leq p < c$*
- (b) $T(P) = \left(\begin{smallmatrix} p & 1 \\ c & N \end{smallmatrix}\right)$ *for some p, c, N such that $N \geq 1$, $1 \leq p < c$.*

We prove this by applying result 5.5 to (K, G, P_1, Z), where we define $Z \subset S_K$ to be the union of the $\widetilde{E}_Q^K$ for all $Q \in K$ satisfying $\widetilde{E}_Q^K \subset B_\infty$. As B_∞ cannot contain a (-1)-curve (cf. 6.3), Z is a proper subset of $\pi_K^{-1}(P_1)$; so condition 5.5(i) is satisfied. As $B_\infty = \widetilde{F}^K \cup \widetilde{\Delta}^K \cup \widetilde{G}^K \cup Z$ is a linear chain, $\widetilde{G}^K \cup Z$ too is a linear chain and hence condition 5.5(ii) holds. Condition (37) implies that $\widetilde{E}_{P_1}^K \subset B$, so 5.5(iii) holds. If P_1 is a maximal element of K_G then $K_G = \{P_1\}$, so $(\widetilde{G}^K)^2 = -1$ in S_K, which contradicts $(K, B) \in \mathcal{P}$; so 5.5(iv) holds. The fact that $(K, B) \in \mathcal{P}$ also implies that 5.5(v) holds, so (K, G, P_1, Z) satisfies all hypotheses of 5.5. By that result, there exists $P \in K$ such that $T(P)$ is as described in (38), and such that $\widetilde{E}_P^K \cap (\widetilde{G}^K \cup Z) \neq \varnothing$ and $\widetilde{E}_P^K \nsubseteq \widetilde{G}^K \cup Z$. This last condition implies that $\widetilde{E}_P^K \nsubseteq B$, so $P \in K \setminus K(B)$. This proves (38).

Observe that if $T(P)$ is as in part (a) of (38) then $p \neq 1$ (if $p = 1$ then $P \in K_G$, so (37) implies that $P \in K(B)$, a contradiction). Then it follows from 2.3 that, for any P satisfying (38), $T(P)$ is not an exact tableau. Consequently,

(39) There exists $P \in K \setminus K(B)$ such that $T(P)$ is not an exact tableau.

This contradicts 7.11 and hence completes the proof that (37) is impossible. The proof of the Proposition is complete. $\square$

Refer to 3.5, 4.9 and 6.2 for the notations $\mathcal{G}_\infty[U]$, K_G and $\mathbb{L}_{(K)}$.

8.4. **Proposition.** *Let $(K, B) \in \mathcal{P}$ and let $U = U_{(K,B)}$. Suppose that*
- (a) *U is affine;*
- (b) *some element of $\mathcal{G}_\infty[U]$ is a linear chain of the form $\overset{0}{\bullet}\!-\!\!-\!\overset{x}{\bullet}\!-\!\!-\!\overset{\omega_1}{\bullet}\!-\cdots-\!\overset{\omega_q}{\bullet}$ where $q \geq 0$, x is any integer and $\omega_1, \dots, \omega_q \in \mathbb{Z}$ are such that $\omega_i \leq -2$ for all i;*
- (c) *$U \setminus \mathrm{Sing}(U)$ has trivial canonical class.*

Then $U \in \mathfrak{D}(\mathbf{k})$.

Proof. Let $(K,B) \in \mathcal{P}$, let $U = U_{(K,B)}$, and suppose that (a), (b) and (c) are satisfied. If $K = \varnothing$ then, by 7.1, $U \cong V \times \mathbb{A}^1$ where V is $\mathbb{P}^1$ minus q points, $q \geq 1$. So the weighted graph (28) belongs to $\mathcal{G}_\infty[U]$; by assumption (b), it follows that $q = 1$, so $U \cong \mathbb{A}^2$ and hence $U \in \mathfrak{D}(\mathbf{k})$. So we are done in this case.

