Families of Affine Fibrations

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To Gerry Schwarz on his sixtieth birthday

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Abstract

This paper gives a method of constructing affine fibrations for polynomial rings. The method can be used to construct the examples of $\mathbb{A}^2$-fibrations in dimension 4 due to Bhatwadekar and Dutta (1994) and Vénéreau (2001). The theory also provides an elegant way to prove many of the known results for these examples.

1 Introduction

One version of the famous Dolgachev-Weisfeiler Conjecture asserts that, if $\varphi : \mathbb{A}^n \to \mathbb{A}^m$ is a flat morphism of affine spaces in which every fiber is isomorphic to $\mathbb{A}^{n-m}$, then it is a trivial fibration [6]. In his 2001 thesis, Vénéreau constructed a family of fibrations $\varphi_n : \mathbb{C}^4 \to \mathbb{C}^2$ ($n \geq 1$) whose status relative to the Dolgachev-Weisfeiler Conjecture could not be determined. These examples attracted wide interest in the intervening years, and have been investigated in several papers, for example, [8, 10, 11, 13]. A less well-known example of an affine fibration, due to Bhatwadekar and Dutta, appeared in 1992 [5], and it turns out that this older example is quite similar to the fibration φ_1 of Vénéreau. Bhatwadekar and Dutta asked if their fibration is trivial, a question which remains open.

In this paper, we prove the following result, which is a tool for building affine fibrations. Here, it should be noted that statements (ii)-(iv) follow by combining (i) with known results.

Proposition 1.1 *Let k be a field, and let $B = k[x, y_1, \ldots, y_r, z_1, \ldots, z_m] = k^{[r+m+1]}$. Suppose that $v_1, \ldots, v_m \in B$ are such that*

$$k[x, y_1, \ldots, y_r, v_1, \ldots, v_m]_x = B_x.$$

Pick any $\phi_1, \ldots, \phi_r \in k[x, v_1, \ldots, v_m]$, and define $f_i = y_i + x\phi_i$ ($1 \leq i \leq r$) and $A = k[x, f_1, ..., f_r]$. Then:

- **(i)** *B is an $\mathbb{A}^m$-fibration over A.*
- **(ii)** *B is a stably polynomial algebra over A, i.e., $B^{[n]} = A^{[m+n]}$ for some $n \in \mathbb{N}$.*

If moreover $m = 2$ and $\operatorname{char}(k) = 0$:

- **(iii)** $\ker D = A^{[1]} = k^{[r+2]}$ *for every non-zero $D \in \operatorname{LND}_A(B)$.*
- **(iv)** *If I is an ideal of A such that A/I is a PID, then $B/IB = (A/I)^{[2]}$.*

As a consequence, we obtain a family of $\mathbb{A}^2$-fibrations in dimension 4 to which the examples of Vénéreau and Bhatwadekar-Dutta belong. In addition, many of the known results about these examples follow from the more general theory, for example, that the fibrations are stably trivial.

An excellent summary of known results for affine fibrations is found in [5], and the reader is referred to this article for further background.

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2 Preliminaries

2.1 Notation and Definitions

Throughout, k denotes a field, and rings are assumed to be commutative. For any field F, affine n-space over F is denoted by $\mathbb{A}^n_F$, or simply $\mathbb{A}^n$ when the ground field is understood. For a ring A and positive integer n, $A^{[n]}$ is the polynomial ring in n variables over A. If B is a domain, and A is a subring of B, then $\text{tr.deg}_A(B)$ denotes the transcendence degree of the field $\text{frac}(B)$ over $\text{frac}(A)$. If $x \in A$ is non-zero, then A_x is the localization of A at the set $\{x^n | \, n \in \mathbb{N}\}$. Likewise, if $\mathfrak{p}$ is a prime ideal of A, then $A_\mathfrak{p}$ is the localization of A determined by $\mathfrak{p}$, and $\kappa(\mathfrak{p})$ denotes the field $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. Here is the definition of the main object under consideration (following [5]).

> **Definition.** Let B be an algebra over a ring A. Then B is an $\mathbb{A}^m$*-fibration over* A if and only if B is finitely generated as an A-algebra, flat as an A-module, and for every $\mathfrak{p} \in \text{Spec}\, A$, $\kappa(\mathfrak{p}) \otimes_A B \cong \kappa(\mathfrak{p})^{[m]}$.