From now-on, assume that $K \neq \varnothing$. In view of 8.1, it suffices to show that the following conditions hold:

(d) $\mathbb{L}_{(K)}$ is a singleton $\{G\}$
(e) $K = K_G$
(f) $B = \widetilde{F}^K \cup \widetilde{\Delta}^K \cup \widetilde{G}^K \cup \big(\bigcup_{P\in K'} \widetilde{E}_P^K\big)$, where

$$K' = \big\{\, x \in K \mid x \text{ is not a maximal element of } K \,\big\}.$$

As $S_K \setminus B \cong U \setminus \operatorname{Sing} U$ by 6.6, $S_K \setminus B$ has trivial canonical class; thus $(K,B) \in \mathcal{P}_0$. It follows from assumption (b) that every minimal element of $\mathcal{G}_\infty[U]$ is a linear chain; hence $\mathcal{G}(S_K, B_\infty)$ is a linear chain by 6.6, so all hypotheses of 8.3 are satisfied. That result implies that (d) and (e) hold, and that $\big\{\, H \in \mathbb{L} \mid \widetilde{H}^K \subseteq B \,\big\} = \{F, G\}$; so, to complete the proof, it only remains to show that $K \setminus K(B)$ is equal to the set $\max K$ of maximal elements of K. We have

$$\big\{\, P \in K \mid \widetilde{E}_P^K \cap \widetilde{G}^K \neq \varnothing \,\big\} = \max K \subseteq K \setminus K(B) \subseteq \big\{\, P \in K \mid \widetilde{E}_P^K \cap B_\infty \neq \varnothing \,\big\}, \tag{40}$$

where the equality follows from $K = K_G$, the first inclusion from $(K,B) \in \mathcal{P}$, and the second inclusion from condition $(*)$ of 7.4 (which must hold, since U is affine). The fact that $\big\{\, P \in K \mid \widetilde{E}_P^K \cap \widetilde{G}^K \neq \varnothing \,\big\} \subseteq K \setminus K(B)$ implies that $B_\infty = \widetilde{F}^K \cup \widetilde{\Delta}^K \cup \widetilde{G}^K$, so in fact we have $\big\{\, P \in K \mid \widetilde{E}_P^K \cap \widetilde{G}^K \neq \varnothing \,\big\} = \big\{\, P \in K \mid \widetilde{E}_P^K \cap B_\infty \neq \varnothing \,\big\}$, so all inclusions in (40) are in fact equalities. In particular we have $\max K = K \setminus K(B)$. So (f) holds, and the proof is complete. $\square$

9. Surfaces with trivial Makar-Limanov invariant

Let us begin by making a list of the facts that we need for proving the main results.

9.1. **Definition.** Let R be an integral domain and an algebra over a field $\mathbf{k}$. We say that R is a *complete intersection over* $\mathbf{k}$ if it is isomorphic to a quotient

$$\mathbf{k}[X_1, \dots, X_n]/(f_1, \dots, f_p)$$

for some $n, p \in \mathbb{N}$, where $(f_1, \dots, f_p)$ is a prime ideal of $\mathbf{k}[X_1, \dots, X_n]$ of height p. If R is a complete intersection over $\mathbf{k}$, we also call $\operatorname{Spec} R$ a complete intersection over $\mathbf{k}$.

9.2. **Lemma.** *Let X be an affine variety over an algebraically closed field $\mathbf{k}$. If X is a complete intersection over $\mathbf{k}$, then $X \setminus \operatorname{Sing}(X)$ has trivial canonical class.*

Proof. Apparently, this is a well-known fact. Being unable to find an appropriate reference, we give some indications of how to prove it.