Geometrically, in this case if $X = \text{Spec}\, B$, $Y = \text{Spec}\, A$, and $\varphi : X \to Y$ is the morphism induced by the inclusion $A \to B$, then φ will be called an $\mathbb{A}^m$-fibration of X over Y. For convenience, we also introduce the following terminology.

> **Definition.** The ring B is an $\mathbb{A}^m$*-prefibration* over the subring A if and only if for every $\mathfrak{p} \in \text{Spec}\, A$, $\kappa(\mathfrak{p}) \otimes_A B \cong \kappa(\mathfrak{p})^{[m]}$.

Note that if B is an $\mathbb{A}^m$-fibration over A then B is faithfully flat over A (by flatness and surjectivity of $\text{Spec}\, B \to \text{Spec}\, A$) so the homomorphism $A \to B$ is injective. Thus, every fibration is a prefibration. The converse is valid in certain situations, for instance we observe:

Lemma 2.1 *Let $A \subset B$ be polynomial rings over a field. If B is an $\mathbb{A}^m$-prefibration over A, then it is an $\mathbb{A}^m$-fibration over A.*

Proof. We have $A = k^{[n]}$ and $B = k^{[n+m]}$ for some field k and some $n \in \mathbb{N}$.

If k is algebraically closed then $\text{Spec}\, B \to \text{Spec}\, A$ is a morphism of nonsingular algebraic varieties such that every fiber has dimension equal to $\dim B - \dim A$, so B is flat over A and B is an $\mathbb{A}^m$-fibration over A.

For the general case, let $\bar{k}$ be the algebraic closure of k, $\bar{A} = \bar{k} \otimes_k A = \bar{k}^{[n]}$ and $\bar{B} = \bar{k} \otimes_k B = \bar{k}^{[n+m]}$. One can see that $\bar{B}$ is an $\mathbb{A}^m$-prefibration over $\bar{A}$; by the first paragraph, it follows that $\bar{B}$ is an $\mathbb{A}^m$-fibration over $\bar{A}$ so in particular $\bar{B}$ is faithfully flat over $\bar{A}$.

$$\begin{array}{ccccc} \bar{k} & \to & \bar{A} & \to & \bar{B} \\ \uparrow & & \uparrow & & \uparrow \\ k & \to & A & \to & B \end{array}$$

It follows that $\bar{B}$ is faithfully flat over B and also over A; consequently B is faithfully flat over A (descent property). Thus B is an $\mathbb{A}^m$-fibration over A. $\square$

Suppose B is an affine fibration over A (i.e., B is an $\mathbb{A}^m$-fibration for some non-negative integer m). This fibration is said to be *trivial* if $B = A^{[m]}$, i.e., B is a polynomial algebra over A. Likewise, the fibration is *stably trivial* if $B^{[n]} = A^{[m+n]}$ for some $n \geq 0$, and we say that B is a stably polynomial algebra over A.

For $i = 1, 2$, suppose B_i is an affine fibration over A_i, with inclusion map $j_i : A_i \to B_i$. These two fibrations are said to be *equivalent* if there exist isomorphisms $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$ such that $\beta j_1 = j_2 \alpha$.

2.2 Some Known Results on Affine Fibrations

This section lays out certain known results on affine fibrations which are needed in the rest of the paper. The module of Kähler differentials of B over A is denoted by $\Omega_{B/A}$.

Theorem 2.1 (Asanuma, [2], Thm. 3.4) *Let A be a noetherian ring and B an $\mathbb{A}^m$-fibration over A. Then $\Omega_{B/A}$ is a projective B-module of rank m, and there exists $n \geq 0$ such that $A \subset B \subset A^{[n]}$. If $\Omega_{B/A}$ is a free B-module, then $B^{[n]} = A^{[m+n]}$.*

In view of the Quillen-Suslin Theorem, there follows:

Corollary 2.1 *Consider $A \subset B$ where A is a noetherian ring and B is a polynomial ring over a field. If B is an $\mathbb{A}^m$-fibration over A, then $B^{[n]} = A^{[m+n]}$ for some $n \geq 0$.*

It was proved by Hamann [9] that if A is a noetherian ring containing $\mathbb{Q}$ then the conditions $A \subset B$ and $B^{[n]} = A^{[n+1]}$ imply $B = A^{[1]}$. Combining this with the above result of Asanuma gives:

Theorem 2.2 ([5], Thm. 3.4) *Let A be a noetherian ring containing a field of characteristic zero, and let B be an $\mathbb{A}^1$-fibration over A. If $\Omega_{B/A}$ is a free B-module, then $B = A^{[1]}$.*

From the results of Sathaye [16] and Bass, Connell, and Wright [4], one derives:

Theorem 2.3 ([5], Cor. 4.8) *Let A be a PID containing a field of characteristic zero. If B is an $\mathbb{A}^2$-fibration over A, then $B = A^{[2]}$.*

Corollary 2.2 *Suppose that B is an $\mathbb{A}^2$-fibration over a ring A which contains $\mathbb{Q}$. If I is an ideal of A such that A/I is a PID, then $B/IB = (A/I)^{[2]}$.*

Proof. As $B/IB = A/I \otimes_A B$, the ring homomorphism $A/I \to B/IB$ makes B/IB an $\mathbb{A}^2$-fibration over A/I, so the desired conclusion follows from *Theorem 2.3*. □

There are many other papers which discuss affine fibrations. For example, results concerning morphisms with $\mathbb{A}^1$-fibers are due to Kambayashi and Miyanishi [14], and to Kambayashi and Wright [15]. Likewise, Asanuma and Bhatwadekar [3], and Kaliman and Zaidenberg [12] give important facts about $\mathbb{A}^2$-fibrations. For an overview of affine fibrations, the reader is referred to the aforementioned survey article [5].

For affine spaces, the first case where few results are known is the case of $\mathbb{A}^2$-fibrations $\mathbb{A}^4 \to \mathbb{A}^2$, and consequently these will receive special attention.

2.3 Locally Nilpotent Derivations

By a *locally nilpotent derivation* of a commutative ring B of characteristic 0, we mean a derivation $D : B \to B$ such that, to each $b \in B$, there is a positive integer n with $D^n b = 0$. The kernel of D is denoted $\ker D$. Let $D : B \to B$ be a non-zero locally nilpotent derivation, where B is an integral domain of characteristic zero. If $K = \ker D$, then it is known that K is factorially closed in B, $B^* \subset K$, and $\text{tr.deg}_K B = 1$. An element $s \in B$ is a *slice* for D if $Ds = 1$, and in this case $B = K[s]$. The notation $\text{LND}(B)$ denotes the set of all locally nilpotent derivations of B. Likewise, if A is a subring of B, then $\text{LND}_A(B)$ denotes the set of locally nilpotent derivations D of B with $D(A) = 0$. If $D \in \text{LND}(B)$, then $\exp D$ is an automorphism of B. A reference for locally nilpotent derivations is [7].

An important fact about locally nilpotent derivations which we need is the following.

Proposition 2.1 *Let B be a UFD of characteristic zero, and let $A \subset B$ be a subring such that $B^{[n]} = A^{[n+2]}$ for some $n \geq 0$. Then $\ker D = A^{[1]}$ for every non-zero $D \in \text{LND}_A(B)$.*

Proof. From $B^{[n]} = A^{[n+2]}$, it follows that A is a UFD and that $\text{tr.deg}_A(B) = 2$.

Let $D \in \text{LND}_A(B)$ be given, $D \neq 0$. As $\ker D$ is factorially closed in B, it is a UFD. We also have $\text{tr.deg}_{\ker D}(B) = 1$, so

$$A \subset \ker D \subset A^{[n+2]}$$

where A and $\ker D$ are UFDs and $\text{tr.deg}_A(\ker D) = 1$. It now follows from a classical result of Abhyankar, Eakin, and Heinzer that $\ker D = A^{[1]}$ ([1], Thm. 4.1). □

By *Prop. 2.1* and *Cor. 2.1*, we obtain:

Corollary 2.3 *Let A and B be polynomial rings over a field k of characteristic 0 such that $A \subset B$, and B is an $\mathbb{A}^2$-fibration over A. Then:*

(i) $\ker D = A^{[1]}$ *for every non-zero $D \in \mathrm{LND}_A(B)$.*

(ii) $B = A^{[2]}$ *if and only if there exists $D \in \mathrm{LND}_A(B)$ with a slice.*

Remark 2.1 The corollary above is of interest since when B is a polynomial ring, the kernel of D is not a polynomial ring for most $D \in \mathrm{LND}(B)$. So the fact that $\ker D = A^{[1]} = k^{[d-1]}$, $d = \dim_k B$, for all non-zero $D \in \mathrm{LND}_A(B)$ when A is a polynomial ring means that these subrings are quite special.