Let $p, q \in \mathbb{N}$ and $f_1, \dots, f_p \in \mathbf{k}[X_1, \dots, X_{p+q}]$ some polynomials. Consider the $\mathbf{k}$-algebra $A = \mathbf{k}[X_1, \dots, X_{p+q}]/(f_1, \dots, f_p)$, the ideal $\mathcal{J}$ of A generated by the $p \times p$ minors of the jacobian matrix $(\partial f_j / \partial X_i)$, and the open subset $U = X \setminus V(\mathcal{J})$ of

$X = \operatorname{Spec} A$. Consider the sheaf of $\mathcal{O}_X$-modules $(\Omega^q_{A/\mathbf{k}})^{\sim}$ associated to the A-module $\Omega^q_{A/\mathbf{k}} = \bigwedge^q \Omega_{A/\mathbf{k}}$, where $\Omega_{A/\mathbf{k}}$ is the sheaf of differentials of A over $\mathbf{k}$. Then we leave it as an exercise to show that there exists an A-linear map $\varphi : \Omega^q_{A/\mathbf{k}} \to A$ with image $\mathcal{J}$ and such that, for each $\mathfrak{p} \in U$, the localized map $\varphi_{\mathfrak{p}} : (\Omega^q_{A/\mathbf{k}})_{\mathfrak{p}} \to A_{\mathfrak{p}}$ is bijective. This means that $\tilde{\varphi} : (\Omega^q_{A/\mathbf{k}})^{\sim} \to \mathcal{O}_X$ restricts to an isomorphism $(\Omega^q_{A/\mathbf{k}})^{\sim}\big|_U \cong \mathcal{O}_X|_U$. Note that, in this generality, it may happen that $U = \varnothing$. However, if we now assume that $(f_1, \dots, f_p)$ is a prime ideal of height p, then $V(\mathcal{J}) = \operatorname{Sing}(X)$ and $(\Omega^q_{A/\mathbf{k}})^{\sim}\big|_U$ is the canonical sheaf of $U = X \setminus \operatorname{Sing}(X)$, so $X \setminus \operatorname{Sing}(X)$ has trivial canonical sheaf. $\square$

9.3. **Lemma.** *Let R be an integral domain and a complete intersection over a field $\mathbf{k}$. If R is regular in codimension one, then R is normal.*

Proof. Since R is a complete intersection, it is Cohen-Macaulay and hence satisfies Serre's condition (S_2). As R is a noetherian domain which is regular in codimension one and satisfies (S_2), it is normal. (Prop. 18.13 of [11] and Theorem 39 of [16].) $\square$

9.4. (Cor. 4.11 of [15]) *Let R be a two-dimensional*[12] *integral domain and a finitely generated algebra over a field $\mathbf{k}$ of characteristic zero. If $\mathrm{ML}(R) = \mathbf{k}$ then R is regular in codimension one.*

9.5. (Lemma 3.7 of [7]) *Let R be an integral domain containing a field $\mathbf{k}$ of characteristic zero. If R is normal and $\mathrm{ML}(R) = \mathbf{k}$, then for any field extension K of $\mathbf{k}$ we have:*

$$K \otimes_{\mathbf{k}} R \text{ is an integral domain and } \mathrm{ML}(K \otimes_{\mathbf{k}} R) = K.$$

9.6. (Theorem 2.3 of [7]) *For an algebra R over a field $\mathbf{k}$ of characteristic zero, the following conditions are equivalent:*

(a) $R \in \mathfrak{D}(\mathbf{k})$
(b) $\mathrm{ML}(R) \neq R$ *and there exists a field extension $K/\mathbf{k}$ such that $K \otimes_{\mathbf{k}} R \in \mathfrak{D}(K)$.*

9.7 (Theorem 2.20 of [10]). *Let U be a normal affine surface over an algebraically closed field $\mathbf{k}$ of characteristic zero. Then $\mathrm{ML}(U) = \mathbf{k}$ if and only if the following conditions are satisfied:*

- *U is rational and completable by rational curves*
- *some element of $\mathcal{G}_\infty[U]$ is a linear chain of the form $\overset{0}{\bullet}\!-\!\overset{x}{\bullet}\!-\!\overset{\omega_1}{\bullet} \cdots \!-\!\overset{\omega_q}{\bullet}$ where $q \geq 0$, x is any integer and $\omega_1, \dots, \omega_q \in \mathbb{Z}$ are such that $\omega_i \leq -2$ for all i.*