3 A Criterion for Affine Fibrations

Lemma 3.1 *Consider a triple (S, B, x) and an $m \in \mathbb{N}$ satisfying:*

(i) *B is a domain, S is a subring of B, and $x \in B$ is transcendental over S;*

(ii) *$S \cap xB = 0$;*

(iii) *B_x is an $\mathbb{A}^m$-prefibration over A_x, where $A = S[x]$;*

(iv) *$\bar{B}$ is an $\mathbb{A}^m$-prefibration over $\bar{A}$, where $\bar{B} = B/xB$, and $\bar{A} \subseteq \bar{B}$ is the image of A via the canonical epimorphism $B \to \bar{B}$.*

Then B is an $\mathbb{A}^m$-prefibration over A.

Proof. Let $\mathfrak{p} \in \operatorname{Spec} A$ and consider the fiber of $f : \operatorname{Spec} B \to \operatorname{Spec} A$ over $\mathfrak{p}$.

If $x \notin \mathfrak{p}$, let $\mathfrak{q} = \mathfrak{p}A_x \in \operatorname{Spec}(A_x)$. Then the fiber of f over $\mathfrak{p}$ is the same thing as that of $\operatorname{Spec}(B_x) \to \operatorname{Spec}(A_x)$ over $\mathfrak{q}$, and by (iii) this is $\mathbb{A}^m_{\kappa(\mathfrak{q})}$ $(= \mathbb{A}^m_{\kappa(\mathfrak{p})})$.

If $x \in \mathfrak{p}$, let $\mathfrak{q} = \mathfrak{p}\bar{A} \in \operatorname{Spec} \bar{A}$. As (ii) implies $A \cap xB = xA$, we may identify $A/xA \to B/xB = \bar{B}$ with $\bar{A} \to \bar{B}$ and consequently the fiber of f over $\mathfrak{p}$ is the same thing as that of $\operatorname{Spec} \bar{B} \to \operatorname{Spec} \bar{A}$ over $\mathfrak{q}$, which is $\mathbb{A}^m_{\kappa(\mathfrak{q})}$ $(= \mathbb{A}^m_{\kappa(\mathfrak{p})})$ by (iv). □

Remark 3.1 If $B_x = A_x^{[m]}$ and $\bar{B} = \bar{A}^{[m]}$ then conditions (iii) and (iv) are satisfied.

Proof of Prop. 1.1. Let $S = k[f_1, \ldots, f_r]$ and note that $A = S[x] = S^{[1]}$. We have $S \cap xB = 0$ and

$$B_x = A_x[v_1, \ldots, v_m] = A_x^{[m]} \quad \text{and} \quad \bar{B} = \bar{A}[z_1, \ldots, z_m] = \bar{A}^{[m]},$$

so (S, B, x) satisfies the hypothesis of *Lemma 3.1.* Consequently B is an $\mathbb{A}^m$-prefibration over A. As A, B are polynomial rings over a field, *Lemma 2.1* implies that B is an $\mathbb{A}^m$-fibration over A. This proves (i).

Part (ii) follows from *Cor. 2.1*; part (iii) follows from *Cor. 2.3*; and part (iv) follows from *Cor. 2.2.* □

Remark 3.2 The example of Vénéreau (details of which are discussed below) uses a subring $A = \mathbb{C}[x, f] \subset B = \mathbb{C}[x, y, z, u] = \mathbb{C}^{[4]}$ of the form hypothesized in the proposition. Vénéreau proved that for every $\gamma(x) \in \mathbb{C}[x]$, $B/(f - \gamma(x)) = \mathbb{C}[x]^{[2]}$. Item (iv) in the proposition is thus a generalization of Vénéreau's result.

Construction of Examples. As above, let $B = k[x, y_1, \ldots, y_r, z_1, \ldots, z_m] = k^{[r+m+1]}$. In addition, let $R = k[x, y_1, \ldots, y_r]$. Here are two ways to choose $v_1, \ldots, v_m \in B$ satisfying

$$R[v_1, ..., v_m]_x = B_x \ .$$

1. Let M be an $m \times m$ matrix with entries in R such that $\det M = x^n$ for some non-negative integer n. In particular, $M \in GL_m(R_x)$. Define $v_1, ..., v_m \in B$ by

$$\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = M \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} .$$

Then $B_x = R_x[z_1, ..., z_m] = R_x[v_1, ..., v_m]$.