Theorems 9.8 and 9.9 may be regarded as the main results of this paper. The proof of 9.8 makes use of the framework developed in sections 6–8, but note that 6.7 and 8.4 are the only results from earlier sections which are used here. The reader should also keep in mind that, in 9.8 (resp. in 9.9), we view $\mathfrak{D}(\mathbf{k})$ as a class of surfaces (resp. of algebras). See the introduction for the definition of $\mathfrak{D}(\mathbf{k})$.

9.8. **Theorem.** *Let U be an affine surface over an algebraically closed field $\mathbf{k}$ of characteristic zero. Then the following are equivalent.*

[12] We mean Krull dimension.

(a) *$U \in \mathfrak{D}(\mathbf{k})$;*
(b) *$\mathrm{ML}(U) = \mathbf{k}$ and U is a complete intersection over $\mathbf{k}$;*
(c) *$\mathrm{ML}(U) = \mathbf{k}$, U is normal and $U \setminus \mathrm{Sing}\, U$ has trivial canonical class.*

Proof. It is well known that if $U \in \mathfrak{D}(\mathbf{k})$ then $\mathrm{ML}(U) = \mathbf{k}$; as U is also a hypersurface of $\mathbb{A}^3$, it is a complete intersection; so (a) implies (b).

Suppose that U satisfies (b). By 9.4, U is regular in codimension 1; so 9.3 implies that U is normal. By 9.2, $U \setminus \mathrm{Sing}\, U$ has trivial canonical class. So (b) implies (c).

Finally, suppose that U satisfies (c). As U is a normal affine surface such that $\mathrm{ML}(U) = \mathbf{k}$, it is well known that there exists a surjective morphism $\rho : U \to \mathbb{A}^1$ whose general fiber is an affine line (see for instance 2.3 of [10]). Then $(U, \rho) \in \mathcal{C}$ and, by 6.7, there exists $(K, B) \in \mathcal{P}$ such that $[U_{(K,B)}, \rho_{(K,B)}] = [U, \rho]$; then $U_{(K,B)} \cong U$. Result 9.7 implies that U satisfies hypothesis (b) of 8.4, so U satisfies all hypotheses (a–c) of 8.4; so $U \in \mathfrak{D}(\mathbf{k})$ by 8.4. □

9.9. **Theorem.** *Let R be a two-dimensional integral domain which contains a field $\mathbf{k}$ of characteristic zero. The following conditions are equivalent.*

(a) *$R \in \mathfrak{D}(\mathbf{k})$*
(b) *$\mathrm{ML}(R) = \mathbf{k}$ and R is 3-generated as a $\mathbf{k}$-algebra*
(c) *$\mathrm{ML}(R) = \mathbf{k}$ and R is a complete intersection over $\mathbf{k}$.*

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are well known and easy to see. We prove that (c) implies (a). Suppose that R satisfies (c) and let K be the algebraic closure of $\mathbf{k}$. By 9.4, R is nonsingular in codimension 1; so 9.3 implies that R is normal. In view of 9.5, we obtain:

(c′) $K \otimes_{\mathbf{k}} R$ is a two-dimensional integral domain, $\mathrm{ML}(K \otimes_{\mathbf{k}} R) = K$ and $K \otimes_{\mathbf{k}} R$ is a complete intersection over K.

Then $U = \mathrm{Spec}(K \otimes_{\mathbf{k}} R)$ satisfies condition (b) of 9.8. By that result, we obtain $U \in \mathfrak{D}(K)$, or equivalently

$$K \otimes_{\mathbf{k}} R \in \mathfrak{D}(K).$$

Since $\mathrm{ML}(R) = \mathbf{k} \neq R$ by assumption, 9.6 implies that $R \in \mathfrak{D}(\mathbf{k})$. □

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