2. For this construction, we need to assume that k has characteristic 0. Choose $D \in \mathrm{LND}_R(B_x)$ and consider the R_x-automorphism $\exp(D) : B_x \to B_x$, which we abbreviate $\alpha : B_x \to B_x$. Choose $i_1, \ldots, i_m \in \mathbb{Z}$ such that $\alpha(x^{i_j} z_j) \in B$ for all $j = 1, \ldots, m$ and define

$$v_j = \alpha(x^{i_j} z_j) \quad (1 \leq j \leq m).$$

Then

$$B_x = R[x^{i_1} z_1, \ldots, x^{i_m} z_m]_x = R[\alpha(x^{i_1} z_1), \ldots, \alpha(x^{i_m} z_m)]_x = R[v_1, \ldots, v_m]_x .$$

4 Dimension Four

In this section, assume k is any field of characteristic zero. We consider a family of $\mathbb{A}^2$-fibrations $\mathbb{A}^4 \to \mathbb{A}^2$ which includes the examples of Bhatwdekar-Dutta and Vénéreau .

Let $B = k[x, y, z, u]$ be a polynomial ring in four variables. We consider the set of polynomials in B of the form $p = yu + \lambda(x, z)$, where $\lambda = z^2 + r(x)z + s(x)$ for some $r, s \in k[x]$. Given $p = yu + \lambda(x, z)$ of this form, define $\theta \in \mathrm{LND}(B_x)$ by $\theta x = \theta y = 0$, $\theta z = x^{-1} y$, and $\theta u = -x^{-1} \lambda_z$ noting that $\theta p = 0$. Set

$$v = \exp(p\theta)(xz) = xz + yp \quad \text{and} \quad w = \exp(p\theta)(x^2 u) = x^2 u - xp\lambda_z - yp^2 .$$

In addition, given $n \geq 1$, define $f_n \in B$ by $f_n = y + x^n v$. Note that $f_n, v, w \in B$, and these depend on our choice of p. Let $\varphi_n(p) : \mathbb{A}^4 \to \mathbb{A}^2$ denote the morphism defined by the inclusion $k[x, f_n] \subset B$. It is easy to see that the hypothesis of *Prop. 1.1* is satisfied (with $r = 1$, $m = 2$), so $\varphi_n(p)$ is an $\mathbb{A}^2$-fibration over $\mathbb{A}^2$ and more precisely:

Corollary 4.1 *For any choice of p and n as above, if $A = k[x, f_n]$ then:*

(i) *B is an $\mathbb{A}^2$-fibration over A.*

(ii) *$B^{[s]} = A^{[s+2]}$ for some $s \in \mathbb{N}$.*

(iii) $\ker D = A^{[1]} = k^{[3]}$ *for every non-zero $D \in \mathrm{LND}_A(B)$.*

(iv) *If $a \in A$ is such that $A/aA = k^{[1]}$, then $B/aB = (A/aA)^{[2]} = k^{[3]}$.*

In addition:

Lemma 4.1 *$\varphi_n(p)$ is trivial for each $n \geq 3$.*

Proof. Assume $n \geq 3$, and define the derivation d of B by

$$d = \frac{\partial(x, \cdot, v, w)}{\partial(x, y, z, u)} .$$

Since $k(x)[y, v, w] = k(x)[y, z, u]$, it follows that d is locally nilpotent. And since $dx = dv = 0$, we have that $x^{n-3} vd$ is also locally nilpotent. In addition, we have by direct calculation that $dy = v_z w_u - v_u w_z = x^3$. Therefore, if $\beta = \exp(x^{n-3} vd)$, then β is an automorphism of B for which $\beta(x) = x$ and $\beta(y) = y + x^{n-3} vd(y) = y + x^n v = f_n$. Therefore, $B = k[x, f_n]^{[2]}$ when $n \geq 3$. □

Example 1: $p = yu + z^2 + z$. This choice of p yields the 1992 example of Bhatwadekar and Dutta [5] (Example 4.13). In particular, the authors work over a DVR (R, π) containing $\mathbb{Q}$, and they define $F \in R[X, Y, Z] = R^{[3]}$ by

$$F = (\pi Y^2)Z + Y + \pi Y(X + X^2) + \pi^2 X \ .$$

By the substitutions

$$\pi \to x \ , \ X \to z \ , \ Y \to y \ , \ Z \to u$$

we see that F becomes exactly the polynomial f_1 for this p, namely,

$$f_1 = y + x(xz + y(yu + z^2 + z)) \ .$$

The authors also list rational co-generators G and H, which under these substitutions become $G \to v$ and $H \to w$. Here, R should be viewed as the localization of $k[x]$ at the prime ideal defined by x. The authors ask (Question 4.14) if $R[X, Y, Z] = R[F]^{[2]}$, and this is equivalent to the question whether $\varphi_1(p)$ is trivial. This question is still open.

Example 2: $p = yu + z^2$. This choice of p yields the 2001 examples of Vénéreau [17]. The polynomials f_n defined using $p = yu + z^2$ are called the *Vénéreau polynomials*. Vénéreau showed that $\varphi_n(p)$ is trivial for $n \geq 3$, which is equivalent to the condition $B = k[x, f_n]^{[2]}$. Later, the second author showed that $\varphi_n(p)$ is stably trivial for all $n \geq 1$ [8]. It remains an open question whether $\varphi_1(p)$ or $\varphi_2(p)$ is trivial. However, Vénéreau proved for all $\gamma(x) \in k[x]$ that the quotient $B/(f_1 - \gamma(x))$ is x-isomorphic to $k[x]^{[2]}$. In particular, f_1 defines a hyperplane in $\mathbb{A}^4$, but the question as to whether f_1 (or f_2) is a variable of B remains open.

Question 1. Are the fibrations $\varphi_1(p) : \mathbb{A}^4 \to \mathbb{A}^2$ equivalent for $p = yu + z^2$ and $p = yu + z^2 + z$?

5 A Remark on Stable Variables

Let $B = k^{[n]}$ for a field k. If $f \in B$ is a variable of $B^{[q]}$, where $q \in \mathbb{N}$, we say that f is a *q-stable variable* of B (or simply a *stable variable* of B). It is not known whether every stable variable is a variable.

Example 3. Let $f \in B = k^{[n]}$, where k is of characteristic zero. By Prop. 3.20 of [7], if there exists $D \in \mathrm{LND}(B)$ such that $D(f) = 1$ then f is a 1-stable variable of B. Using this fact, one can show that if every 1-stable variable is a variable then the Cancellation Problem for affine spaces has an affirmative answer.

Example 4. Consider the situation of *Example 2*: Let $B = k[x, y, z, u] = k^{[4]}$, $p = yu + z^2$ and $n \in \{1, 2\}$. Define $f_n \in B$ as in Section 4 (Vénéreau polynomials) and let $A_n = k[x, f_n]$. By *Cor. 4.1* we have $B^{[q]} = A_n^{[q+2]}$ for some q, but in fact the second author showed in [8] that $B^{[1]} = A_n^{[3]}$. It follows in particular that f_n is a variable of $B^{[1]} = k^{[5]}$, i.e., f_n is a 1-stable variable of B. It is not known whether f_n is a variable of B, or whether there exists $D \in \mathrm{LND}(B)$ satisfying $D(f_n) = 1$.

We now explain how stable variables can be used to construct rings which are of interest in relation to the Cancellation Problem.

We begin with a general observation. Let k be a field and suppose that $R \subset S$ are k-algebras satisfying $S^{[q]} = R^{[q+m]}$. If $R \to R'$ is any k-homomorphism and if we define $S' = R' \otimes_R S$, then ${S'}^{[q]} = {R'}^{[q+m]}$; so in the case where $R' = k^{[r]}$, we have ${S'}^{[q]} = k^{[q+m+r]}$ and hence S' is a potential counterexample to the Cancellation Problem.

Now suppose that f is a q-stable variable of $B = k^{[n]}$ and let $R = k[f]$. As $B^{[q]} = R^{[n+q-1]}$, it follows that if $R \to R'$ is any k-homomorphism such that $R' = k^{[r]}$ for some r, then the algebra $B' = R' \otimes_R B$ satisfies ${B'}^{[q]} = k^{[n+q+r-1]}$.

For instance let $d \geq 2$ be an integer, W an indeterminate, $R' = k[W] = k^{[1]}$ and let $R \to R'$ be the k-homomorphism which maps f to W^d. This gives $R' \otimes_R B = B\left[\sqrt[d]{f}\right]$, so we obtain the following observation.

For a field k, suppose that f is a q-stable variable of $B = k^{[n]}$ and define $B_d = B\left[\sqrt[d]{f}\right]$. Then $B_d^{[q]} = k^{[n+q]}$ for every positive integer d.

In particular let $f_n \in B = k^{[4]}$ be as in *Example 2* and let $B_d = B\left[\sqrt[d]{f_n}\right]$ (with $n \in \{1,2\}$ and $d \geq 2$); then $B_d^{[1]} = k^{[5]}$ but we don't know whether B_d is $k^{[4]}$.

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